GENERALIZED UNI-SOFT INTERIOR IDEALS IN ORDERED SEMIGROUPS

R. KHAN*, A. KHAN, B. AHMAD AND R. GUL

Abstract. For all $M, N \in P(U)$ such that $M \subseteq N$, we first introduced the definitions of $(M,N)$-uni-soft ideals and $(M,N)$-uni-soft interior ideals of an ordered semigroup and studied them. When $M = \emptyset$ and $N = U$, we meet the ordinary soft ones. Then we proved that in regular and in intra-regular ordered semigroups the concept of $(M,N)$-uni-soft ideals and the $(M,N)$-uni-soft interior ideals coincide. Finally, we introduced $(M,N)$-uni-soft simple ordered semigroup and characterized the simple ordered semigroups in terms of $(M,N)$-uni-soft interior ideals.

1. Introduction

An ideal of a semigroup is a special subsemigroup satisfying certain conditions. The best way to know an algebraic structure is to begin with a special substructure of it. There are plenty of papers on ideals. After Zadeh’s introduction of fuzzy set in 1965 [20], the fuzzy sets have been used in the reconsideration of classical mathematics. For example, Meng and Guo [15] researched fuzzy ideals of BCK/BCI-algebras, Koguep [13] researched fuzzy ideals of hyperlattices, and Kehayopulu and Tsingelis [9] researched fuzzy interior ideals of ordered semigroups.

This inadequacy is removed by Molodtsov [16], by the invention of soft set theory in 1999. He introduced parameterization tools to tackle various uncertainties. Due to the beauty of parameterization tools,
several researchers attracted towards this direction. Many papers have been published in this regard. In [14], Maji et al. studied various operations on soft sets. Some new operations on soft sets have been introduced by Ali et al. in [2]. Aktas and Cagman [1], compared soft sets to the related concepts of fuzzy sets and rough sets. Also, Feng and Li [4], considered soft product operations. Jun et al., [6], applied the concept of soft set theory to ordered semigroups. Khan et al. [10, 11, 12], characterized different classes of ordered semigroups by using soft-union quasi-ideals and soft-union ideals.

In this paper, we introduced the concept of \((M, N)\)-uni-soft ideals and \((M, N)\)-uni-soft interior ideals of an ordered semigroup and studied them. We also proved that in regular and in intra-regular ordered semigroups the concept of \((M, N)\)-uni-soft ideals and the \((M, N)\)-uni-soft interior ideals coincide. Lastly, we introduced \((M, N)\)-uni-soft simple ordered semigroup and characterized the simple ordered semigroups in terms of \((M, N)\)-uni-soft interior ideals.

2. Basic definitions and preliminaries

An ordered semigroup \((S, \cdot, \leq)\) is a Poset \((S, \leq)\) equipped with a binary operation “\(\cdot\)” such that

1. \((S, \cdot)\) is a semigroup,
2. If \(x, a, b \in S\), then \(a \leq b \implies \begin{cases} xa \leq xb \\ ax \leq bx. \end{cases}\)

Let \((S, \cdot, \leq)\) be an ordered semigroup. For subsets \(A\) and \(B\) of an ordered semigroup \(S\), we denote

\[ AB := \{ab \mid a \in A, b \in B\}. \]

If \(A\) is a subset of \(S\), we denote by \((A)\) the subset of \(S\) defined as follows

\[ (A) := \{t \in S \mid t \leq h \text{ for some } h \in A\}. \]

For \(a \in S\), we write \((a)\) instead of \((\{a\})\). For subsets \(A\) and \(B\) of an ordered semigroup \(S\), we have \(A \subseteq (A)\). If \(A \subseteq B\), then \((A) \subseteq (B)\), \((A)(B) \subseteq (AB)\), \(((A)) = (A)\) and \(((A)(B)) \subseteq (AB)\).

Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \(A\) of \(S\) is called a subsemigroup of \(S\) if \(A^2 \subseteq A\).

A non-empty subset \(A\) of \(S\) is called a right (resp., left) ideal of \(S\) if:

1. \(AS \subseteq A\) (resp., \(SA \subseteq A\)) and
2. if \(a \in A\) and \(S \ni b \leq a\), then \(b \in A\).

If \(A\) is both a right and a left ideal of \(S\), then it is called an ideal of \(S\).

A non-empty subset \(A\) of \(S\) is called an interior ideal of \(S\) if:

1. \(SAS \subseteq A\)
(2) if \( a \in A \) and \( S \ni b \leq a \), then \( b \in A \).

An ordered semigroup \( S \) is said to be regular if for every \( x \in S \) there exist \( a \in S \) such that \( a \leq axa \).

An ordered semigroup \( S \) is said to be intra-regular if for all \( a \in S \) there exists \( x, y \in S \) such that \( a \leq xay \).

An ordered semigroup \( S \) is said to be left (resp., right) simple if it contains no proper left (resp., right) ideal.

An ordered semigroup \( S \) is said to be simple if it contains no proper two-sided ideal.

In the following, we assume that \( U \) is an initial universe set, \( E \) is a set of parameters, \( P(U) \) denotes the power set of \( U \) and \( A, B, C, \ldots \subseteq E \). And we will assume that \( \emptyset \subseteq \mathcal{M} \subseteq \mathcal{N} \subseteq U \).

A soft set theory is introduced by Molodstov [16], and Çağman [3] provided new definitions and various results on soft set theory.

**Definition 2.1.** [16, 3] A soft set \( f_A \) over \( U \) is defined to be the set of ordered pairs

\[
\left\{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\right\},
\]

where \( f_A : E \rightarrow P(U) \) such that \( f_A(x) = \emptyset \) if \( x \notin A \).

The function \( f_A \) is also called an approximation function.

It is clear from Definition 2.1, that a soft set is a parameterized family of subsets of \( U \). Note that the set of all soft sets over \( U \) will be denoted \( S(U) \).

Define an ordered relation \( \subseteq_{(\mathcal{M}, \mathcal{N})} \) on \( P(U) \) as follows:

For any \( f_A, f_B \in S(U) \), \( \emptyset \subseteq \mathcal{M} \subseteq \mathcal{N} \subseteq U \), we defined

\[
f_A \subseteq_{(\mathcal{M}, \mathcal{N})} f_B \iff f_A(x) \cap \mathcal{N} \subseteq f_B(x) \cup \mathcal{M},
\]

and we define a relation \( =_{(\mathcal{M}, \mathcal{N})} \) as follows:

\[
f_A =_{(\mathcal{M}, \mathcal{N})} f_B \iff f_A \subseteq_{(\mathcal{M}, \mathcal{N})} f_B \text{ and } f_B \subseteq_{(\mathcal{M}, \mathcal{N})} f_A.
\]

The soft union of \( f_A \) and \( f_B \), denoted by \( f_A \cup f_B = f_{A \cup B} \), is defined by

\[
(f_A \cup f_B)(x) = f_A(x) \cup f_B(x) \text{ for all } x \in E.
\]

The soft intersection of \( f_A \) and \( f_B \), denoted by \( f_A \cap f_B = f_{A \cap B} \), is defined by

\[
(f_A \cap f_B)(x) = f_A(x) \cap f_B(x) \text{ for all } x \in E.
\]

For a soft \( f_A \) over \( U \) and \( \delta \subseteq U \). The \( \delta \)-exclusive set of \( (f_A, S) \), denoted by \( e_A(f_A; \delta) \), is defined as

\[
e_A(f_A; \delta) = \{x \in L \mid f_A(x) \subseteq \delta\}
\]

For a non-empty subset \( A \) of \( S \), the characteristic soft set \( (x_A, S) \) over \( U \) is a soft set defined as follows:
χ_A : S → P(U), x ↦ \begin{cases} U, & \text{if } x \in A, \\ \emptyset, & \text{if } x \in S \setminus A. \end{cases}

For the characteristic soft set (χ_A, S) over U, the soft set (χ^c_A, S) over U is given as follows:

χ^c_A : S → P(U), x ↦ \begin{cases} \emptyset, & \text{if } x \in A, \\ U, & \text{if } x \in S \setminus A. \end{cases}

3. (M, N)-uni-soft interior ideals of ordered semigroups

In this section, we define (M, N)-uni-soft interior ideals of ordered semigroups and study their properties as regards soft set operations and soft uni-product. Also, it is shown that every (M, N)-uni-soft ideal is a (M, N)-uni-soft interior ideal.

Definition 3.1. Let (S, ⋅, ≤) be an ordered semigroup. A soft set (f_S, S) over U is called (M, N)-uni-soft left ideal over U if:

1. x ≤ y ⇒ f_S(x) ∩ N ⊆ f_S(y) ∪ M for all x, y ∈ S and
2. f_S(xy) ∩ N ⊆ f_S(y) ∪ M for all x, y ∈ S.

A soft set (f_S, S) over U is called (M, N)-uni-soft right ideal over U if:

1. x ≤ y ⇒ f_S(x) ∩ N ⊆ f_S(y) ∪ M for all x, y ∈ S and
2. f_S(xy) ∩ N ⊆ f_S(x) ∪ M for all x, y ∈ S.

A soft set f_S of S over U is called a (M, N)-uni-soft ideal of S over U if it is both a (M, N)-uni-soft left and a (M, N)-uni-soft right ideal of S over U.

Example 3.2. Let S = {e, a} be an ordered semigroup defined by the order relation e ≤ a with the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>e</td>
</tr>
</tbody>
</table>

Define a soft set (f_S, S) over U as follows:

f_S : S → P(U), x ↦ f_S(x) = \begin{cases} \gamma, & \text{if } x = e, \\ \gamma, & \text{if } x = a. \end{cases}

Where M ⊆ γ ⊆ N. Then f_S(xy) ∩ N = f_S(e) ∩ N = γ ∩ N = γ = γ ∪ M = f_S(e) ∪ M = f_S(y) ∪ M, for every x, y ∈ S. Therefore, (f_S, S) is a (M, N)-uni-soft ideal (resp., right) ideal over U.

Definition 3.3. Let (S, ⋅, ≤) be an ordered semigroup. A soft set (f_S, S) over U is called (M, N)-uni-soft interior ideal over U if:
Let a a a a a b a a b c d

Suppose that $M$ is an exclusive set of an interior ideal over $U$.

**Example 3.4.** [11] Let $S = \{a, b, c, d\}$ be an ordered semigroup with the following multiplication table and the ordered relation:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<tr>
<td>a</td>
<td>a</td>
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<td>a</td>
<td>a</td>
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<tr>
<td>b</td>
<td>a</td>
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<td>a</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>d</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, d)\}.$

Let $(f_S, S)$ be a soft set over $U = \mathbb{Z}$ defined by

$$f_S : S \rightarrow P(U), \ x \mapsto f_S(x) = \begin{cases} 6\mathbb{N} & \text{if } x = a, \\ 3\mathbb{Z} & \text{if } x \in \{b, d\}, \\ 3\mathbb{N} & \text{if } x = c. \end{cases}$$

Where $\mathcal{M} \subseteq 6\mathbb{N} \subset 3\mathbb{N} \subset 3\mathbb{Z} \subseteq \mathcal{N}$. Then $(f_S, S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over $U$.

**Theorem 3.5.** Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $(f_s, S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of $S$ over $U$ if and only if the $\delta$-exclusive set of $(f_S, S)$ is an interior ideal of $S$ for all $\delta \in P(U)$, where $\mathcal{M} \subseteq \delta \subseteq \mathcal{N}$.

**Proof.** Suppose that $(f_S, S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of $S$ over $U$ and $\delta \in P(U)$. First, we need to show that $xay \in e_S(f_S; \delta)$, for all $a \in e_S(f_S; \delta)$, $x, y \in S$. By hypothesis, we have $f_S(xay) \cap \mathcal{N} \subseteq f_S(a) \cup \mathcal{M} \subseteq \delta \cup \mathcal{M} = \delta$ and $\delta \subseteq \mathcal{N}$, which means that $f_S(xay) \subseteq \delta$, that is $xay \in e_S(f_S; \delta)$. Now, for all $x \in S$ and $y \in e_S(f_S; \delta)$ such that $x \leq y$, since $(f_S, S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of $S$ over $U$, so from $x \leq y$ we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \subseteq \delta \cup \mathcal{M} = \delta$, we conclude that $f_S(y) \subseteq \delta$, that is $y \in e_S(f_S; \delta)$. Hence the $\delta$-exclusive set of $(f_S, S)$ is an interior ideal of $S$ for all $\delta \in P(U)$ where $\mathcal{M} \subseteq \delta \subseteq \mathcal{N}$.

Conversely, assume that the $\delta$-exclusive set of $(f_S, S)$ is an interior ideal of $S$ for all $\delta \in P(U)$ where $\mathcal{M} \subseteq \delta \subseteq \mathcal{N}$ and $a, x, y \in S$. Let $f_S(xay) \cap \mathcal{N} \supset \delta = f_S(a) \cup \mathcal{M}$ for $\delta \in P(U)$. It follows that $f_S(a) \subseteq \delta$ and $f_S(xay) \supset \delta$, that is $a \in e_S(f_S; \delta)$ and $xay \notin e_S(f_S; \delta)$. Which is a contradiction to the fact that $e_S(f_S; \delta)$ is an interior ideal of $S$. Hence $f_S(xay) \cap \mathcal{N} \subseteq f_S(a) \cup \mathcal{M}$ for all $a, x, y \in S$. If there are $x, y \in S$...
such that \(x \leq y\). Let \(f_S(x) \cap N \supset \delta = f_S(y) \cup M\) then \(\delta \subseteq P(U)\), which means that \(f_S(y) \subseteq \delta\) and \(f_S(x) \supset \delta\), that is \(y \in e_S(f_S; \delta)\) and \(x \notin e_S(f_S; \delta)\). Which is again a contradiction to the fact that \(e_S(f_S; \delta)\) is an interior ideal of \(S\). Hence if \(x \leq y\), then \(f_S(x) \cap N \subseteq f_S(y) \cup M\), for all \(x, y \in S\). Therefore \((f_S, S)\) is a \((M, N)\)-uni-soft interior ideal of \(S\) over \(U\).

Lemma 3.6. Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \(I\) of \(S\) is an ideal of \(S\) if and only if the characteristic function \((\chi_I, S)\) is a \((M, N)\)-uni-soft ideal over \(U\).

Proof. It follows from Theorem 3.5. \(\square\)

Theorem 3.7. Let \((S, \cdot, \leq)\) be an ordered semigroup. Then the soft union of two \((M, N)\)-uni-soft interior (resp., left, right) ideals over \(U\) is also a \((M, N)\)-uni-soft interior (resp., left, right) ideal over \(U\).

Proof. Let \((f_S, S)\) and \((g_S, S)\) be \((M, N)\)-uni-soft interior ideals over \(U\). For any \(x, a, y \in S\), we have

\[
(f_S \cup g_S)(xy) \cap N = (f_S(xy) \cup g_S(xy)) \cap N \\
= (f_S(xy) \cap N) \cup (g_S(xy) \cap N) \\
\subseteq (f_S(a) \cup M) \cup (g_S(a) \cup M) \\
= (f_S(a) \cup g_S(a)) \cup M \\
= (f_S \cup g_S)(a) \cup M.
\]

Furthermore, let \(x, y \in S\) such that \(x \leq y\). Since \((f_S, S)\) and \((g_S, S)\) are \((M, N)\)-uni-soft interior ideals over \(U\), we have

\[
(f_S \cup g_S)(x) \cap N = (f_S(x) \cup g_S(x)) \cap N \\
= (f_S(y) \cap N) \cup (g_S(y) \cap N) \\
\subseteq (f_S(y) \cup M) \cup (g_S(y) \cup N) \\
= (f_S(y) \cup g_S(y)) \cup M \\
= (f_S \cup g_S)(y) \cup M.
\]

Therefore \((f_S \cup g_S, S)\) is a \((M, N)\)-uni-soft ideal over \(U\). In a similar way, \((f_S \cup g_S, S)\) is a \((M, N)\)-uni-soft left (resp., right) ideal over \(U\). \(\square\)

Let \((S, \cdot, \leq)\) and \((T, \cdot, \leq)\) be two ordered semigroups. Under the coordinatewise multiplication, i.e.,

\[ (x, a)(y, b) = (xy, ab) \]

where \((x, a), (y, b) \in S \times T\), the Cartesian product

\[ S \times T = \{(x, a) \mid x \in S, a \in T\} \]
is a semigroup. Define a partial order $\leq$ on $S \times T$ by

$$(x, a) \leq (y, b) \text{ if and only if } x \leq y \text{ and } a \leq b,$$

where $(x, a), (y, b) \in S \times T$. Then, $(S \times T, \cdot, \leq)$ is an ordered semigroup.

For uni-soft sets $(f_S, S)$ and $(f_T, T)$ over $U$, we consider a uni-soft set $(f_{S\cup T}, S \times T)$ over $U$ in which $f_{S\cup T}$ is defined as follows:

$$f_{S\cup T} : S \times T \rightarrow P(U), \quad (x, a) \mapsto f_S(x) \cup f_T(a).$$

**Theorem 3.8.** Let $(S, \cdot, \leq)$ be an ordered semigroup. If $(f_S, S)$ and $(f_T, T)$ are $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideals over $U$, then $(f_{S\cup T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior (resp., left, right) ideal over $U$.

**Proof.** Let $(x, a), (y, b), (z, c) \in S \times T$. Then

$$f_{S\cup T}((x, a), (y, b), (z, c)) \cap \mathcal{N} = f_{S\cup T}(xyz, abc) \cap \mathcal{N} = (f_S(xyz) \cup f_T(abc)) \cap \mathcal{N} = (f_S(xyz) \cap \mathcal{N}) \cup (f_T(abc) \cap \mathcal{N}) \quad (#).$$

Since $(f_S, S)$ and $(f_T, T)$ are $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over $U$, we have $f_S(xyz) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$ and $f_T(abc) \cap \mathcal{N} \subseteq f_T(b) \cup \mathcal{M}$. Hence from equation (#) we have

$$(f_S(xyz) \cap \mathcal{N}) \cup (f_T(abc) \cap \mathcal{N}) \subseteq (f_S(y) \cup \mathcal{M}) \cup (f_T(b) \cup \mathcal{M}) = (f_S(y) \cup f_T(b)) \cup \mathcal{M} = f_{S\cup T}(y, b) \cup \mathcal{M}.$$

Furthermore, let $(x, a), (y, b) \in S \times T$ be such that $(x, a) \leq (y, b)$. Then

$$f_{S\cup T}(x, a) \cap \mathcal{N} = (f_S(x) \cup f_T(a)) \cap \mathcal{N} = (f_S(x) \cap \mathcal{N}) \cup (f_T(a) \cap \mathcal{N}) \subseteq (f_S(y) \cup \mathcal{M}) \cup (f_T(b) \cup \mathcal{M}) = (f_S(y) \cup f_T(b)) \cup \mathcal{M} = f_{S\cup T}(y, b) \cup \mathcal{M}.$$

Therefore $(f_{S\cup T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over $U$. Similarly, we show that if $(f_S, S)$ and $(f_T, T)$ are $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideals over $U$, then $(f_{S\cup T}, S \times T)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft left (resp., right) ideal over $U$. \qed

**Theorem 3.9.** Let $\varphi : S \rightarrow T$ be a homomorphism of an ordered semigroup. If $(f_S, S)$ is a uni-soft interior (resp., left, right) ideal over
Let \( \varphi^{-1}(f_S), S \) of \((f_S, T)\) under \( \varphi \) is a uni-soft interior (resp., left, right) ideal over \( U \), where \( \varphi^{-1}(f_S) \) is given as follows:

\[
\varphi^{-1}(f_S): S \rightarrow P(U), \ x \mapsto f_S(\varphi(x)).
\]

**Proof.** Let \((S, \cdot, \leq)\) be an ordered semigroup and \( \varphi: S \rightarrow T \) be a homomorphism. Let \( x, y \in S \) and \( x \leq y \). Since \( \varphi \) is a homomorphism of ordered semigroups from \( S \) to \( T \), we have \( \varphi(x) \leq \varphi(y) \). Since \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \( U \), we have \( f_S(\varphi(x)) \cap \mathcal{N} \subseteq f_S(\varphi(y)) \cup \mathcal{M} \). Hence

\[
\varphi^{-1}(f_S)(x) \cap \mathcal{N} = f_S(\varphi(x)) \cap \mathcal{N} \subseteq f_S(\varphi(y)) \cup \mathcal{M} = \varphi^{-1}(f_S)(y) \cup \mathcal{M}.
\]

Furthermore, for any \( x, y, z \in S \), we have

\[
\varphi^{-1}(f_S)(xyz) \cap \mathcal{N} = f_S(\varphi(xyz)) \cap \mathcal{N} = f_S(\varphi(x)\varphi(y)\varphi(z)) \cap \mathcal{N} \subseteq f_S(\varphi(y)) \cup \mathcal{M} = \varphi^{-1}(f_S)(y) \cup \mathcal{M}.
\]

Therefore \((\varphi^{-1}(f_S), S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \( U \). Similarly, we can show that \((\varphi^{-1}(f_S), S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft left (resp., right) ideal over \( U \).

**Lemma 3.10.** Let \((S, \cdot, \leq)\) be an ordered semigroup. Then every \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \( U \) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \( U \).

**Proof.** Let \( x, y, a \in S \). Since \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft right ideal over \( U \), we have,

\[
f_S(xay) \cap \mathcal{N} = f_S((xa)y) \cap \mathcal{N} \subseteq f_S(xa) \cup \mathcal{M}, \quad (1)
\]

and since \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft right ideal over \( U \), we have

\[
f_S(xay) \cap \mathcal{N} = f_S(x(ay)) \cap \mathcal{N} \subseteq f_S(ay) \cup \mathcal{M}. \quad (2)
\]

From (1) and (2), we get

\[
f_S(xay) \cap \mathcal{N} = f_S(x(ay) \cap \mathcal{N}) \cap \mathcal{N} \subseteq (f_S(ay) \cup \mathcal{M}) \cap \mathcal{N} = (f_S(ay) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \subseteq (f_S(a) \cup \mathcal{M}) \cup \mathcal{M} = f_S(a) \cup \mathcal{M}.
\]

Let \( x, y \in S \) such that \( x \leq y \). Then \( f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \), because \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \( U \). Thus \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \( U \).

The following example shows that the converse of the Lemma 3.10, is not true in general.
Example 3.11. In Example 3.4, soft set \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\). But it is not a uni-soft left ideal over \(U\), since 
\[ f_S(dc) \cap \mathcal{N} = f_S(b) \cap \mathcal{N} = 3\mathbb{Z} \cap \mathcal{N} = 3\mathbb{Z} \not\subseteq 3\mathbb{N} = 3\mathbb{N} \cup \mathcal{M} = f_S(c) \cup \mathcal{M} , \]
for every \(c, d \in S\), and hence it is not a uni-soft two-sided ideal over \(U\).

4. \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideals of regular/intra-regular ordered semigroups

In this section, we prove that in regular and in intra-regular ordered semigroups, the concepts of \((\mathcal{M}, \mathcal{N})\)-uni-soft ideals and \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideals coincide.

Theorem 4.1. Let \((S, \cdot, \leq)\) be a regular ordered semigroup. Then every \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \(U\).

Proof. Let \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\) and let \(x, y \in S\). Since \(S\) is a regular, then there exist \(a, b \in S\) such that 
\[ x \leq xax \quad \text{and} \quad y \leq yby. \]
We have
\[ f_S(xy) \cap \mathcal{N} \subseteq (f_S((xax) y) \cup \mathcal{M}) \cap \mathcal{N} \]
\[ = (f_S((xa) xy) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \]
\[ \subseteq (f_S (x) \cup \mathcal{M}) \cup \mathcal{M} \]
\[ = f_S (x) \cup \mathcal{M}, \]
and
\[ f_S(xy) \cap \mathcal{N} \subseteq (f_S(x(yby)) \cup \mathcal{M}) \cap \mathcal{N} \]
\[ = (f_S(x(by)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \]
\[ \subseteq (f_S (y) \cup \mathcal{M}) \cup \mathcal{M} \]
\[ = f_S (y) \cup \mathcal{M}. \]
Now, let \(x, y \in S\) such that \(x \leq y\). Then \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\), because \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal of \(S\) over \(U\). Therefore \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \(U\). \(\Box\)

By Lemma 3.10 and Theorem 4.1, we have the following:

Remark 4.2. In regular ordered semigroups the concepts of \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideals and \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideals coincide.

Theorem 4.3. Let \((S, \cdot, \leq)\) be an intra-regular ordered semigroup and \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\). Then \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \(U\).
Proof. Let \((f_S, S)\) be a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\). Let \(x, y \in S\), since \(S\) is a intra-regular then there exists \(a, b \in S\) such that \(x \leq ax^2a\) and \(y \leq by^2b\). Since \((f_S, S)\) is an uni-soft interior ideal of \(S\), we have

\[
\begin{align*}
  f_S(xy) \cap \mathcal{N} & \subseteq (f_S((ax^2a) y) \cup \mathcal{M}) \cap \mathcal{N} \\
  &= (f_S((ax) x (ay)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\
  &\subseteq (f_S(x) \cup \mathcal{M}) \cup \mathcal{M} \\
  &= f_S(x) \cup \mathcal{M},
\end{align*}
\]

and

\[
\begin{align*}
  f_S(xy) \cap \mathcal{N} & \subseteq (f_S(by^2b) \cup \mathcal{M}) \cap \mathcal{N} \\
  &= (f_S((xb) y (yb)) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\
  &\subseteq (f_S(y) \cup \mathcal{M}) \cup \mathcal{M} \\
  &= f_S(y) \cup \mathcal{M}.
\end{align*}
\]

Let \(x, y \in S\) be such that \(x \leq y\). Then \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\), because \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal of \(S\) over \(U\). Thus \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal of \(S\) over \(U\).

By Lemma 3.10 and Theorem 4.3, we have the following:

Remark 4.4. In intra regular ordered semigroups, the concepts of \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideals and \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideals coincide.

Theorem 4.5. If \(S\) is a monoid with identity \(e\). Then every \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two sided ideal over \(U\).

Proof. Let \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \(U\) and \(x, y \in S\). Then \(f_S(xy) \cap \mathcal{N} = f_S(xye) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\) and \(f_S(xy) \cap \mathcal{N} = f_S(exy) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}\). Furthermore, let \(x, y \in S\) be such that \(x \leq y\). Then \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\), because \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal of \(S\) over \(U\). Therefore \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft two-sided ideal over \(U\).

5. \((\mathcal{M}, \mathcal{N})\)-uni-soft simple ordered semigroups

In this section, we define \((\mathcal{M}, \mathcal{N})\)-uni-soft simple ordered semigroups and characterize simple ordered semigroups in terms of \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideals.

Definition 5.1. An ordered semigroup \(S\) is called \((\mathcal{M}, \mathcal{N})\)-uni-soft simple if for any \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal of \(S\), we have \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\), for all \(x, y \in S\).
Theorem 5.2. Let \((S, \leq)\) be an ordered semigroup. Then \(S\) is \((\mathcal{M}, \mathcal{N})\)-uni-soft simple if and only if for any \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal \((f_S, S)\) of \(S\), if \(e_S(f_S, S) \neq \emptyset\), then \(e_S(f_S, \delta) = S\), for all \(\delta \in P(U)\) where \(\mathcal{M} \subseteq \delta \subseteq \mathcal{N}\).

Proof. Let \((f_S, S)\) be a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal of \(S\) over \(U\), and \(e_S(f_S, \delta) \neq \emptyset\). We need to show that \(x \in e_S(f_S, \delta)\) for all \(x \in S\), where \(\mathcal{M} \subseteq \delta \subseteq \mathcal{N}\). Since \(e_S(f_S, \delta) \neq \emptyset\), then there exists \(y \in e_S(f_S, \delta)\), that is \(f_S(y) \subseteq \delta\). By hypothesis we have,

\[
f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \subseteq \delta \cup \mathcal{M} = \delta.
\]

Notice that \(\delta \subseteq \mathcal{N}\), which means that \(f_S(x) \subseteq \delta\), that is \(x \in e_S(f_S, \delta)\).

Conversely, suppose that for any \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal \((f_S, S)\) over \(U\), we have \(e_S(f_S, \delta) = S\) for all \(\mathcal{M} \subseteq \delta \subseteq \mathcal{N}\). We need to show that \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\) for all \(x, y \in S\). Let if

\[
f_S(x) \cap \mathcal{N} \supset \delta = f_S(y) \cup \mathcal{M}.
\]

Then \(f_S(y) \subseteq \delta\) and \(f_S(x) \supset \delta\), which means that \(x \notin e_S(f_S, \delta) = S\), a contradiction. So \(f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}\) for all \(x, y \in S\). \(\square\)

Theorem 5.3. Let \((S, \cdot, \leq)\) be an ordered semigroup and soft set \((f_S, S)\) a \((\mathcal{M}, \mathcal{N})\)-uni-soft right (resp., left) ideal over \(U\). Then \(I_x = \{y \in S \mid f_S(y) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M}\}\) is a right (resp., left) ideal of \(S\), for all \(x \in S\).

Proof. Let \(x \in S\). Then \(I_x \neq \emptyset\) since \(x \in I_x\). Suppose that \(y \in I_x\) and \(s \in S\), Then \(ys \in I_x\). Since \((f_S, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal over \(U\) and \(y, s \in S\), we have

\[
f_S(ys) \cap \mathcal{N} = f_S(ys) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \cap \mathcal{N} \subseteq f_S(y) \cap \mathcal{N} \cup (\mathcal{M} \cap \mathcal{N}) \subseteq f_S(x) \cup \mathcal{M} \cup \mathcal{M} \subseteq f_S(x) \cup \mathcal{M}.
\]

Hence \(ys \in I_x\).
Furthermore, let \( s, y \in S \) be such that \( s \leq y \). If \( y \in I_x \), then \( s \in I_x \). Since \( (f_S, S) \) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal over \( U \) and \( s \leq y \), we have
\[
\begin{align*}
  f_S(s) \cap \mathcal{N} &= (f_S(s) \cap \mathcal{N}) \cap \mathcal{N} \\
  &\subseteq (f_S(y) \cup \mathcal{M}) \cap \mathcal{N} \\
  &= (f_S(y) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \\
  &\subseteq (f_S(x) \cup \mathcal{M}) \cup \mathcal{M} \quad \text{since } y \in I_x \\
  &= f_S(x) \cup \mathcal{M}.
\end{align*}
\]
So \( s \in I_x \). Therefore, \( I_x \) is a right ideal of \( S \), for all \( x \in S \). \( \square \)

By left right dual of Theorem 5.3, we have the following result.

**Theorem 5.4.** Let \((S, \cdot, \leq)\) be an ordered semigroup and soft set \((f_S, S)\) a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal over \( U \). Then for all \( x \in S \), the set
\[
I_x = \{ y \in S \mid f_S(y) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M} \}
\]
is a right ideal of \( S \).

**Theorem 5.5.** An ordered semigroup \( S \) is simple if and only if it is \((\mathcal{M}, \mathcal{N})\)-uni-soft simple.

**Proof.** Assume that \( S \) is simple. Let \((f_S, S)\) be a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal over \( U \) and \( x, y \in S \). By Theorem 5.4, the set \( I_x \) is an ideal of \( S \). Since \( S \) is simple, we have \( I_x = S \). Then \( b \in I_x \), from which we have that \( f_S(y) \cap \mathcal{N} \subseteq f_S(x) \cup \mathcal{M} \). Thus \( S \) is \((\mathcal{M}, \mathcal{N})\)-uni-soft simple.

Conversely, suppose \( S \) contains proper ideals and let \( I \) be such ideal of \( S \). By Lemma 3.6, we know that \((\chi_I^f, S)\) is a \((\mathcal{M}, \mathcal{N})\)-uni-soft ideal of \( S \). We have that \( S \subseteq I \). Indeed, let \( x \in S \). Since \( S \) is \((\mathcal{M}, \mathcal{N})\)-uni-soft simple, \( \chi_I^f(x) \cap \mathcal{N} \subseteq \chi_I^f(y) \cup \mathcal{M} \) for all \( y \in S \). Now, let \( z \in I \). Then we have
\[
\chi_I^f(x) \cap \mathcal{N} \subseteq \chi_I^f(z) \cup \mathcal{M} = U \cup \mathcal{M} = U.
\]
Notice that \( \mathcal{M} \subseteq \mathcal{N} \), we conclude that \( \chi_I^f(x) \subseteq U \), which implies that \( \chi_I^f(x) = U \), that is \( x \in I \). Thus we have that \( S \subseteq I \), and so \( S = I \). We get a contradiction to hypothesis that \( S \) contains proper ideals. \( \square \)

**Lemma 5.6.** [8, 9] An ordered semigroup \( S \) is simple if and only if for every \( a \in S \), we have \( S = (SaS) \).

**Theorem 5.7.** Let \((S, \cdot, \leq)\) be an ordered semigroup. Then \( S \) is simple if and only if for every \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal \((f_S, S)\) of \( S \), we have \( f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M} \), for all \( x, y \in S \).

**Proof.** Suppose \( S \) is simple. Let \((f_S, S)\) be a \((\mathcal{M}, \mathcal{N})\)-uni-soft interior ideal over \( U \) and \( x, y \in S \). Since \( S \) is simple and \( y \in S \), by Lemma 5.6, we have \( S = (SyS) \). Since \( x \in S \), we have \( x \in (SyS) \), then \( x \leq ayb \) for
some $a, b \in S$. Since $(f_S, S)$ is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal over $U$, we have

$$f_S(x) \cap \mathcal{N} = (f_S(x) \cap \mathcal{N}) \cap \mathcal{N} \subseteq (f_S(ayb) \cup \mathcal{M}) \cap \mathcal{N} = (f_S(ayb) \cap \mathcal{N}) \cup (\mathcal{M} \cap \mathcal{N}) \subseteq (f_S(y) \cup \mathcal{M}) \cup \mathcal{M} = f_S(y) \cup \mathcal{M}.$$ 

Conversely, suppose that for every $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal $(f_S, S)$ over $U$, we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$. Now let $(f_S, S)$ be any $(\mathcal{M}, \mathcal{N})$-uni-soft ideal of $S$ over $U$, then it is a $(\mathcal{M}, \mathcal{N})$-uni-soft interior ideal of $S$ over $U$. So we have $f_S(x) \cap \mathcal{N} \subseteq f_S(y) \cup \mathcal{M}$, for all $x, y \in S$. Thus $S$ is $(\mathcal{M}, \mathcal{N})$-uni-soft simple, then by Theorem 5.5, we conclude that $S$ is simple.

\[\square\]

**References**


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GENERALIZED UNI-SOFT INTERIOR IDEALS IN ORDERED SEMIGROUPS

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اﯾﺪه آل ﻫﺎیداﺧﻠﯽ اﺟﺘﻤﺎع-نرم تعمیم‌یافته در ﻧﯿﮕروههای مرتب

برای هر (M, N) اجتماع-نرم و ایدآل‌های داخلي (M, N) مفهوم ایدآل های مرتب را معرفی کرده و آن‌ها را مورد مطالعه قرار می‌دهد. در حالت خاص که N = 0 و M = P(U) مفاهیم فوق با مفاهیم موجود قبلی ایدآل‌نرم و ایدآل داخلي نرم، سازگار می‌باشند. سپس ثابت می‌کنیم که در ﻧﯿﮕروههای مرتب ﻧﯿﮕروههای مرتب درون-منتظم، مفاهیم تعریف شده فوق برهم منطبق می‌باشند. در انتهای، ضمن معرفی مفهوم ﻧﯿﮕروه مرتب ساده بر حسب ایدآل‌های داخلي (M, N) اجتماع-نرم می‌پردازیم.

کلمات کلیدی: مجموعه نرم، ایدآل‌های (M, N) اجتماع-نرم، ﻧﯿﮕروههای مرتب درون-منتظم، ایدآل‌های داخلي (M, N) اجتماع-نرم، ﻧﯿﮕروههای مرتب-نرم، ﻧﯿﮕروههای مرتب ساده.