Journal of Algebraic Systems
Vol. 7, No. 2, (2020), pp 105-130

SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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ABSTRACT. In this article we consider groups G such that G admits at least one nontrivial subgroup (recall that a subgroup H of G is said to be nontrivial if $H \notin \{G, \{e\}\}$). Let G be a group. Recall that the intersection graph of subgroups of G, denoted by $\Gamma(G)$, is an undirected graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. Let G be a finite group. The aim of this article is to investigate the interplay between the group-theoretic properties of a finite group G and the graph-theoretic properties of the complement of $\Gamma(G)$.

1. Introduction

Let G be a group which admits at least one nontrivial subgroup. Recall that the *intersection graph of* G, denoted by $\Gamma(G)$ is an undirected simple graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. The intersection graphs of groups have been investigated by several algebraists (for example, refer the articles [1, 4, 7, 8, 9, 11, 12]). Let $G = (V, E)$ be a simple graph. Recall from [2, Definition 1.1.13] that the *complement of* G, denoted by G^c is a graph whose vertex set is V and distinct vertices u, v are joined by an

MSC(2010): Primary: 20D99; Secondary: 05C25, 05C12.
Keywords: Complement of the intersection graph of subgroups of a finite group, finite abelian group, connected graph, girth of a graph.
Received: 19 June 2017, Accepted: 26 December 2018.
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edge in G^c if and only if there is no edge joining u and v in G. Thus for a group G which admits at least one nontrivial subgroup, $(\Gamma(G))^c$ is a graph whose vertex set is the set of all nontrivial subgroups of G and distinct vertices H, K are joined by an edge in $(\Gamma(G))^c$ if and only if $H \cap K = \{e\}$. The groups considered in this article are finite which admit at least one nontrivial subgroup. Let G be a finite group. The purpose of this article is to investigate the effect of certain graph parameters of $(\Gamma(G))^c$ on the group structure of G.

It is useful to recall the following definitions and results from graph theory before we give an account of results that are proved on $(\Gamma(G))^c$, where G is a finite group which admits at least one nontrivial subgroup. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. Recall from [2] that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b if such a path exists in G. Otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. A graph $G = (V, E)$ is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Let $G = (V, E)$ be a connected graph. Recall from [2, Definition 4.2.1] that the *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b) : a, b \in V\}$. Let $a \in V$. The *eccentricity* of a, denoted by $e(a)$ is defined as $e(a) = sup\{d(a, b) : b \in V\}$. The *radius* of G, denoted by $r(G)$ is defined as $r(G) = min\{e(a) : a \in V\}$.

Let $G = (V, E)$ be a graph. Suppose that G contains a cycle. Recall from [2, p. 159] that the *girth* of G, denoted by $girth(G)$ is the length of a shortest cycle in G. If G does not contain any cycle, then we set $girth(G) = \infty$. A complete graph on n vertices is denoted by K_n. Recall from [2, Definition 1.2.2] that a *clique* of G is a complete subgraph of G. Let $G = (V, E)$ be a simple graph. Suppose that there exists $k \in \mathbb{N}$ such that any clique of G is a clique on at most k vertices. Then the *clique number* of G, denoted by $\omega(G)$ is defined as the largest positive integer n such that G contains a clique on n vertices. If G contains a clique on n vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from [2, p.129] that a *vertex coloring* of G is a mapping $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper* if adjacent vertices of G receive distinct colors of S; that is, if u and v are adjacent in G, then $f(u) \neq f(v)$. The *chromatic number* of G, denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. It is clear that for any graph G, $\omega(G) \leq \chi(G)$.

Let G be a group. Recall that a nontrivial subgroup H of G is said to be a *minimal subgroup* of G if there is no nontrivial subgroup K of

G such that K is properly contained in H. A nontrivial subgroup H of G is said to be a *maximal subgroup* of G if there is no nontrivial subgroup K of G such that H is properly contained in K. If G is a finite group with at least one nontrivial subgroup, then it is clear that G admits at least one minimal (respectively, one maximal) subgroup of G. Let G be a finite group with at least one nontrivial subgroup. Let $\mathcal{C} = \{H : H \text{ is a minimal subgroup of } G\}$. As in [9], we denote the subgroup of G generated by $\cup_{H\in\mathcal{C}} H$ by N_G. In Section 2 of this article, we discuss some results regarding the connectedness of $(\Gamma(G))^c$. Let G be a finite group with at least two nontrivial subgroups. It is shown in Proposition 2.1 that $(\Gamma(G))^c$ is connected if and only if $N_G = G$. And in the case $(\Gamma(G))^c$ is connected, it is verified in Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$. In Lemma 2.5 and Proposition 2.6, we characterize finite groups G which admit at least two nontrivial subgroups such that $(\Gamma(G))^c$ is complete. Let G be a finite abelian group which admits at least two nonrivial subgroups. With the help of fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] and Proposition 2.1, we are able to determine the structure of finite abelian groups G such that $(\Gamma(G))^c$ is connected (see Propositions 2.8 and 2.9). Moreover, in the case when $(\Gamma(G))^c$ is connected, we characterize finite abelian groups G such that $diam((\Gamma(G))^c) = 1, 2$ or 3 (see Propositions 2.8 and 2.11). Furthermore, in the case when $(\Gamma(G))^c$ is connected, we determine $r((\Gamma(G))^c)$ (see Proposition 2.8 and Remark 2.13).

Let $n \geq 3$ and let S_n denote the *symmetric group of degree* n. With the help of Proposition 2.1, it is verified in Proposition 2.14 that $(\Gamma(S_n))^c$ is connected. Moreover, it is shown that $diam((\Gamma(S_3))^c) = 1$ and for $n \geq 4$, it is proved that $diam((\Gamma(S_n))^c) = r((\Gamma(S_n))^c) = 2$ (see Proposition 2.14 and Remark 2.15). Let $n \geq 4$ and let A_n denote the *alternating group of degree* n. It is shown in Proposition 2.17 that $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$. It is observed in Proposition 2.18(i) that $r((\Gamma(A_4))^c) = 1$ and for any $n \geq 5$, it is shown in Proposition 2.18(ii) that H is any minimal subgroup of A_n with either $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod 4)$, then $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Let $n \geq 3$ and let D_n denote the *dihedral group of degree* n. It is shown that $(\Gamma(D_n))^c$ is connected and moreover, the values of n are classified according as $diam((\Gamma(D_n))^c)$ is either 1, 2 or 3 (see Remark 2.19 and Proposition 2.20). Let $n \geq 4$ be such that n is not a prime number. It is proved in Remark 2.21 that $r((\Gamma(D_n))^c) = 2$.

In Section 3 of this article, we discuss some results regarding the girth of $(\Gamma(G))^c$, where G is a finite group which admits at least one nontrivial subgroup. It is proved in Proposition 3.1 that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G.

It is noted in Proposition 3.2 that $girth((\Gamma(G))^c) = 3$ if and only if G has at least three minimal subgroups. It is observed in Remark 3.4 that if $o(G)$ is divisible by at least three distinct prime numbers, then $girth((\Gamma(G))^c) = 3$. Let G be a finite abelian group such that $o(G)$ is divisible by exactly t distinct prime numbers. Then it is shown in Proposition 3.3 that $\omega((\Gamma(G))^c) = t$ if and only if G is cyclic. Let G be a finite group with $o(G) = p_1^{n_1}p_2^{n_2}$, where p_1, p_2 are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2\}$. If $n_i = 1$ for each $i \in \{1, 2\}$, then it is proved in Proposition 3.5 that $girth((\Gamma(G))^c) \in \{3, \infty\}$. If G is cyclic and if $n_i > 1$ for each $i \in \{1, 2\}$, then it is shown in Proposition 3.7 that $girth((\Gamma(G))^c) = 4$. If G is cyclic and if $n_1 > 1$ and $n_2 = 1$, then it is verified in Proposition 3.8 that the subgraph of $(\Gamma(G))^c$ induced on its nonisolated vertices is a star graph and hence, $girth((\Gamma(G))^c) = \infty$. If G is abelian but not cyclic, then it is proved in Proposition 3.9 that $girth((\Gamma(G))^c) = 3$.

Whenever a set A is a subset of a set B and $A \neq B$, we denote it symbolically by $A \subset B$.

2. MAIN RESULTS

Let G be a finite group admitting at least two nontrivial subgroups. The aim of this section is to characterize G such that $(\Gamma(G))^c$ is connected and also to determine $diam((\Gamma(G))^c)$ in the case when $(\Gamma(G))^c$ is connected.

Proposition 2.1. *Let G be a finite group which admits at least two nontrivial subgroups. Then the following statements are equivalent:*
(i) $(\Gamma(G))^c$ is connected.
(ii) $N_G = G$.
Moreover, if either (i) or (ii) holds, then $diam((\Gamma(G))^c) \leq 3$.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. Let H be a nontrivial subgroup of G. Since G is finite, there exists a minimal subgroup K of G such that $H \supseteq K$. Hence, $H \cap N_G \supseteq K$ and so, $H \cap N_G \neq \{e\}$. If $N_G \neq G$, then we obtain that N_G is an isolated vertex of $(\Gamma(G))^c$. This is impossible since G has at least two nontrivial subgroups and $(\Gamma(G))^c$ is connected. Therefore, $N_G = G$.
$(ii) \Rightarrow (i)$ Assume that $N_G = G$. Let H_1, H_2 be nontrivial subgroups of G with $H_1 \neq H_2$. We now verify that there exists a path of length at most three between H_1 and H_2 in $(\Gamma(G))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. If H is any nontrivial subgroup of G, then as $N_G = G$, it follows that there exists a minimal subgroup K of G such that $K \not\subseteq H$.

Case(1): There exists a minimal subgroup K of G such that $K \nsubseteq H_1$ and $K \nsubseteq H_2$.

Observe that $H_1 \cap K = H_2 \cap K = \{e\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$.

Case(2): There exists a minimal subgroup W_1 of G such that $W_1 \nsubseteq H_1$ but $W_1 \subseteq H_2$ and there exists a minimal subgroup W_2 of G such that $W_2 \nsubseteq H_2$ but $W_2 \subseteq H_1$.

It is clear that $H_1 \cap W_1 = H_2 \cap W_2 = W_1 \cap W_2 = \{e\}$ and so, $H_1 - W_1 - W_2 - H_2$ is a path of length three between H_1 and H_2 in $(\Gamma(G))^c$.

This proves that $(\Gamma(G))^c$ is connected and $diam((\Gamma(G))^c) \leq 3$.

The proof of the moreover part is contained in the proof of $(ii) \Rightarrow (i)$ of this Proposition. □

Let G be a finite group which admits at least two nontrivial subgroups. We next try to characterize G such that $(\Gamma(G))^c$ is complete.

Remark 2.2. Let G be a group. It is not hard to verify that G has a unique nontrivial subgroup if and only if G is a finite cyclic group with $o(G) = p^2$, where p is a prime number.

Lemma 2.3. *Let G be a finite group which admits at least one nontrivial subgroup. Then $(\Gamma(G))^c$ is complete if and only if every nontrivial subgroup of G is minimal.*

Proof. Assume that $(\Gamma(G))^c$ is complete. Let H be a nontrivial subgroup of G. Let K be a nontrivial subgroup of G such that $K \subseteq H$. If $K \neq H$, then as H, K are adjacent in $(\Gamma(G))^c$, we obtain that $H \cap K = \{e\}$. This implies that $K = H \cap K = \{e\}$. This is a contradiction and so, H is a minimal subgroup of G.

Conversely, assume that any nontrivial subgroup of G is minimal. Let H_1, H_2 be nontrivial subgroups of G such that $H_1 \neq H_2$. Then $H_1 \cap H_2 = \{e\}$ and so, H_1 and H_2 are adjacent in $(\Gamma(G))^c$. This shows that $(\Gamma(G))^c$ is complete. □

Remark 2.4. Let G be a finite group which admits at least one nontrivial subgroup. If K is any minimal subgroup of G, then $o(K)$ is a prime number.

Proof. Suppose that $o(K)$ is composite. Let p be a prime number such that p divides $o(K)$. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup H of K such that $o(H) = p$. It is clear that $\{e\} \subset H \subset K$. This implies that K is not a minimal subgroup of G. This is a contradiction. Therefore, $o(K)$ is a prime number. □

Lemma 2.5. *Let G be a finite group with at least two nontrivial subgroups. Suppose that $o(G) = p^n$, where p is a prime number and $n \geq 2$. Then the following statements are equivalent:*
(*i*) $(\Gamma(G))^c$ *is complete.*
(*ii*) $n = 2$ *and G is not cyclic.*

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is complete. Then we know from Lemma 2.3 that any nontrivial subgroup of G is minimal. Suppose that $n \geq 3$. Note that p^2 is a divisor of $o(G)$. Hence, we obtain from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p^2$. We know from Remark 2.4 that H is not a minimal subgroup of G. This is a contradiction. Therefore, $n \leq 2$. Since G has at least two nontrivial subgroups, we obtain that $n \geq 2$ and so, $n = 2$. This shows that $o(G) = p^2$. As a cyclic group of order p^2 has a unique nontrivial subgroup, it follows that G is not cyclic.
$(ii) \Rightarrow (i)$ Assume that $o(G) = p^2$, where p is a prime number and G is not cyclic. We know from [6, Corollary, p.86] that G is abelian. Let $g \in G, g \neq e$. It follows as a consequence of Lagrange's theorem [6, Corollary 1, p.41] that $o(g)$ is a divisor of $o(G) = p^2$. Since G is not cyclic, we obtain that $o(g) = p$. Hence, it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups A_1, A_2 of G such that $o(A_i) = p$ for each $i \in \{1, 2\}$ and G is the internal direct product of A_1 and A_2. It is clear from Lagrange's theorem [6, Theorem 2.4.1, p.41] that any nontrivial subgroup of G is of order p and it is well-known that there are exactly $p+1$ subgroups of G each of order p. Therefore, $(\Gamma(G))^c$ is K_{p+1}. □

Let G be a finite group such that $o(G)$ is divisible by at least two distinct prime numbers. In Proposition 2.6, we characterize G such that $(\Gamma(G))^c$ is complete.

Proposition 2.6. *Let G be a finite group such that $o(G)$ is divisible by at least two distinct prime numbers. Then the following statements are equivalent:*
(*i*) $(\Gamma(G))^c$ *is complete.*
(*ii*) $o(G) = p_1p_2$, *where p_1 and p_2 are distinct prime numbers.*

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is complete. We know from Lemma 2.3 that each nontrivial subgroup of G is minimal. Let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$ be the factorization of $o(G)$ into product of prime numbers (here $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \in \mathbb{N}$ for each $i \in \{1, 2, \ldots, k\}$). We claim that $n_1 = n_2 = \cdots = n_k = 1$. Suppose that $n_i > 1$ for some $i \in \{1, 2, \ldots, k\}$. We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p_i^{n_i}$.

We know from Remark 2.4 that H is not a minimal subgroup of G. This is a contradiction. Therefore, $n_i = 1$ for each $i \in \{1, 2, \ldots, k\}$.

We next verify that $k = 2$. By hypothesis, $k \geq 2$. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that for each $i \in \{1, 2, \ldots, k\}$, there exists a subgroup P_i of G such that $o(P_i) = p_i$. We claim that P_i is normal in G for at least one $i \in \{1, 2, \ldots, k\}$. Suppose that P_i is not normal in G for each $i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, \ldots, k\}$. Observe that $N(P_i) \supseteq P_i$, where $N(P_i)$ is the normalizer of P_i in G. Since P_i is not normal in G, it follows that $N(P_i) \neq G$. Hence, $N(P_i)$ is a nontrivial subgroup of G. As any nontrivial subgroup of G is minimal, we obtain that $N(P_i) = P_i$. Note that P_i is a p_i-Sylow subgroup of G. We know from [6, Lemma 2.12.6, p.99] that the number of p_i-Sylow subgroups in G equals $\frac{o(G)}{o(N(P_i))} = \frac{o(G)}{o(P_i)} = \frac{o(G)}{p_i}$. Let $\{P_i = P_{i1}, P_{i2}, \ldots, P_{i\frac{o(G)}{p_i}}\}$ be the set of all p_i-Sylow subgroups of G. As any element g of a p_i-Sylow subgroup with $g \neq e$ is of order p_i, it follows that G has exactly $\frac{o(G)}{p_i}(p_i - 1)$ elements of order p_i. As any nontrivial subgroup of G is minimal, it follows that if $x \in G$ with $x \neq e$, then $o(x) = p_i$ for some $i \in \{1, 2, \ldots, k\}$. It is now clear from the above discussion that $o(G) = \frac{o(G)}{p_1}(p_1 - 1) + \frac{o(G)}{p_2}(p_2 - 1) + \cdots + \frac{o(G)}{p_k}(p_k - 1) + 1$. This implies that $1 = k - (\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}) + \frac{1}{o(G)}$. We can assume that $2 \leq p_1 < p_2 < \cdots < p_k$. Hence, we obtain that $k - 1 + \frac{1}{o(G)} < \frac{k}{2}$. This is a contradiction. Therefore, P_i is normal in G for at least one $i \in \{1, 2, \ldots, k\}$. Fix $i \in \{1, 2, \ldots, k\}$ such that P_i is normal in G. Suppose that $k \geq 3$. Let $j \in \{1, 2, \ldots, k\} \backslash \{i\}$. Observe that $P_i P_j$ is a subgroup of G and as $P_i \cap P_j = \{e\}$, it follows from [6, Theorem 2.5.1, p.45] that $o(P_i P_j) = p_i p_j$. Note that $P_i P_j$ is a nontrivial subgroup of G and is not minimal. This is in contradiction to the assumption that $(\Gamma(G))^c$ is complete. Therefore, $k = 2$. Hence, $o(G) = p_1 p_2$, where p_1, p_2 are distinct prime numbers.

$(ii) \Rightarrow (i)$ Assume that $o(G) = p_1 p_2$, where p_1 and p_2 are distinct prime numbers. It follows from Lagrange's theorem that any nontrivial subgroup of G is of order either p_1 or p_2. Hence, any nontrivial subgroup of G is minimal and so, we obtain from Lemma 2.3 that $(\Gamma(G))^c$ is complete. □

Remark 2.7. Let G be a finite group with $o(G) = p_1 p_2$, where p_1, p_2 are distinct primes. In this remark, we mention some well-known facts about the structure of G. If G is abelian, then G is necessarily cyclic and in such a case, $(\Gamma(G))^c$ is K_2. Suppose that G is not abelian. We can assume that $p_1 < p_2$. We know from [6, Theorem 2.12.3 and Lemma 2.12.6, p.100, p.99] that G has a unique subgroup H with $o(H) = p_2$

and has exactly p_2 subgroups of G each of order p_1. Hence, $(\Gamma(G))^c$ is K_{p_2+1}.

Let G be a finite abelian group which admits at least two nontrivial subgroups. We next proceed to discuss regarding the characterization of G such that $(\Gamma(G))^c$ is connected and determine its diameter when it is connected. First, we consider finite abelian groups with $o(G) = p^n$, where p is prime number and $n \geq 2$.

Proposition 2.8. *Let G be a finite abelian group with $o(G) = p^n$, where p is a prime number and $n \geq 2$. Then the following statements are equivalent:*

(i) *$(\Gamma(G))^c$ is connected.*

(ii) *G is the internal direct product of cyclic subgroups $A_1, A_2, \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$.*

Moreover, in the case when $(\Gamma(G))^c$ is connected, $diam((\Gamma(G))^c) = 1$ if $n = 2$ and $diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2$ if $n \geq 3$.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. We know from Proposition 2.1 that $N_G = G$. Since $o(G) = p^n$ where p is a prime number, we obtain that any minimal subgroup of G is of order p. As G is the subgroup of G generated by all its minimal subgroups, it follows that each element $g \in G$ with $g \neq e$ is of order p. We know from [10, Example 2.5, p.146] that there exist cyclic subgroups $A_1, A_2, \ldots, A_n$ of G satisfying the following properties: $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$ and G is the internal direct product of $A_1, A_2, \ldots, A_n$. This shows that G is the internal direct product of cyclic subgroups $A_1, A_2, \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$.
$(ii) \Rightarrow (i)$ Assume that there exist cyclic subgroups $A_1, A_2 \ldots, A_n$ with $o(A_i) = p$ for each $i \in \{1, 2, \ldots, n\}$ and G is the internal direct product of $A_1, A_2, \ldots, A_n$.

Suppose that $n = 2$. Then we know from the proof of $(ii) \Rightarrow (i)$ of Lemma 2.5 that $(\Gamma(G))^c$ is K_{p+1}. Therefore, $diam((\Gamma(G))^c) = 1$.

Let us next suppose that $n \geq 3$. Let H_1, H_2 be two distinct nontrivial subgroups of G with $H_1 \neq H_2$. We show that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(G))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. If H_1, H_2 are not comparable under the inclusion relation, then it is clear that $H_1 \cup H_2$ is not a subgroup of G and therefore, $H_1 \cup H_2 \neq G$. Let $g \in G$ be such that $g \notin H_! \cup H_2$. Let $K =< g >$. Note that $o(K) = p$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. Suppose that H_1 and H_2 are comparable under the inclusion relation. We can assume without loss of generality that $H_1 \subset H_2$. Since $H_2 \neq G$, it follows that $A_i \not\subseteq H_2$

for some $i \in \{1, 2, \ldots, n\}$. As $o(A_i) = p$, it follows that $H_2 \cap A_i = \{e\}$ and so, $H_1 \cap A_i = \{e\}$. Hence, $H_1 - A_i - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. This shows that $(\Gamma(G))^c$ is connected and $diam((\Gamma(G))^c) \leq 2$. We next verify that $e(S) \geq 2$ in $(\Gamma(G))^c$ for any nontrivial subgroup S of G. Note that $o(S) = p^i$ for some i with $1 \leq i < n$. Observe that there exists a subgroup W of S with $o(W) = p$. If $i > 1$, then $W \neq S$ and S and W are not adjacent in $(\Gamma(G))^c$ and so, $d(S, W) \geq 2$ in $(\Gamma(G))^c$. Suppose that $i = 1$. Now, $A_k \not\subseteq S$ for some $k \in \{1, 2, \ldots, n\}$. Hence, $A_k \cap S = \{e\}$. Observe that SA_k is a subgroup of G and it follows from [6, Theorem 2.5.1, p.45] that $o(SA_k) = p^2$. As $o(G) = p^n$ with $n \geq 3$, it is clear that SA_k is a nontrivial subgroup of G. Since $S \cap SA_k \neq \{e\}$, we get that S and SA_k are not adjacent in $(\Gamma(G))^c$. Therefore, $d(S, SA_k) \geq 2$ in $(\Gamma(G))^c$. This proves that $e(S) \geq 2$ in $(\Gamma(G))^c$ for each nontrivial subgroup S of G. This proves that $diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2$.

The proof of the moreover part is contained in the proof of $(ii) \Rightarrow (i)$ of this Proposition. ☐

Let G be a finite abelian group with $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. We next proceed to characterize G such that $(\Gamma(G))^c$ is connected and determine its diameter when it is connected.

Proposition 2.9. *Let G be a finite abelian group such that $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. For each $i \in \{1, 2, \ldots, k\}$, let P_i denote the unique p_i-Sylow subgroup of G. Then the following statements are equivalent:*

(i) $(\Gamma(G))^c$ is connected.

(ii) Given $i \in \{1, 2, \ldots, k\}$, either $o(P_i) = p_i$ or $(\Gamma(P_i))^c$ is connected.

Proof. $(i) \Rightarrow (ii)$ Assume that $(\Gamma(G))^c$ is connected. Since $k \geq 2$, G has at least two nontrivial subgroups. Indeed, P_i is a nontrivial subgroup of G for each $i \in \{1, 2, \ldots, k\}$ and $o(P_i) = p_i^{n_i}$ for each $i \in \{1, 2, \ldots, k\}$. It is well-known that G is the internal direct product of $P_1, P_2, \ldots, P_k$. As $(\Gamma(G))^c$ is connected, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Let $g \in G$, $g \neq e$. It follows from $N_G = G$ that $o(g) = \prod_{j \in A} p_j$ for some nonempty subset A of $\{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots, k\}$. Suppose that $o(P_i) \neq p_i$. Hence, $n_i \geq 2$. As any element x of P_i with $x \neq e$ is of order p_i, it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups $A_{i1}, A_{i2}, \ldots, A_{in_i}$ of P_i such that $o(A_{ij}) = p_i$ for each $j \in \{1, 2, \ldots, n_i\}$ and P_i is the

internal direct product of $A_{i1}, A_{i2}, \ldots, A_{in_i}$. Now, it follows from $(ii) \Rightarrow (i)$ of Proposition 2.8 that $(\Gamma(P_i))^c$ is connected.
$(ii) \Rightarrow (i)$ It is well-known that G is the internal direct product of $P_1, P_2, \ldots, P_k$. Let $g \in G$, $g \neq e$. Now, there exist unique elements $x_1, x_2, \ldots x_k$ with $x_i \in P_i$ for each $i \in \{1, 2, \ldots, k\}$ such that $g = \prod_{i=1}^{k} x_i$. As $g \neq e$, it follows that $x_i \neq e$ for at least one $i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots . k\}$ be such that $x_i \neq e$. By assumption, either $o(P_i) = p_i$ or $(\Gamma(P_i))^c$ is connected. If $o(P_i) = p_i$, then $o(x_i) = p_i$. Suppose that $(\Gamma(P_i))^c$ is connected. Then it follows from $(i) \Rightarrow (ii)$ of Proposition 2.8 that $o(x_i) = p_i$. Hence, in any case $o(x_i) = p_i$. Now, it follows from $g = \prod_{i=1}^{k} x_i$ that $g \in N_G$ and so, $N_G = G$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(G))^c$ is connected. □

Let G be a finite abelian group and let $o(G)$ be as in the statement of Proposition 2.9. Suppose that $(\Gamma(G))^c$ is connected. In Proposition 2.11, we determine $diam((\Gamma(G))^c)$. We use Lemma 2.10 in the proof of Proposition 2.11.

Lemma 2.10. *Let G be a finite abelian group such that G has at least two nontrivial subgroups. Suppose that $(\Gamma(G))^c$ is connected. Then the following hold:*

(i) $diam((\Gamma(G))^c) = 2$ if and only if G admits a nontrivial subgroup which is not a minimal subgroup of G and if H_1, H_2 are distinct maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$, then H_1 and H_2 are isomorphic.

(ii) $diam((\Gamma(G))^c) = 3$ if and only if there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$.

Proof. Since $(\Gamma(G))^c$ is connected, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$.
(i) Assume that $diam((\Gamma(G))^c) = 2$. We know from Lemma 2.3 that G admits at least one nontrivial subgroup which is not a minimal subgroup of G. Let H_1, H_2 be distinct maximal subgroups of G such that $H_1 \cap H_2 \neq \{e\}$. Note that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. As $diam((\Gamma(G))^c) = 2$, there exists a nontrivial subgroup K of G such that $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. Hence, $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. As H_i is a maximal subgroup of G, we obtain that $H_i K = G$. Therefore, we obtain from the second isomorphism theorem of groups [3, Theorem 2.3, p.98] that $\frac{G}{K} = \frac{H_i K}{K} \cong \frac{H_i}{H_i \cap K} = H_i$ for each $i \in \{1, 2\}$.

Conversely, assume that G admits at least one nontrivial subgroup which is not a minimal subgroup and any two distinct maximal subgroups of G which are not adjacent in $(\Gamma(G))^c$ are isomorphic. As there exists at least one nontrivial subgroup of G which is not a minimal subgroup of G, it follows from Lemma 2.3 that $diam((\Gamma(G))^c) \geq 2$. Let W_1, W_2 be nontrivial subgroups of G. We prove that there exists a path of length at most two between W_1 and W_2 in $(\Gamma(G))^c$. We can assume that W_1 and W_2 are not adjacent in $(\Gamma(G))^c$. That is, $W_1 \cap W_2 \neq \{e\}$. Let H_i be a maximal subgroup of G such that $W_i \subseteq H_i$ for each $i \in \{1, 2\}$. Observe that $H_1 \cap H_2 \neq \{e\}$. It can happen that $H_1 = H_2$. And in the case, $H_1 \neq H_2$, we know from the assumption that H_1 and H_2 are isomorphic. Thus in any case, $o(H_1) = o(H_2)$. Hence, we obtain that $o(\frac{G}{H_1}) = o(\frac{G}{H_2})$. Since $\frac{G}{H_i}$ is an abelian simple group, we get that $\frac{G}{H_i}$ is a cyclic group for each $i \in \{1, 2\}$ with $o(\frac{G}{H_1}) = o(\frac{G}{H_2}) = p$, where p is a prime number. As $(\Gamma(G))^c$ is connected, we know from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Let $i \in \{1, 2\}$. As $H_i \neq G$, there exists a minimal subgroup M_i of G such that $M_i \not\subseteq H_i$. We know from Remark 2.4 that $o(M_i)$ is a prime number. Since H_i is a maximal subgroup of G, we obtain that $H_iM_i = G$. It follows from $H_i \cap M_i = \{e\}$ and [6, Theorem 2.5.1, p.45] that $o(G) = o(H_i)o(M_i)$. Therefore, we obtain that $o(M_i) = \frac{o(G)}{o(H_i)} = o(\frac{G}{H_i}) = p$. If $M_2 \not\subseteq H_1$, then it follows from $H_1 \cap M_2 = H_2 \cap M_2 = \{e\}$ that $W_i \cap M_2 = \{e\}$ for each $i \in \{1, 2\}$ and so, $W_1 - M_2 - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Similarly, if $H_2 \cap M_1 = \{e\}$, then it follows that $W_1 - M_1 - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Suppose that $M_2 \subseteq H_1$ and $M_1 \subseteq H_2$. Note that M_i is a cyclic group with $o(M_i) = p$ for each $i \in \{1, 2\}$. Let $g_1 \in M_1 \backslash M_2$ and let $g_2 \in M_2 \backslash M_1$. Observe that $o(g_1g_2) = p$ and let us denote $< g_1g_2 >$ by M. It is clear that M is a minimal subgroup of G and $M \not\subseteq H_i$ for each $i \in \{1, 2\}$. Therefore, $W_i \cap M \subseteq H_i \cap M = \{e\}$ for each $i \in \{1, 2\}$. Hence, we obtain that $W_1 - M - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(G))^c$. Therefore, we get that $diam((\Gamma(G))^c) = 2$.

(ii) Assume that $diam((\Gamma(G))^c) = 3$. Let W_1, W_2 be distinct nontrivial subgroups of G such that $d(W_1, W_2) = 3$ in $(\Gamma(G))^c$. Let $i \in \{1, 2\}$. Let H_i be a maximal subgroup of G such that $W_i \subseteq H_i$. From $W_1 \cap W_2 \neq \{e\}$, it follows that $H_1 \cap H_2 \neq \{e\}$. If $H_1 \cong H_2$ as groups, then it follows from the proof of the if part of (i) that $d(W_1, W_2) = 2$ in $(\Gamma(G))^c$. This is in contradiction to the assumption that $d(W_1, W_2) = 3$ in $(\Gamma(G))^c$. Therefore, H_1 and H_2 are nonisomorphic. This proves that there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$.

Conversely, assume that there exist nonisomorphic maximal subgroups H_1, H_2 of G such that $H_1 \cap H_2 \neq \{e\}$. It follows from the proof of the only if part of (i) that $d(H_1, H_2) \geq 3$ in $(\Gamma(G))^c$ and so, $diam((\Gamma(G))^c) \geq 3$. Therefore, we obtain that $diam((\Gamma(G))^c) = 3$. $\square$

Proposition 2.11. *Let G be a finite abelian group. Let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers, and $n_i \geq 1$ for each $i \in \{1, 2, \ldots, k\}$. Suppose that $(\Gamma(G))^c$ is connected. Then the following hold.*

If $k = 2$, then $diam((\Gamma(G))^c) = 1$ if and only if $n_1 = n_2 = 1$. If $n_i \geq 2$ for some $i \in \{1, 2\}$, then $diam((\Gamma(G))^c) = 3$.

If $k \geq 3$, then $diam((\Gamma(G))^c) = 3$.

Proof. Suppose that $(\Gamma(G))^c$ is connected. For each $i \in \{1, 2, \ldots, k\}$, let P_i denote the unique p_i-Sylow subgroup of G. We know that G is the internal direct product of $P_1, P_2, \ldots, P_k$.

Suppose that $k = 2$. If $n_1 = n_2 = 1$, then P_1, P_2 are the only nontrivial subgroups of G and $(\Gamma(G))^c$ is K_2 and so, $diam((\Gamma(G))^c) = 1$. Suppose that $n_i \geq 2$ for some $i \in \{1, 2\}$. Without loss of generality, we can assume that $n_1 \geq 2$. Let W_i be a subgroup of P_i with $o(W_i) = p_i^{n_i - 1}$ for each $i \in \{1, 2\}$. Let H_1 be the internal direct product of W_1 and P_2 and H_2 be the internal direct product of P_1 and W_2. Observe that $o(H_1) = p_1^{n_1-1} p_2^{n_2}$ and $o(H_2) = p_1^{n_1} p_2^{n_2-1}$. It is clear that H_1 and H_2 are nonisomorphic maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$. Hence, it follows from Lemma 2.10(ii) that $diam((\Gamma(G))^c) = 3$.

Suppose that $k \geq 3$. Let W_1 be the internal direct product of $P_1, P_2, \ldots, P_{k-1}$. Let W_2 be the internal direct product of $P_2, \ldots, P_k$. Let U be a subgroup of P_1 with $o(U) = p_1^{n_1-1}$ and let W be a subgroup of P_k with $o(W) = p_k^{n_k-1}$. Let H_1 be the internal direct product of W_1 and W and let H_2 be the internal direct product of W_2 and U. It is clear that $o(H_1) = (\prod_{i=1}^{k-1} p_i^{n_i}) p_k^{n_k-1}$, $o(H_2) = p_1^{n_1-1}(\prod_{j=2}^{k} p_j^{n_j})$, H_1 and H_2 are nonisomorphic maximal subgroups of G with $H_1 \cap H_2 \neq \{e\}$. Therefore, we obtain from Lemma 2.10(ii) that $diam((\Gamma(G))^c) = 3$. $\square$

Remark 2.12. Let G be a finite group which admits at least two nontrivial subgroups. If $(\Gamma(G))^c$ is connected, then $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G.

Proof. Let H be a minimal subgroup of G. Let W be any nontrivial subgroup of G with $W \neq H$. We claim that $d(H, W) \leq 2$ in $(\Gamma(G))^c$. We can assume that H and W are not adjacent in $(\Gamma(G))^c$. Hence, $H \cap W \neq \{e\}$. As H is a minimal subgroup of G, it follows that $H \subset W$. Since $(\Gamma(G))^c$ is connected, we know from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. It follows from $W \neq G$ that there exists a minimal subgroup S of G such that $S \not\subseteq W$. Observe that $H \cap S = W \cap S = \{e\}$. Therefore, $H - S - W$ is a path of length two between H and W in $(\Gamma(G))^c$. This proves that $d(H,W) \leq 2$ in $(\Gamma(G))^c$ for any nontrivial subgroup W of G and so, $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. □

Remark 2.13. Let G be a finite abelian group and let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $n_i \geq 1$for each $i \in \{1, 2, \ldots, k\}$. Suppose that $(\Gamma(G))^c$ is connected and in the case $k = 2$, either $n_1 > 1$ or $n_2 > 1$. Then $r((\Gamma(G))^c) = 2$.

Proof. Let H be any minimal subgroup of G. We know from Remark 2.12 that $e(H) \leq 2$ in $(\Gamma(G))^c$.

In the case $k \geq 3$, it is clear that if H is a minimal subgroup of G, then there exists at least one nontrivial subgroup W of G such that $H \subset W$ and so, H and W are not adjacent in $(\Gamma(G))^c$. In the case $k = 2$, we are assuming that either $n_1 > 1$ or $n_2 > 1$. Hence, in this case also, given a minimal subgroup H of G, there exists a nontrivial subgroup W of G such that $H \subset W$ and so, H and W are not adjacent in $(\Gamma(G))^c$. Therefore, $d(H,W) \geq 2$ in $(\Gamma(G))^c$. It is already shown that $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. This proves that $e(H) = 2$ in $(\Gamma(G))^c$ for any minimal subgroup H of G. As for a given nontrivial subgroup W of G, there exists a minimal subgroup H of G such that $H \subseteq W$, it follows that $e(W) \geq 2$ in $(\Gamma(G))^c$. Therefore, we obtain that $r((\Gamma(G))^c) = 2$. □

Let $n \geq 3$. Let S_n denote the symmetric group of degree n. We know from [6, Lemma 2.10.2, p.78] that any $\sigma \in S_n$ is a product of transpositions. If $\tau = (i, j)$ is any transposition, then $o(\tau) = 2$ in S_n. Therefore, $N_{S_n} = S_n$ and so, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(S_n))^c$ is connected. In Proposition 2.14, we determine $diam((\Gamma(S_n))^c)$.

Proposition 2.14. *Let $n \geq 3$. Then $(\Gamma(S_n))^c$ is connected and $diam((\Gamma(S_3))^c) = 1$, whereas $diam((\Gamma(S_n))^c) = 2$ for all $n \geq 4$.*

Proof. It is already noted above that $(\Gamma(S_n))^c$ is connected. Observe that $o(S_3) = 6 = 2 \times 3$ and S_3 is not abelian. We know from Remark 2.7 that $(\Gamma(S_3))^c$ is a complete graph on four vertices. Therefore, we obtain that $diam((\Gamma(S_3))^c) = 1$. Let $n \geq 4$. Let $\sigma = (1, 2, 3, 4)$. Let $H =< \sigma >$ and let $K =< \sigma^2 >$. Observe that $o(H) = 4$ and $o(K) = 2$ and $H \cap K = K$ is nontrivial. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$. Therefore, $diam((\Gamma(S_n))^c) \geq 2$. We next verify that $diam((\Gamma(S_n))^c) \leq 2$. Let H_1, H_2 be any nontrivial subgroups of S_n with $H_1 \neq H_2$. We claim that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(S_n))^c$. We can assume that H_1 and H_2

are not adjacent in $(\Gamma(S_n))^c$. It is well-known that S_n is generated by the set of 2-cycles $\{(1,i) : i \in \{2,3,\ldots,n\}\}$. Since $H_1 \neq S_n$, it follows that $(1,i) \notin H_1$ for some $i \in \{2,3,\ldots,n\}$. If $(1,i) \notin H_2$, then with $H =< (1,i) >$, we get that $H_i \cap H = \{e\}$ for each $i \in \{1,2\}$. Hence, $H_1 - H - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. Suppose that $(1,i) \in H_2$. As $H_2 \neq S_n$, we obtain that there exists $j \in \{2,3,\ldots,n\}$ such that $(1,j) \notin H_2$. It is clear that $i \neq j$. If $(1,j) \notin H_1$, then $H_1 - < (1,j) > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. Suppose that $(1,j) \in H_1$. Thus $(1,j) \in H_1 \backslash H_2$ and $(1,i) \in H_2 \backslash H_1$. Let $\rho = (1,i)(1,j)$. Note that $\rho = (1,j,i)$ is a cycle of length 3 and let $H_3 =< (1,j,i) >$. It is clear that $H_3 = \{e,\rho,\rho^2\}$ and $\rho \notin H_1 \cup H_2$. Hence, we get that $H_i \cap H_3 = \{e\}$ for each $i \in \{1,2\}$. Therefore, $H_1 - H_3 - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(S_n))^c$. From the above discussion, it is clear that $diam((\Gamma(S_n))^c) \leq 2$ and so, $diam((\Gamma(S_n))^c) = 2$. □

Remark 2.15. Let $n \geq 4$. Then $r((\Gamma(S_n))^c) = 2$.

Proof. Let $n \geq 4$. We know from Proposition 2.14 that $(\Gamma(S_n))^c$ is connected and $diam((\Gamma(S_n))^c) = 2$. Therefore, $e(H) \leq 2$ in $(\Gamma(S_n))^c$ for each nontrivial subgroup H of S_n. Hence, to prove this remark, it is enough to show that $e(H) \geq 2$ in $(\Gamma(S_n))^c$ for any nontrivial subgroup H of S_n. Let H be any nontrivial subgroup of S_n. If H is not a minimal subgroup of S_n, then it is clear that $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Hence, we can assume that H is a minimal subgroup of S_n. Note that either $H \subseteq A_n$ or $H \nsubseteq A_n$, where A_n is the alternating group of degree n. It is known that $o(A_n) = \frac{o(S_n)}{2}$ [6, Lemma 2.10.3, p.80] Thus, A_n is a maximal subgroup of S_n and is a normal subgroup of S_n. If $H \subseteq A_n$, then as A_n is not a minimal subgroup of S_n, it follows that $H \neq A_n$. Therefore, it follows from $H \cap A_n = H \neq \{e\}$ that H and A_n are not adjacent in $(\Gamma(S_n))^c$. Hence, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that $H \nsubseteq A_n$. Then $A_nH = S_n$ and $H \cap A_n = \{e\}$. We know from [6, Theorem 2.5.1, p.45] that $o(S_n) = o(A_n)o(H)$ and so, $o(H) = 2$. Let $\sigma \in S_n$ be such that $H = \{e,\sigma\}$. Note that $o(\sigma) = 2$. It follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $\tau_1,\ldots,\tau_k$ such that $\sigma = \prod_{i=1}^{k} \tau_i$. As σ is an odd permutation, k must be odd. Suppose that $k = 1$. Let $\sigma = \tau_1 = (i_1,i_2)$. As $n \geq 4$, there exist distinct symbols i_3,i_4 such that $i_3,i_4 \in \{1,2,\ldots,n\} \backslash \{i_1,i_2\}$. Let K be the subgroup of S_n generated by $\{\sigma,(i_1,i_3)\}$. It is clear that $(i_1,i_4) \notin K$ and so, $K \neq S_n$. Since $(i_1,i_3) \notin H$, it follows that $H \subset K$. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that k is odd and $k \geq 3$. Let $\tau_1 = (i_1,i_2), \tau_2 = (i_3,i_4),\ldots,\tau_k = (i_{2k-1},i_{2k})$. Let K be the subgroup of S_n generated by $\{\sigma,(i_1,i_2)\}$. It is clear that

$(i_1, i_3) \notin K$ and so, $K \neq S_n$. Since $(i_1, i_2) \in K \backslash H$, it follows that $H \subset K$. Hence, H and K are not adjacent in $(\Gamma(S_n))^c$. Therefore, we get that $e(H) \geq 2$ in $(\Gamma(S_n))^c$. This proves that $r((\Gamma(S_n))^c) = 2$. □

Remark 2.16. Let $n \geq 4$. It is well-known that A_n is generated by the set of 3-cycles $\{(1, 2, i) : i \in \{3, 4, \ldots, n\}\}$ [10, Proposition 4.5.1, p.55]. If $\sigma \in S_n$ is any 3-cycle, then $o(\sigma) = 3$ and so, $<\sigma> = \{e, \sigma, \sigma^2\}$. It follows from the above given arguments that $N_{A_n} = A_n$ and so, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(A_n))^c$ is connected. We prove in Proposition 2.17 that $diam((\Gamma(A_n))^c) = 2$.

Proposition 2.17. *Let $n \geq 4$. Then $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$.*

Proof. It is noted in Remark 2.16 that $(\Gamma(A_n))^c$ is connected. Let $H = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$ and $K = \{e, (1, 2)(3, 4)\}$. It is clear that H, K are subgroups of A_n and $H \cap K = K$ is nontrivial. Hence, H and K are not adjacent in $(\Gamma(A_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(A_n))^c$ and so, $diam((\Gamma(A_n))^c) \geq 2$. We next verify that $diam((\Gamma(A_n))^c)) \leq 2$. Let H_1, H_2 be nontrivial subgroups of A_n with $H_1 \neq H_2$. We show that there exists a path of length at most two between H_1 and H_2 in $(\Gamma(A_n))^c$. We can assume that H_1 and H_2 are not adjacent in $(\Gamma(A_n))^c$. Since $H_1 \neq A_n$, $(1, 2, i) \notin H_1$ for some $i \in \{3, 4, \ldots, n\}$. If $(1, 2, i) \notin H_2$, then $H_1 - < (1, 2, i) > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. Suppose that $(1, 2, i) \in H_2$. As $H_2 \neq A_n$, there exists $j \in \{3, 4, \ldots, n\}$ such that $(1, 2, j) \notin H_2$. If $(1, 2, j) \notin H_1$, then $H_1 - < (1, 2, j) > - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. Suppose that $(1, 2, j) \in H_1$. Now, $(1, 2, j) \in H_1 \backslash H_2$ and $(1, 2, i) \in H_2 \backslash H_1$. Let $\rho = (1, 2, i)(1, 2, j)$. Observe that $\rho \notin H_1 \cup H_2$ and $\rho = (1, i)(2, j)$. Let $H_3 = <\rho>$. As $H_3 = \{e, \rho\}$, we obtain that $H_i \cap H_3 = \{e\}$ for each $i \in \{1, 2\}$ and so, $H_1 - H_3 - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(A_n))^c$. This proves that $diam((\Gamma(A_n))^c) \leq 2$ and so, we obtain that $diam((\Gamma(A_n))^c) = 2$. □

Proposition 2.18. *Let $n \geq 4$. Then the following hold.*

(i) $r((\Gamma(A_4))^c) = 1$.

(ii) Let $n \geq 5$. Let H be a minimal subgroup of A_n. If $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod4)$, then there exists a nontrivial subgroup W of A_n such that $H \subset W$.

Proof. Let $n \geq 4$. It is proved in Proposition 2.17 that $(\Gamma(A_n))^c$ is connected and $diam((\Gamma(A_n))^c) = 2$.
(i) We verify that $r((\Gamma(A_4))^c) = 1$. Let $\sigma = (1, 2, 3)$ and let $H = <\sigma>$. Observe that $o(H) = 3$. We claim that $e(H) = 1$ in $(\Gamma(A_4))^c$. Let

K be any nontrivial subgroup of A_4 with $K \neq H$. We assert that $H \cap K = \{e\}$. Suppose that $H \cap K \neq \{e\}$. Then as H is a minimal subgroup of A_4, it follows that $H \subset K$. It follows from Lagrange's theorem [6, Theorem 2.4.1, p.41] that $o(K) = 3t$ for some $t \in \mathbb{N}$ with $t \geq 2$. As $o(K)$ is a divisor of $o(A_4) = 12$ and $K \neq A_4$, it follows that $o(K) = 6$. This is impossible since it is well-known that A_4 has no subgroup of order 6 [10, Example 3.3.6, p.75] Therefore, $H \cap K = \{e\}$ and so, H and K are adjacent in $(\Gamma(A_4))^c$. This proves that $e(H) = 1$ in $(\Gamma(A_4))^c$ and therefore, $r((\Gamma(A_4))^c) = 1$.
(ii) Let $n \geq 5$. Let H be a minimal subgroup of A_n. Note that $o(H) = p$, where p is a prime number and $H =< \sigma >$ for any $\sigma \in H\backslash\{e\}$. We discuss two cases.
Case(1): $p = 2$.
Let $\sigma \in H\backslash\{e\}$. As $\sigma \in A_n$ and $o(\sigma) = 2$, it follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $(i_1, i_2), (i_3, i_4), \ldots, (i_{2k-1}, i_{2k})$ with $k \geq 2$ is even and is such that $\sigma = \prod_{s=1}^{k}(i_{2s-1}, i_{2s})$. If $k = 2$, then $\sigma = (i_1, i_2)(i_3, i_4)$. Observe that $W = \{e, \sigma, (i_1, i_3)(i_2, i_4), (i_1, i_4)(i_2, i_3)\}$ is a nontrivial subgroup of A_n such that $o(W) = 4$ and $H \subset W$. As H and W are not adjacent in $(\Gamma(A_n))^c$, it follows that $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $k \geq 4$. Let $\sigma_1 = (i_1, i_2)(i_3, i_4)$ and let $\sigma_2 = \prod_{s=3}^{k}(i_{2s-1}, i_{2s})$. Note that $\sigma_1, \sigma_2 \in A_n$, $o(\sigma_i) = 2$ for each $i \in \{1, 2\}$ and $\sigma_1\sigma_2 = \sigma = \sigma_2\sigma_1$ and $W_1 = \{e, \sigma_1, \sigma_2, \sigma\}$ is a nontrivial subgroup of A_n with $o(W_1) = 4$ and $H \subset W_1$. Hence, H and W_1 are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.
Case(2): p is odd.
Let $\sigma \in H\backslash\{e\}$. Note that $\sigma \in A_n$ and $o(\sigma) = p$. Hence, it follows from [6, Lemma 2.10.1, p.78] that there exists $t \in \mathbb{N}$ and disjoint cycles $C_1, \ldots, C_t$ such that C_i is of length p for each $i \in \{1, \ldots, t\}$ and $\sigma = \prod_{i=1}^{t} C_i$. Suppose that $t = 1$. Then $\sigma = C_1 = (i_1, i_2, i_3, \ldots, i_p)$. Observe that either $p = 3$ or $p \geq 5$. Assume that $p = 3$. Let $i_4 \in \{1, 2, 3, \ldots, n\}\backslash\{i_1, i_2, i_3\}$. Let $W = \{e, (i_1, i_2, i_3), (i_1, i_3, i_2), (i_1, i_2, i_4), (i_1, i_4, i_2), (i_1, i_3, i_4), (i_1, i_4, i_3), (i_2, i_3, i_4), (i_2, i_4, i_3), (i_1, i_2)(i_3, i_4), (i_1, i_3)(i_2, i_4), (i_1, i_4)(i_2, i_3)\}$. Note that W is a nontrivial subgroup of A_n with $o(W) = 12$, $W \cong A_4$ as groups, and $H \cap W = H$. Therefore, H and W are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $p \geq 5$. Note that $\sigma = (i_1, i_2, i_3 \ldots, i_p)$. Suppose that $p \equiv 1(mod 4)$. Let $\tau \in S_n$ be given by $\tau(i_1) = i_1, \tau(i_j) = i_{p-j+2}$ for each $j \in \{2, 3, \ldots, p\}$. Observe that $\tau = \prod_{j=2}^{\frac{p+1}{2}}(i_j, i_{p-j+2})$ is the product of $\frac{p-1}{2}$ disjoint transpositions. As $\frac{p-1}{2}$ is even, we obtain that $\tau \in A_n$. Observe

that $\sigma^p = e, \tau^2 = e, \sigma^{p-1}\tau = \tau\sigma$. Let W be the subgroup of A_n generated by $\{\sigma, \tau\}$. Note that $W = \{e, \sigma, \sigma^2, \ldots, \sigma^{p-1}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{p-1}\tau = \tau\sigma\}$ and $W \cong D_p$ as groups, where D_p is the dihedral group of degree p. It is clear that W is a nontrivial subgroup of A_n and $H \subset W$. Hence, $H \cap W = H \neq \{e\}$ and so, H and W are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Suppose that p is an odd prime number and σ is the product of t $(t \geq 2)$ disjoint cycles $C_1, C_2, \ldots, C_t$ such that C_i is of length p for each $i \in \{1, 2, \ldots, t\}$. Let $\sigma_1 = C_1$ and let $\sigma_2 = \prod_{j=2}^{t} C_j$. Note that $\sigma_i \in A_n$ for each $i \in \{1, 2\}$, $o(\sigma_1) = o(\sigma_2) = p$, and $\sigma = \sigma_1\sigma_2 = \sigma_2\sigma_1$. Let W be the subgroup of A_n generated by $\{\sigma_1, \sigma_2\}$. Let $H_1 =< \sigma_1 >$ and let $H_2 =< \sigma_2 >$. It is clear that $o(H_1) = o(H_2) = p$, $H_1 \cap H_2 = \{e\}$, and $W = H_1H_2$. It follows from [6, Theorem 2.5.1, p.45] that $o(W) = o(H_1)o(H_2) = p^2$. Observe that $H =< \sigma >\subset W$ and so, $H \cap W = H \neq \{e\}$. Hence, H and W are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Thus for any $n \geq 5$, it is shown that if H is any minimal subgroup of A_n with $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod 4)$, then there exists a nontrivial subgroup W of A_n such that $H \subset W$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. $\square$

Remark 2.19. Let $n \geq 3$. Recall from [3, Theorem 5.2, p.87 and p.88] that the *dihedral group of degree* n denoted by D_n is the subgroup of S_n generated by σ and τ, where σ is the cycle given by $\sigma = (1, 2, 3, \ldots, n)$ and τ is given by $\tau(1) = 1, \tau(i) = n - i + 2$ for each $i \in \{2, 3, \ldots, n\}$. Note that $o(\sigma) = n$, $o(\tau) = 2$, $\sigma^{n-1}\tau = \tau\sigma$, $o(D_n) = 2n$ and indeed, $D_n = \{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{n-1}\tau = \tau\sigma\}$. Observe that $D_3 = S_3$ and it is already shown in Proposition 2.14 that $(\Gamma(S_3))^c$ is connected and $diam((\Gamma(S_3))^c) = 1$. Hence, in discussing the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$. Suppose that n is a prime number. Then n is odd and $o(D_n) = 2n$ is the product of two distinct prime numbers. As D_n is not abelian, it follows from Remark 2.7 that $(\Gamma(D_n))^c$ is a complete graph on $n + 1$ vertices. Therefore, in our discussion regarding the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$ and n is not a prime number. We prove in Proposition 2.20 that $(\Gamma(D_n))^c$ is connected and moreover, we determine $diam((\Gamma(D_n))^c)$.

Proposition 2.20. *Let $n \geq 4$ and suppose that n is not a prime number. Then $(\Gamma(D_n))^c$ is connected. Moreover, the following hold.*

(i) $diam((\Gamma(D_n))^c) = 2$ if either n is odd or $n = 2m$, where $m \geq 3$ is odd.

(ii) $diam((\Gamma(D_n))^c) = 3$ if $n = 2^k t$, where $k \geq 2$ and $t \geq 1$ is odd.

Proof. We know that D_n is the subgroup of S_n generated by σ and τ, where σ and τ are mentioned as above in Remark 2.19. Note that $o(\sigma) = n, o(\tau) = o(\sigma^i\tau) = 2$ for each $i \in \{1, 2, \ldots, n-1\}$. It is clear that D_n has at least two nontrivial subgroups and as D_n is generated by $\sigma\tau$ and τ, it follows that $N_{D_n} = D_n$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(D_n))^c$ is connected. Moreover, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(D_n))^c) \leq 3$.

Let $n \geq 4$ and suppose that n is not a prime number. Then $< \sigma >$ is not a minimal subgroup of D_n. Let H be a nontrivial subgroup of $< \sigma >$ such that $H \subset < \sigma >$. Observe that H and $< \sigma >$ are not adjacent in $(\Gamma(D_n))^c$. Hence, $d(H, < \sigma >) \geq 2$ in $(\Gamma(D_n))^c$ and so, we obtain that $diam((\Gamma(D_n))^c) \geq 2$. Let H_1, H_2 be nontrivial subgroups of D_n with $H_1 \neq H_2$. Suppose that $\tau \notin H_1 \cup H_2$. Note that $K =< \tau >$ is a subgroup of D_n with $o(K) = 2$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. As $o(\sigma\tau) = 2$, it follows that if $\sigma\tau \notin H_1 \cup H_2$, then $H_1- < \sigma\tau > -H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Hence, in finding $d(H_1, H_2)$ in $(\Gamma(D_n))^c$, we can assume that $\tau, \sigma\tau \in H_1 \cup H_2$. Since D_n is generated by $\sigma\tau$ and τ, both τ and $\sigma\tau$ cannot be in H_i for each $i \in \{1, 2\}$. Without loss of generality, we can assume that $\tau \in H_1 \backslash H_2$ and $\sigma\tau \in H_2 \backslash H_1$. Since D_n is generated by τ and $\tau\sigma$ and as $\tau \in H_1$, it follows that $\tau\sigma \notin H_1$.
(i) Suppose that $n \geq 4$ and n is odd. We claim that $\tau\sigma \notin H_2$. For, if $\tau\sigma \in H_2$, then $(\sigma\tau)(\tau\sigma) = \sigma^2 \in H_2$. As n is odd, $o(\sigma) = o(\sigma^2) = n$. This implies that $\sigma \in H_2$ and so, $H_2 = D_n$. This is a contradiction. Therefore, $\tau\sigma \notin H_2$. It is now clear that $H_1- < \tau\sigma > -H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. This proves that for any nontrivial subgroups H_1, H_2 of D_n with $H_1 \neq H_2$, $d(H_1, H_2) \leq 2$ in $(\Gamma(D_n))^c$. Therefore, we get that $diam((\Gamma(D_n))^c) \leq 2$ and so, $diam((\Gamma(D_n))^c) = 2$.

Suppose that $n = 2m$, where $m \geq 3$ and m is odd. If $\tau\sigma \notin H_2$, then $H_1- < \tau\sigma > -H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that $\tau\sigma \in H_2$. Then it follows that $(\sigma\tau)(\tau\sigma) = \sigma^2 \in H_2$. Thus $\sigma^2, \sigma\tau \in H_2$ and so, $< \sigma^2, \sigma\tau > \subseteq H_2$. As $\sigma \notin H_2$, $\sigma^2 \in H_2$, and m is odd we obtain that $\sigma^m \notin H_2$. Suppose that $\sigma^m \notin H_1$. Let $K =< \sigma^m >$. Note that $o(K) = 2$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that $\sigma^m \in H_1$. As $\sigma \notin H_1$, it follows that $\sigma^2 \notin H_1$. Since $\tau \in H_1$, we obtain that $\sigma^2\tau \notin H_1$. As $\sigma^2 \in H_2$ and $\tau \notin H_2$, we obtain that $\sigma^2\tau \notin H_2$. Thus $\sigma^2\tau \notin H_1 \cup H_2$. As $o(< \sigma^2\tau >) = 2$, we obtain that $H_i\cap < \sigma^2\tau >= \{e\}$ for each $i \in \{1, 2\}$. Therefore, $H_1- < \sigma^2\tau > -H_2$ is a path of length two between H_1

and H_2 in $(\Gamma(D_n))^c$. It follows from the above given arguments that $diam((\Gamma(D_n))^c) \leq 2$ and so, $diam((\Gamma(D_n))^c) = 2$.
(ii) Suppose that $n = 2^k t$, where $k \geq 2$ and $t \geq 1$ is odd. Let H_1 be the subgroup of D_n generated by σ^2 and τ and let H_2 be the subgroup of D_n generated by σ^2 and $\sigma\tau$. Observe that $< \sigma^2 >$ is a characteristic subgroup of $< \sigma >$. Since $[D_n :< \sigma >] = 2$, it follows that $< \sigma >$ is a normal subgroup of D_n. Therefore, we obtain from [6, Problem 9, p.70] that $< \sigma^2 >$ is a normal subgroup of D_n. Therefore, $H_1 =< \sigma^2 >< \tau >$ and $H_2 =< \sigma^2 >< \sigma\tau >$. Note that $o(< \sigma^2 >) = 2^{k-1}t$ and $o(< \tau >) = o(\sigma\tau >) = 2$ and $< \sigma^2 > \cap < \tau >=< \sigma^2 > \cap < \sigma\tau >= \{e\}$. Therefore, we obtain from [6, Theorem 2.5.1, p.45] that $o(H_1) = o(H_2) = (2^{k-1}t)(2) = 2^k t$. Hence, H_1 and H_2 are maximal subgroups of D_n and they are also normal subgroups of D_n. Since $\sigma^2 \in H_1 \cap H_2$, it follows that H_1 and H_2 are not adjacent in $(\Gamma(D_n))^c$. We claim that there exists no path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Suppose that there exists a path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Let H_3 be a nontrivial subgroup of D_n such that $H_1 - H_3 - H_2$ is a path of length two in $(\Gamma(D_n))^c$. Then $H_i \cap H_3 = \{e\}$ for each $i \in \{1, 2\}$. Note that $H_3 \not\subseteq H_1$ and H_1 is a maximal and a normal subgroup of D_n. Therefore, we obtain that $H_1 H_3 = D_n$. Hence, $o(H_1)o(H_3) = o(D_n)$ and so, $o(H_3) = 2$. Observe that $S = \{\sigma^{2^{k-1}t}, \tau, \sigma\tau, \sigma^2\tau, \ldots, \sigma^{n-1}\tau\}$ is the set of all elements of order 2 in D_n. Hence, $H_3 =< s >$ for some $s \in S$. Note that $\{\sigma^{2^{k-1}t}, \tau, \sigma^2\tau, \ldots, \sigma^{n-2}\tau\} \subseteq H_1$ and $\{\sigma\tau, \sigma^3\tau, \ldots, \sigma^{n-1}\tau\} \subseteq H_2$. This implies that $S \subseteq H_1 \cup H_2$ and so, either $H_3 \subseteq H_1$ or $H_3 \subseteq H_2$. This is a contradiction. Therefore, there exists no path of length two between H_1 and H_2 in $(\Gamma(D_n))^c$. Hence, we obtain that $diam((\Gamma(D_n))^c) \geq 3$ and as $diam((\Gamma(D_n))^c) \leq 3$, it follows that $diam((\Gamma(D_n))^c) = 3$. □

Remark 2.21. Let $n \geq 4$ be such that n is not a prime number. Then $r((\Gamma(D_n))^c) = 2$.

Proof. It is already noted in Remark 2.19 that $(\Gamma(D_n))^c$ is connected and $diam((\Gamma(D_n))^c)$ is determined in Proposition 2.20. Let σ, τ be as mentioned in Remark 2.19. Let H be any minimal subgroup of D_n. We know from Remark 2.12 that $e(H) \leq 2$ in $(\Gamma(D_n))^c$. We next verify that $e(H) \geq 2$ in $(\Gamma(D_n))^c$. We consider the following cases.
Case(1): $H \subseteq< \sigma >$.

Note that $H =< \sigma^{\frac{n}{p}} >$ for some prime number p such that p is a divisor of n. Observe that $o(H) = p$. Since H is a characteristic subgroup of $< \sigma >$ and $< \sigma >$ is a normal subgroup of D_n, we obtain from [6, Problem 9, p.70] that H is a normal subgroup of D_n. Let K be the subgroup of D_n generated by $\sigma^{\frac{n}{p}}$ and τ. Observe that $K = H <$

$\tau >$. As $o(\tau) = 2$ and $H\cap < \tau >= \{e\}$, it follows from [6, Theorem 2.5.1, p.45] that $o(K) = 2p < 2n = o(D_n)$. Hence, K is a nontrivial subgroup of D_n. Since $H \cap K = H \neq \{e\}$, we get that H and K are not adjacent in $(\Gamma(D_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(D_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(D_n))^c$.

Case(2): $H \nsubseteq < \sigma >$.

In this case $H =< \sigma^i\tau >$ for some $i \in \{0, 1, \ldots, n-1\}$. Note that $o(H) = 2$. Let p be a prime number such that p is a divisor of n. Let K be the subgroup of D_n generated by $\sigma^{\frac{n}{p}}$ and $\sigma^i\tau$. Using the same arguments as in Case(1), we obtain that $o(K) = 2p < 2n = o(D_n)$. Hence, K is a nontrivial subgroup of D_n. It is clear that H and K are not adjacent in $(\Gamma(D_n))^c$. Therefore, $d(H, K) \geq 2$ in $(\Gamma(D_n))^c$. This proves that $e(H) \geq 2$ in $(\Gamma(D_n))^c$.

Therefore, $e(H) = 2$ in $(\Gamma(D_n))^c$ for any minimal subgroup H of D_n. It is clear that if K is any nontrivial subgroup of D_n which is not minimal, then $e(K) \geq 2$ in $(\Gamma(D_n))^c$. Therefore, we obtain that $r((\Gamma(D_n))^c) = 2$. $\square$

Proposition 2.22. *Let $G, \overline{G}$ be finite groups such that both of them admit at least two nontrivial subgroups. Let $\phi : G \to \overline{G}$ be a surjective homomorphism of groups. If $(\Gamma(G))^c$ is connected, then $(\Gamma(\overline{G}))^c$ is also connected. Moreover, if $diam((\Gamma(G))^c) \leq 2$, then $diam((\Gamma(\overline{G}))^c) \leq 2$.*

Proof. Let e denote the identity element of G and let us denote the identity element of $\overline{G}$ by $\overline{e}$. Let us denote $Ker\phi$ by N. It is clear that $N \neq G$. If $N = \{e\}$, then $G \cong \overline{G}$ as groups. Hence, the graphs $(\Gamma(G))^c$ and $(\Gamma(\overline{G}))^c$ are isomorphic. Therefore, there is nothing to prove in this case. So, we can assume that $N \neq \{e\}$. Let $y \in \overline{G}$, $y \neq \overline{e}$. Since ϕ is a surjective homomorphism from G onto $\overline{G}$, there exists $x \in G\backslash\{e\}$ such that $y = \phi(x)$. We are assuming that $(\Gamma(G))^c$ is connected. Therefore, we obtain from $(i) \Rightarrow (ii)$ of Proposition 2.1 that $N_G = G$. Note that there exist $k \geq 1$ and elements $g_1, \ldots, g_k \in G$ such that $o(g_i)$ is a prime number for each $i \in \{1, \ldots, k\}$ and $x = \prod_{i=1}^{k} g_i$. Hence, $y = \phi(x) = \prod_{i=1}^{k} \phi(g_i)$. Since $y \neq \overline{e}$, it follows that $\phi(g_i) \neq \overline{e}$ for at least one $i \in \{1, \ldots, k\}$ and for such an i, $o(\phi(g_i)) = o(g_i)$ is a prime number. The above discussion implies that $N_{\overline{G}} = \overline{G}$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(\overline{G}))^c$ is connected.

We next prove the moreover part. Suppose that $diam((\Gamma(G))^c) \leq 2$. We show that $diam((\Gamma(\overline{G}))^c) \leq 2$. Let W_1, W_2 be nontrivial subgroups of $\overline{G}$ with $W_1 \neq W_2$. We now show that there exists a path of length at most two between W_1 and W_2 in $(\Gamma(\overline{G}))^c$. We can assume that W_1 and W_2 are not adjacent in $(\Gamma(\overline{G}))^c$. We know from [6, Lemma 2.7.5, p.63] that there exist nontrivial subgroups H_1, H_2 of G with $N \subset H_i$ for each $i \in \{1,2\}$ and $W_i = \phi(H_i)$ for each $i \in \{1,2\}$. It is clear that $H_1 \neq H_2$ and as $H_1 \cap H_2 \neq \{e\}$, we obtain that H_1 and H_2 are not adjacent in $(\Gamma(G))^c$. We are assuming that $diam((\Gamma(G))^c) \leq 2$. Hence, there exists a nontrivial subgroup K of G such that $H_1 - K - H_2$ is a path of length two between H_1 and H_2 in $(\Gamma(G))^c$. We assert that $W_i \cap \phi(K) = \{\overline{e}\}$. for each $i \in \{1,2\}$. Let $i \in \{1,2\}$. Let $z \in W_i \cap \phi(K)$. Then $z = \phi(h_i) = \phi(k)$ for some $h_i \in H_i$ and $k \in K$. Hence, $kh_i^{-1} \in N \subset H_i$ and so, $k \in H_i \cap K = \{e\}$. Therefore, $z = \phi(k) = \phi(e) = \overline{e}$. This shows that $W_i \cap \phi(K) = \{\overline{e}\}$ for each $i \in \{1,2\}$. From $H_1 \cap K = \{e\}$ and $N \subset H_1$, it follows that $\phi(K) \neq \{\overline{e}\}$. Hence, $W_1 - \phi(K) - W_2$ is a path of length two between W_1 and W_2 in $(\Gamma(\overline{G}))^c$. This proves that $diam((\Gamma(\overline{G}))^c) \leq 2$. □

Remark 2.23. Let $G, \overline{G}$ be finite groups such that both G and $\overline{G}$ admit at least two nontrivial subgroups. Let $\phi : G \to \overline{G}$ be a surjective homomorphism of groups. Suppose that $(\Gamma(G))^c$ is connected. Then $(\Gamma(\overline{G}))^c$ is connected. If $diam((\Gamma(\overline{G}))^c) = 3$, then $diam((\Gamma(G))^c) = 3$.

Proof. We know from Proposition 2.22 that $(\Gamma(\overline{G}))^c$ is connected. If $diam((\Gamma(\overline{G}))^c) = 3$, then it follows from Proposition 2.22 that $diam((\Gamma(G))^c) \geq 3$. We know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $diam((\Gamma(G))^c) \leq 3$ and so, we get that $diam((\Gamma(G))^c) = 3$. □

3. SOME MORE RESULTS

Let G be a finite group which admits at least one nontrivial subgroup. The aim of this section is to determine $\omega((\Gamma(G))^c)$ and $girth((\Gamma(G))^c)$.

Proposition 3.1. *Let G be a finite group. Then $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G.*

Proof. Since G is a finite group with at least one nontrivial subgroup, G has at least one minimal subgroup and G has only a finite number of minimal subgroups. Let k be the number of minimal subgroups of G. Let $\{W_1, \ldots, W_k\}$ be the set of all minimal subgroups of G. Since $W_i \cap W_j = \{e\}$ for all distinct $i, j \in \{1, 2, \ldots, k\}$, it follows that the subgraph of $(\Gamma(G))^c$ induced on $\{W_1, \ldots, W_k\}$ is a clique on k vertices. Therefore, we get that $\omega((\Gamma(G))^c) \geq k$. We next verify that the vertices of $(\Gamma(G))^c$ can be properly colored using a set of k distinct colors. Let $\{c_1, \ldots, c_k\}$ be a set of k distinct colors. Now, color W_i with c_i for each $i \in \{1, \ldots, k\}$. Let H be any nontrivial subgroup of G. It is clear that H contains a minimal subgroup of G. Let $i \in \{1, \ldots, k\}$ be least with the property that $H \supseteq W_i$. Then color H using c_i. We claim that the above assignment of colors is a proper vertex coloring of $(\Gamma(G))^c$. Let H_1, H_2 be nontrivial subgroups of G such that H_1 and H_2 are adjacent in $(\Gamma(G))^c$. Hence, $H_1 \cap H_2 = \{e\}$. Let $i \in \{1, \ldots, k\}$ be least with the property that $H_1 \supseteq W_i$ and let $j \in \{1, \ldots, k\}$ be least with the property that $H_2 \supseteq W_j$. Note that H_1 receives color c_i and H_2 receives color c_j. As $H_1 \cap H_2 = \{e\}$, it is clear that $i \neq j$ and so, $c_i \neq c_j$. This shows that $(\Gamma(G))^c$ can be properly colored using a set of k distinct colors. Therefore, we obtain that $\chi((\Gamma(G))^c) \leq k \leq \omega((\Gamma(G))^c) \leq \chi((\Gamma(G))^c)$. This proves that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$. □

Proposition 3.2. *Let G be a finite group. Then $girth((\Gamma(G))^c) = 3$ if and only if G has at least three minimal subgroups.*

Proof. Assume that $girth((\Gamma(G))^c) = 3$. Then there exist nontrivial subgroups H_1, H_2, H_3 such that $H_1 - H_2 - H_3 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Note that $H_1 \cap H_2 = H_2 \cap H_3 = H_3 \cap H_1 = \{e\}$. Let $i \in \{1, 2, 3\}$. Let W_i be a minimal subgroup of G such that $W_i \subseteq H_i$ for each $i \in \{1, 2, 3\}$. Observe that $W_1 \cap W_2 = W_2 \cap W_3 = W_3 \cap W_1 = \{e\}$. Hence, $W_i \neq W_j$ for all distinct $i, j \in \{1, 2, 3\}$. Therefore, G has at least three minimal subgroups.

Conversely, assume that G has at least three minimal subgroups. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. As $k \geq 3$, it follows that $girth((\Gamma(G))^c) = 3$. □

Proposition 3.3. *Let G be a finite group. Let $o(G) = \prod_{i=1}^{t} p_i^{n_i}$ be the factorization of $o(G)$ into product of prime numbers (here, $p_1, \ldots, p_t$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, \ldots, t\}$ and in the case $t = 1$, $n_1 > 1$). Then $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, G has only one subgroup W_i with $o(W_i) = p_i$. Moreover, if G is abelian, then $\omega((\Gamma(G))^c) = t$ if and only if G is cyclic.*

Proof. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. Therefore, $\omega((\Gamma(G))^c) = t$ if and only if G has exactly t minimal subgroups. Let $i \in \{1, \ldots, t\}$. Since p_i is a divisor of $o(G)$, we know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup W_i of G with $o(W_i) = p_i$. It is clear that W_i is a minimal subgroup of G for each $i \in \{1, \ldots, t\}$. Observe that if W is any minimal subgroup of G, then $o(W) = p_i$ for some $i \in \{1, \ldots, t\}$. Hence, $\omega((\Gamma(G))^c) = t$ if and only if $\{W_1, \ldots, W_t\}$ is the set of all minimal subgroups of G. Therefore, we obtain that $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, there exists only one subgroup W_i of G with $o(W_i) = p_i$.

We next verify the moreover part of this Proposition. If G is cyclic, then for each divisor d of $o(G)$, there exists a unique subgroup H of G with $o(H) = d$. Hence, for each $i \in \{1, \dots, t\}$, W_i is the only subgroup of G with $o(W_i) = p_i$. Therefore, $\omega((\Gamma(G))^c) = t$. Conversely, assume that G is abelian and $\omega((\Gamma(G))^c) = t$. For each $i \in \{1, \dots, t\}$, let P_i be the unique p_i-Sylow subgroup of G. Note that $o(P_i) = p_i^{n_i}$ for each $i \in \{1, \dots, t\}$ and G is the internal direct product of $P_1, \dots, P_t$. It is clear that W_i is the only subgroup of P_i with $o(W_i) = p_i$. We assert P_i is cyclic for each $i \in \{1, \dots, t\}$. Suppose that P_i is not cyclic for some $i \in \{1, \dots, t\}$. Then $n_i > 1$ and we know from the proof of the fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] that there exist $s \geq 2$ and cyclic subgroups $A_1, A_2, \dots, A_s$ of P_i such that $o(A_1) = p_i^{n_{i1}}, o(A_2) = p_i^{n_{i2}}, \dots, o(A_s) = p_i^{n_{is}}$ with $n_{i1} \geq n_{i2} \cdots \geq n_{is} \geq 1$ and P_i is the internal direct product of $A_1, A_2, \dots, A_s$. We know from [3, Problem 6, p.154] that the number of minimal subgroups of P_i equals $\frac{p_i^s - 1}{p_i - 1} = 1 + p_i + \cdots + p_i^{s-1} \geq 2$, since $s \geq 2$. This is impossible as W_i is the only minimal subgroup of P_i. This proves that P_i is cyclic for each $i \in \{1, \dots, t\}$. As $(o(P_i), o(P_j)) = 1$ for all distinct $i, j \in \{1, \dots, t\}$, it follows from [6, Problem 6, p.108] that G is cyclic. □

Remark 3.4. Let G be a finite group such that $o(G)$ is divisible by at least three distinct prime numbers p_1, p_2, and p_3. We know from Cauchy's theorem [6, Theorem 2.11.3, p.87] that for each $i \in \{1, 2, 3\}$, there exists a subgroup W_i of G such that $o(W_i) = p_i$. It is clear that W_i is a minimal subgroup of G for each $i \in \{1, 2, 3\}$ and hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

Proposition 3.5. *Let G be a finite group such that $o(G) = p_1 p_2$, where p_1 and p_2 are distinct prime numbers. Then $girth((\Gamma(G))^c) \in \{3, \infty\}$.*

Proof. We can assume without loss of generality that $p_1 < p_2$. It is already noted in Remark 2.7 that $(\Gamma(G))^c$ is either K_2 or K_{p_2+1}. Therefore, we obtain that $girth((\Gamma(G))^c) \in \{3, \infty\}$. □

Lemma 3.6. *Let G be a finite group such that $o(G) = p_1^{n_1} p_2^{n_2}$, where p_1 and p_2 are distinct prime numbers and $n_i > 1$ for each $i \in \{1, 2\}$. Then $girth((\Gamma(G))^c) \leq 4$.*

Proof. Let $i \in \{1, 2\}$. Let $k \in \mathbb{N}$ be such that $k \leq n_i$. We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup H of G such that $o(H) = p_i^k$. Let V_i denote the set of all subgroups H of G such that $o(H) = p_i^k$ for some $k \in \mathbb{N}$ with $k \leq n_i$ for each $i \in \{1, 2\}$. It is clear that each member of V_i is a nontrivial subgroup of G and V_i contains at least n_i elements for each $i \in \{1, 2\}$. As $n_i \geq 2$, it follows that V_i

contains at least two elements for each $i \in \{1,2\}$. Since $(p_1, p_2) = 1$, it follows from Lagrange's theorem that $H \cap W = \{e\}$ for any $H \in V_1$ and $W \in V_2$. If there exist $H_1, H_2 \in V_1$ such that $H_1 \cap H_2 = \{e\}$, then for any $W \in V_2$, we obtain that $H_1 - W - H_2 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Similarly, if there exist $W_1, W_2 \in V_2$ such that $W_1 \cap W_2 = \{e\}$, then for any $H \in V_1$, we get that $W_1 - H - W_2 - W_1$ is a cycle of length three in $(\Gamma(G))^c$. Hence, we can assume that no two distinct members of V_i are adjacent in $(\Gamma(G))^c$ for each $i \in \{1,2\}$. Let $H_1, H_2 \in V_1$ with $H_1 \neq H_2$ and let $W_1, W_2 \in V_2$ with $W_1 \neq W_2$. Note that $H_1 - W_1 - H_2 - W_2 - H_1$ is a cycle of length four in $(\Gamma(G))^c$. This proves that $girth((\Gamma(G))^c) \leq 4$. □

Proposition 3.7. *Let G be a finite cyclic group with $o(G) = p_1^{n_1} p_2^{n_2}$, where p_1, p_2 are distinct prime numbers and $n_i > 1$ for each $i \in \{1,2\}$. Then $girth((\Gamma(G))^c) = 4$.*

Proof. We know from Lemma 3.6 that $girth((\Gamma(G))^c) \leq 4$. Since G is a cyclic group with $o(G) = p_1^{n_1} p_2^{n_2}$, it follows that G has exactly two minimal subgroups. Hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) \neq 3$ and therefore, $girth((\Gamma(G))^c) = 4$. □

Proposition 3.8. *Let G be a finite cyclic group with $o(G) = p_1^n p_2$, where p_1 and p_2 are distinct prime numbers and $n > 1$. Then $girth((\Gamma(G))^c) = \infty$.*

Proof. Let P_1 be the subgroup of G with $o(P_1) = p_1^n$ and let P_2 be the subgroup of G with $o(P_2) = p_2$. Let V_1 denote the set of all subgroups H of P_1 with $H \neq \{e\}$ and let $V_2 = \{P_2\}$. Since P_1 is cyclic, it is clear that V_1 contains exactly n elements. As is noted in the proof of Lemma 3.6, $H \cap P_2 = \{e\}$ for any $H \in V_1$ and hence, H and P_2 are adjacent in $(\Gamma(G))^c$. Let W_1, W_2 be any two distinct nontrivial subgroups of G such that $W_i \notin V_1 \cup V_2$. Observe that $W_i = H_i P_2$ for some subgroup $H_i \in V_1$ such that $H_i \neq P_1$ for each $i \in \{1,2\}$. It is clear that $W_i \cap H \neq \{e\}, W_i \cap P_2 \neq \{e\}, W_1 \cap W_2 \neq \{e\}$ for each $i \in \{1,2\}$ and for any subgroup $H \in V_1$. From the above discussion, we obtain that $V_1 \cup V_2$ is the set of all nonisolated vertices of $(\Gamma(G))^c$ and the subgraph of $(\Gamma(G))^c$ induced on $V_1 \cup V_2$ is a star graph. Indeed, it is $K_{1,n}$. Therefore, we get that $girth((\Gamma(G))^c) = \infty$. □

Proposition 3.9. *Let G be a finite abelian group with $o(G) = p_1^{n_1} p_2^{n_2}$, where p_1 and p_2 are distinct prime numbers. Suppose that G is not cyclic. Then $girth((\Gamma(G))^c) = 3$.*

Proof. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where k is the number of minimal subgroups of G. It is clear that $k \geq 2$.

Since G is abelian but not cyclic, we obtain from Proposition 3.3 that $\omega((\Gamma(G))^c) \geq 3$ and therefore, $girth((\Gamma(G))^c) = 3$. $\square$

We mention an example in Example 3.10 to illustrate that the hypothesis that the group G is abelian cannot be omitted in Proposition 3.9. For any $n \geq 2$, we denote the additive group of integers modulo n by $\mathbb{Z}_n$.

Example 3.10. Let Q_8 be the *quaternion group* of order 8 given in [5, Exercise 44, p.187]. Let $G = Q_8 \times \mathbb{Z}_9$ be the external direct product of Q_8 and $\mathbb{Z}_9$. Observe that $o(G) = 2^3 3^2$. Note that $\{1, -1\} \times \{0\}$ and $\{1\} \times \{0, 3, 6\}$ are the only minimal subgroups of G. Hence, we obtain from Proposition 3.2 that $girth((\Gamma(G)))^c) \neq 3$. We know from Lemma 3.6 that $girth((\Gamma(G))^c) \leq 4$ and therefore, $girth((\Gamma(G))^c) = 4$.

Remark 3.11. Let G be a finite group with $o(G) = p^n$, where p is a prime number and $n \geq 2$. If G is cyclic, then G has only one minimal subgroup and so, $girth((\Gamma(G))^c) = \infty$. If G is abelian but not cyclic, then it is already noted in the proof of Proposition 3.3 that G has at least three minimal subgroups and so, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

Acknowledgments

We are very much thankful to the referee and the Editorial Board Members of JAS for their support.

References

1. S. Akbari, F. Heydari and M. Maghasedi, The intersection graph of a group, *J. Algebra Appl.*, **5** (2015), Article ID: 1550665, 9 pages.
2. R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, New York, 2000.
3. P. B. Bhattacharya, S. K. Jain and R. Nagpaul, *Basic Abstract Algebra*, Cambridge University Press, 1994.
4. B. Csakany and G. Pollak, The graph of subgroups of a finite group (Russian), *Czech. Math. J.*, **19** (1969), 241–247.
5. J. A. Gallian, *Contemporary Abstract Algebra*, Fourth Edition, Narosa Publishing House, New Delhi, 1999.
6. I. N. Herstein, *Topics in Algebra*, Second Edition, Wiley India Private Limited, New Delhi, 2010.
7. S. H. Jafari and N. Jafari Rad, Intersection graphs of normal subgroups of a groups, *Quasigroups and related systems*, **18** (2010), 137–142.
8. S. Kayacan, $K_{3,3}$-free intersection graphs of finite groups, *Comm. Algebra*, **45**(6) (2017), 2466–2477.

9. S. Kayacan, Dominating sets in intersection graphs of finite groups, *Rocky Mountain J. Math.*, **48**(7) (2018), 2311–2335.
10. I. S. Luthar and I. B. S. Passi, *Algebra Volume 1: Groups*, Narosa Publishing House, New Delhi, 1999.
11. R. Rajkumar and P. Devi, Intersection graphs of cyclic subgroups of groups, *Electronic Notes in Discrete Math.*, **5** (2016), 15–24.
12. R. Rajkumar and P. Devi, Intersection graph of subgroups of some non-abelian groups, *Malaya J. Math.*, **4**(2) (2016), 239–242.

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Journal of Algebraic Systems

SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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نتایجی درباره مکمل گراف اشتراکی زیرگروهای یک گروه متناهی

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در این مقاله گروه‌هایی مانند G را در نظر می‌گیریم که دارای حداقل یک زیرگروه غیربدیهی می‌باشند (یادآوری می‌کنیم که زیرگروه H از گروه G غیربدیهی نامیده می‌شود هرگاه $H \notin \{G, e\}$). فرض کنیم G یک گروه باشد. در این صورت گراف اشتراکی زیرگروه‌های G، که با نماد $\Gamma(G)$ نمایش داده می‌شود، یک گراف بدون جهت می‌باشد که مجموعه‌ی راسی آن مجموعه‌ی تمام زیرگروه‌های غیربدیهی G بوده و رئوس متمایز H و K توسط یک یال در این گراف مجاور می‌باشند اگر و تنها اگر $H \cap K \neq \{e\}$. فرض کنیم G یک گروه متناهی باشد. هدف اصلی این مقاله بررسی ارتباط بین خواص گروهی G و خواص گرافی مکمل گراف $\Gamma(G)$ می‌باشد.

کلمات کلیدی: مکمل گراف اشتراکی زیرگروه‌های یک گروه متناهی، گروه آبلی متناهی، گراف همبند، کمر گراف.