SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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Abstract. In this article we consider groups $G$ such that $G$ admits at least one nontrivial subgroup (recall that a subgroup $H$ of $G$ is said to be nontrivial if $H \notin \{G, \{e\}\}$). Let $G$ be a group. Recall that the intersection graph of subgroups of $G$, denoted by $\Gamma(G)$, is an undirected graph whose vertex set is the set of all non-trivial subgroups of $G$ and distinct vertices $H, K$ are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. Let $G$ be a finite group. The aim of this article is to investigate the interplay between the group-theoretic properties of a finite group $G$ and the graph-theoretic properties of the complement of $\Gamma(G)$.

1. Introduction

Let $G$ be a group which admits at least one nontrivial subgroup. Recall that the intersection graph of $G$, denoted by $\Gamma(G)$, is an undirected simple graph whose vertex set is the set of all non-trivial subgroups of $G$ and distinct vertices $H, K$ are joined by an edge in this graph if and only if $H \cap K \neq \{e\}$. The intersection graphs of groups have been investigated by several algebraists (for example, refer the articles [1, 4, 7, 8, 9, 11, 12]). Let $G = (V, E)$ be a simple graph. Recall from [2, Definition 1.1.13] that the complement of $G$, denoted by $G^c$ is a graph whose vertex set is $V$ and distinct vertices $u, v$ are joined by an

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edge in $G^c$ if and only if there is no edge joining $u$ and $v$ in $G$. Thus for a group $G$ which admits at least one nontrivial subgroup, $(\Gamma(G))^c$ is a graph whose vertex set is the set of all nontrivial subgroups of $G$ and distinct vertices $H, K$ are joined by an edge in $(\Gamma(G))^c$ if and only if $H \cap K = \{e\}$. The groups considered in this article are finite which admit at least one nontrivial subgroup. Let $G$ be a finite group. The purpose of this article is to investigate the effect of certain graph parameters of $(\Gamma(G))^c$ on the group structure of $G$.

It is useful to recall the following definitions and results from graph theory before we give an account of results that are proved on $(\Gamma(G))^c$, where $G$ is a finite group which admits at least one nontrivial subgroup. The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Let $a, b \in V, a \neq b$. Recall from [2] that the distance between $a$ and $b$, denoted by $d(a, b)$ is defined as the length of a shortest path in $G$ between $a$ and $b$ if such a path exists in $G$. Otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. A graph $G = (V, E)$ is said to be connected if for any distinct $a, b \in V$, there exists a path in $G$ between $a$ and $b$. Let $G = (V, E)$ be a connected graph. Recall from [2, Definition 4.2.1] that the diameter of $G$, denoted by $\text{diam}(G)$ is defined as $\text{diam}(G) = \sup\{d(a, b) : a, b \in V\}$. Let $a \in V$. The eccentricity of $a$, denoted by $e(a)$ is defined as $e(a) = \sup\{d(a, b) : b \in V\}$. The radius of $G$, denoted by $r(G)$ is defined as $r(G) = \min\{e(a) : a \in V\}$.

Let $G = (V, E)$ be a graph. Suppose that $G$ contains a cycle. Recall from [2, p. 159] that the girth of $G$, denoted by $\text{girth}(G)$ is the length of a shortest cycle in $G$. If $G$ does not contain any cycle, then we set $\text{girth}(G) = \infty$. A complete graph on $n$ vertices is denoted by $K_n$. Recall from [2, Definition 1.2.2] that a clique of $G$ is a complete subgraph of $G$. Let $G = (V, E)$ be a simple graph. Suppose that there exists $k \in \mathbb{N}$ such that any clique of $G$ is a clique on at most $k$ vertices. Then the clique number of $G$, denoted by $\omega(G)$ is defined as the largest positive integer $n$ such that $G$ contains a clique on $n$ vertices. If $G$ contains a clique on $n$ vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from [2, p.129] that a vertex coloring of $G$ is a mapping $f : V \to S$, where $S$ is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be proper if adjacent vertices of $G$ receive distinct colors of $S$; that is, if $u$ and $v$ are adjacent in $G$, then $f(u) \neq f(v)$. The chromatic number of $G$, denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of $G$. It is clear that for any graph $G$, $\omega(G) \leq \chi(G)$.

Let $G$ be a group. Recall that a nontrivial subgroup $H$ of $G$ is said to be a minimal subgroup of $G$ if there is no nontrivial subgroup $K$ of
Let $G$ be a finite group with at least two nontrivial subgroups. It is shown in Proposition 2.1 that $(\Gamma(G))^c$ is connected if and only if $N_G = G$. And in the case $(\Gamma(G))^c$ is connected, it is verified in Proposition 2.1 that $\text{diam}((\Gamma(G))^c) \leq 3$. In Lemma 2.5 and Proposition 2.6, we characterize finite groups $G$ which admit at least two nontrivial subgroups such that $(\Gamma(G))^c$ is complete. Let $G$ be a finite abelian group which admits at least two nontrivial subgroups. With the help of fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] and Proposition 2.1, we are able to determine the structure of finite abelian groups $G$ such that $(\Gamma(G))^c$ is connected (see Propositions 2.8 and 2.9). Moreover, in the case when $(\Gamma(G))^c$ is connected, we characterize finite abelian groups $G$ such that $\text{diam}((\Gamma(G))^c) = 1, 2$ or $3$ (see Propositions 2.8 and 2.11). Furthermore, in the case when $(\Gamma(G))^c$ is connected, we determine $r((\Gamma(G))^c)$ (see Proposition 2.8 and Remark 2.13).

Let $n \geq 3$ and let $S_n$ denote the symmetric group of degree $n$. With the help of Proposition 2.1, it is verified in Proposition 2.14 that $(\Gamma(S_n))^c$ is connected. Moreover, it is shown that $\text{diam}((\Gamma(S_3))^c) = 1$ and for $n \geq 4$, it is proved that $\text{diam}((\Gamma(S_n))^c) = r((\Gamma(S_n))^c) = 2$ (see Proposition 2.14 and Remark 2.15). Let $n \geq 4$ and let $A_n$ denote the alternating group of degree $n$. It is shown in Proposition 2.17 that $(\Gamma(A_n))^c$ is connected and $\text{diam}((\Gamma(A_n))^c) = 2$. It is observed in Proposition 2.18(i) that $r((\Gamma(A_4))^c) = 1$ and for any $n \geq 5$, it is shown in Proposition 2.18(ii) that $H$ is any minimal subgroup of $A_n$ with either $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod4)$, then $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Let $n \geq 3$ and let $D_n$ denote the dihedral group of degree $n$. It is shown that $(\Gamma(D_n))^c$ is connected and moreover, the values of $n$ are classified according as $\text{diam}((\Gamma(D_n))^c)$ is either 1, 2 or 3 (see Remark 2.19 and Proposition 2.20). Let $n \geq 4$ be such that $n$ is not a prime number. It is proved in Remark 2.21 that $r((\Gamma(D_n))^c) = 2$.

In Section 3 of this article, we discuss some results regarding the girth of $(\Gamma(G))^c$, where $G$ is a finite group which admits at least one nontrivial subgroup. It is proved in Proposition 3.1 that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where $k$ is the number of minimal subgroups of $G$.
It is noted in Proposition 3.2 that $girth((\Gamma(G))^c) = 3$ if and only if $G$ has at least three minimal subgroups. It is observed in Remark 3.4 that if $o(G)$ is divisible by at least three distinct prime numbers, then $girth((\Gamma(G))^c) = 3$. Let $G$ be a finite abelian group such that $o(G)$ is divisible by exactly $t$ distinct prime numbers. Then it is shown in Proposition 3.3 that $\omega((\Gamma(G))^c) = t$ if and only if $G$ is cyclic. Let $G$ be a finite group with $o(G) = p_1^{n_1}p_2^{n_2}$, where $p_1, p_2$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, 2\}$. If $n_i = 1$ for each $i \in \{1, 2\}$, then it is proved in Proposition 3.5 that $girth((\Gamma(G))^c) \in \{3, \infty\}$. If $G$ is cyclic and if $n_i > 1$ for each $i \in \{1, 2\}$, then it is shown in Proposition 3.7 that $girth((\Gamma(G))^c) = 4$. If $G$ is cyclic and if $n_1 > 1$ and $n_2 = 1$, then it is verified in Proposition 3.8 that the subgraph of $(\Gamma(G))^c$ induced on its nonisolated vertices is a star graph and hence, $girth((\Gamma(G))^c) = \infty$. If $G$ is abelian but not cyclic, then it is proved in Proposition 3.9 that $girth((\Gamma(G))^c) = 3$.

Whenever a set $A$ is a subset of a set $B$ and $A \neq B$, we denote it symbolically by $A \subset B$.

2. MAIN RESULTS

Let $G$ be a finite group admitting at least two nontrivial subgroups. The aim of this section is to characterize $G$ such that $(\Gamma(G))^c$ is connected and also to determine $diam((\Gamma(G))^c)$ in the case when $(\Gamma(G))^c$ is connected.

**Proposition 2.1.** Let $G$ be a finite group which admits at least two nontrivial subgroups. Then the following statements are equivalent:

(i) $(\Gamma(G))^c$ is connected.

(ii) $N_G = G$.

Moreover, if either (i) or (ii) holds, then $diam((\Gamma(G))^c) \leq 3$.

**Proof.** (i) $\Rightarrow$ (ii) Assume that $(\Gamma(G))^c$ is connected. Let $H$ be a nontrivial subgroup of $G$. Since $G$ is finite, there exists a minimal subgroup $K$ of $G$ such that $H \supseteq K$. Hence, $H \cap N_G \supseteq K$ and so, $H \cap N_G \neq \{e\}$. If $N_G \neq G$, then we obtain that $N_G$ is an isolated vertex of $(\Gamma(G))^c$.

This is impossible since $G$ has at least two nontrivial subgroups and $(\Gamma(G))^c$ is connected. Therefore, $N_G = G$.

(ii) $\Rightarrow$ (i) Assume that $N_G = G$. Let $H_1, H_2$ be nontrivial subgroups of $G$ with $H_1 \neq H_2$. We now verify that there exists a path of length at most three between $H_1$ and $H_2$ in $(\Gamma(G))^c$. We can assume that $H_1$ and $H_2$ are not adjacent in $(\Gamma(G))^c$. If $H$ is any nontrivial subgroup of $G$, then as $N_G = G$, it follows that there exists a minimal subgroup $K$ of $G$ such that $K \not\subseteq H$. 

**Case (1):** There exists a minimal subgroup $K$ of $G$ such that $K \nsubseteq H_1$ and $K \nsubseteq H_2$.

Observe that $H_1 \cap K = H_2 \cap K = \{e\}$. Hence, $H_1 - K - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(G))^c$.

**Case (2):** There exists a minimal subgroup $W_1$ of $G$ such that $W_1 \nsubseteq H_1$ but $W_1 \subseteq H_2$ and there exists a minimal subgroup $W_2$ of $G$ such that $W_2 \nsubseteq H_2$ but $W_2 \subseteq H_1$.

It is clear that $H_1 \cap W_1 = H_2 \cap W_2 = W_1 \cap W_2 = \{e\}$ and so, $H_1 - W_1 - W_2 - H_2$ is a path of length three between $H_1$ and $H_2$ in $(\Gamma(G))^c$.

This proves that $(\Gamma(G))^c$ is connected and $diam((\Gamma(G))^c) \leq 3$.

The proof of the moreover part is contained in the proof of $(ii) \Rightarrow (i)$ of this Proposition. \qed

Let $G$ be a finite group which admits at least two nontrivial subgroups. We next try to characterize $G$ such that $(\Gamma(G))^c$ is complete.

**Remark 2.2.** Let $G$ be a group. It is not hard to verify that $G$ has a unique nontrivial subgroup if and only if $G$ is a finite cyclic group with $o(G) = p^2$, where $p$ is a prime number.

**Lemma 2.3.** Let $G$ be a finite group which admits at least one nontrivial subgroup. Then $(\Gamma(G))^c$ is complete if and only if every nontrivial subgroup of $G$ is minimal.

**Proof.** Assume that $(\Gamma(G))^c$ is complete. Let $H$ be a nontrivial subgroup of $G$. Let $K$ be a nontrivial subgroup of $G$ such that $K \subseteq H$. If $K \neq H$, then as $H, K$ are adjacent in $(\Gamma(G))^c$, we obtain that $H \cap K = \{e\}$. This implies that $K = H \cap K = \{e\}$. This is a contradiction and so, $H$ is a minimal subgroup of $G$.

Conversely, assume that any nontrivial subgroup of $G$ is minimal. Let $H_1, H_2$ be nontrivial subgroups of $G$ such that $H_1 \neq H_2$. Then $H_1 \cap H_2 = \{e\}$ and so, $H_1$ and $H_2$ are adjacent in $(\Gamma(G))^c$. This shows that $(\Gamma(G))^c$ is complete. \qed

**Remark 2.4.** Let $G$ be a finite group which admits at least one nontrivial subgroup. If $K$ is any minimal subgroup of $G$, then $o(K)$ is a prime number.

**Proof.** Suppose that $o(K)$ is composite. Let $p$ be a prime number such that $p$ divides $o(K)$. We know from Cauchy’s theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup $H$ of $K$ such that $o(H) = p$. It is clear that $\{e\} \subseteq H \subseteq K$. This implies that $K$ is not a minimal subgroup of $G$. This is a contradiction. Therefore, $o(K)$ is a prime number. \qed
Lemma 2.5. Let \( G \) be a finite group with at least two nontrivial subgroups. Suppose that \( o(G) = p^n \), where \( p \) is a prime number and \( n \geq 2 \). Then the following statements are equivalent:

(i) \((\Gamma(G))^c\) is complete.

(ii) \( n = 2 \) and \( G \) is not cyclic.

Proof. (i) \( \Rightarrow \) (ii) Assume that \((\Gamma(G))^c\) is complete. Then we know from Lemma 2.3 that any nontrivial subgroup of \( G \) is minimal. Suppose that \( n \geq 3 \). Note that \( p^2 \) is a divisor of \( o(G) \). Hence, we obtain from [6, Theorem 2.12.1, p.92] that there exists a subgroup \( H \) of \( G \) such that \( o(H) = p^2 \). We know from Remark 2.4 that \( H \) is not a minimal subgroup of \( G \). This is a contradiction. Therefore, \( n \leq 2 \). Since \( G \) has at least two nontrivial subgroups, we obtain that \( n \geq 2 \) and so, \( n = 2 \). This shows that \( o(G) = p^2 \). As a cyclic group of order \( p^2 \) has a unique nontrivial subgroup, it follows that \( G \) is not cyclic.

(ii) \( \Rightarrow \) (i) Assume that \( o(G) = p^2 \), where \( p \) is a prime number and \( G \) is not cyclic. We know from [6, Corollary, p.86] that \( G \) is abelian. Let \( g \in G, g \neq e \). It follows as a consequence of Lagrange’s theorem [6, Corollary 1, p.41] that \( o(g) \) is a divisor of \( o(G) = p^2 \). Since \( G \) is not cyclic, we obtain that \( o(g) = p \). Hence, it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups \( A_1, A_2 \) of \( G \) such that \( o(A_i) = p \) for each \( i \in \{1, 2\} \) and \( G \) is the internal direct product of \( A_1 \) and \( A_2 \). It is clear from Lagrange’s theorem [6, Theorem 2.4.1, p.41] that any nontrivial subgroup of \( G \) is of order \( p \) and it is well-known that there are exactly \( p + 1 \) subgroups of \( G \) each of order \( p \). Therefore, \((\Gamma(G))^c\) is \( K_{p+1} \).

Let \( G \) be a finite group such that \( o(G) \) is divisible by at least two distinct prime numbers. In Proposition 2.6, we characterize \( G \) such that \((\Gamma(G))^c\) is complete.

Proposition 2.6. Let \( G \) be a finite group such that \( o(G) \) is divisible by at least two distinct prime numbers. Then the following statements are equivalent:

(i) \((\Gamma(G))^c\) is complete.

(ii) \( o(G) = p_1p_2 \), where \( p_1 \) and \( p_2 \) are distinct prime numbers.

Proof. (i) \( \Rightarrow \) (ii) Assume that \((\Gamma(G))^c\) is complete. We know from Lemma 2.3 that each nontrivial subgroup of \( G \) is minimal. Let \( o(G) = \prod_{i=1}^{k} p_i^{n_i} \) be the factorization of \( o(G) \) into product of prime numbers (here \( p_1, p_2, \ldots, p_k \) are distinct prime numbers and \( n_i \in \mathbb{N} \) for each \( i \in \{1, 2, \ldots, k\} \)). We claim that \( n_1 = n_2 = \cdots = n_k = 1 \). Suppose that \( n_i > 1 \) for some \( i \in \{1, 2, \ldots, k\} \). We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup \( H \) of \( G \) such that \( o(H) = p_i^{n_i} \).
We know from Remark 2.4 that \( H \) is not a minimal subgroup of \( G \). This is a contradiction. Therefore, \( n_i = 1 \) for each \( i \in \{1, 2, \ldots, k\} \).

We next verify that \( k = 2 \). By hypothesis, \( k \geq 2 \). We know from Cauchy’s theorem [6, Theorem 2.11.3, p.87] that for each \( i \in \{1, 2, \ldots, k\} \), there exists a subgroup \( P_i \) of \( G \) such that \( o(P_i) = p_i \).

We claim that \( P_i \) is normal in \( G \) for at least one \( i \in \{1, 2, \ldots, k\} \). Suppose that \( P_i \) is not normal in \( G \) for each \( i \in \{1, 2, \ldots, k\} \). Let \( i \in \{1, \ldots, k\} \). Observe that \( N(P_i) \supseteq P_i \), where \( N(P_i) \) is the normalizer of \( P_i \) in \( G \). Since \( P_i \) is not normal in \( G \), it follows that \( N(P_i) \neq G \). Hence, \( N(P_i) \) is a nontrivial subgroup of \( G \). As any nontrivial subgroup of \( G \) is minimal, we obtain that \( N(P_i) = P_i \). Note that \( P_i \) is a \( p_i \)-Sylow subgroup of \( G \). We know from [6, Lemma 2.12.6, p.99] that the number of \( p_i \)-Sylow subgroups in \( G \) equals \( \frac{o(G)}{o(N(P_i))} = \frac{o(G)}{o(P_i)} = \frac{o(G)}{p_i} \).

Let \( \{P_i = P_{i1}, P_{i2}, \ldots, P_{i, \frac{o(G)}{p_i}}\} \) be the set of all \( p_i \)-Sylow subgroups of \( G \). As any element \( g \) of a \( p_i \)-Sylow subgroup with \( g \neq e \) is of order \( p_i \), it follows that \( G \) has exactly \( \frac{o(G)}{p_i} \) \((p_i - 1)\) elements of order \( p_i \). As any nontrivial subgroup of \( G \) is minimal, it follows that if \( x \in G \) with \( x \neq e \), then \( o(x) = p_i \) for some \( i \in \{1, 2, \ldots, k\} \). It is now clear from the above discussion that \( o(G) = \frac{o(G)}{p_1} \)(\( p_1 - 1 \)) + \( \frac{o(G)}{p_2} \)(\( p_2 - 1 \)) + \cdots + \( \frac{o(G)}{p_k} \)(\( p_k - 1 \)) + 1. This implies that \( 1 = k - \left( \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right) + \frac{1}{o(G)} \). We can assume that \( 2 \leq p_1 < p_2 < \cdots < p_k \). Hence, we obtain that \( k - 1 + \frac{1}{o(G)} < \frac{k}{2} \). This is a contradiction. Therefore, \( P_i \) is normal in \( G \) for at least one \( i \in \{1, 2, \ldots, k\} \). Fix \( i \in \{1, 2, \ldots, k\} \) such that \( P_i \) is normal in \( G \). Suppose that \( k \geq 3 \). Let \( j \in \{1, 2, \ldots, k\} \setminus \{i\} \). Observe that \( P_i P_j \) is a subgroup of \( G \) and as \( P_i \cap P_j = \{e\} \), it follows from [6, Theorem 2.5.1, p.45] that \( o(P_i P_j) = p_ip_j \). Note that \( P_i P_j \) is a nontrivial subgroup of \( G \) and is not minimal. This is in contradiction to the assumption that \((\Gamma(G))^c \) is complete. Therefore, \( k = 2 \). Hence, \( o(G) = p_1p_2 \), where \( p_1, p_2 \) are distinct prime numbers.

\((ii) \Rightarrow (i)\) Assume that \( o(G) = p_1p_2 \), where \( p_1 \) and \( p_2 \) are distinct prime numbers. It follows from Lagrange’s theorem that any nontrivial subgroup of \( G \) is of order either \( p_1 \) or \( p_2 \). Hence, any nontrivial subgroup of \( G \) is minimal and so, we obtain from Lemma 2.3 that \((\Gamma(G))^c \) is complete. \( \square \)

Remark 2.7. Let \( G \) be a finite group with \( o(G) = p_1p_2 \), where \( p_1, p_2 \) are distinct primes. In this remark, we mention some well-known facts about the structure of \( G \). If \( G \) is abelian, then \( G \) is necessarily cyclic and in such a case, \((\Gamma(G))^c \) is \( K_2 \). Suppose that \( G \) is not abelian. We can assume that \( p_1 < p_2 \). We know from [6, Theorem 2.12.3 and Lemma 2.12.6, p.100, p.99] that \( G \) has a unique subgroup \( H \) with \( o(H) = p_2 \).
Let\, that\, that\, 2.5\, (\text{that each element} G\, is the subgroup of number, we obtain that any minimal subgroup of \, G\, is connected and determine its diameter when it is connected. First, we consider finite abelian groups with \, o(G) = p^n\, where \, p\, is prime number and \, n \geq 2.\\

\textbf{Proposition 2.8.} Let \, G\, be a finite abelian group with \, o(G) = p^n\, where \, p\, is a prime number and \, n \geq 2. Then the following statements are equivalent:\n
(i) \, (\Gamma(G))^c\, is connected.\\
(ii) \, G\, is the internal direct product of cyclic subgroups \, A_1, A_2, \ldots, A_n\, with \, o(A_i) = p\, for each \, i \in \{1,2,\ldots,n\}.\\

Moreover, in the case when \, (\Gamma(G))^c\, is connected, \, diam((\Gamma(G))^c) = 1\, if \, n = 2\, and \, diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2\, if \, n \geq 3.\\

\textbf{Proof.} (i) \Rightarrow (ii) Assume that \, (\Gamma(G))^c\, is connected. We know from Proposition 2.1 that \, N_G = G.\, Since \, o(G) = p^n\, where \, p\, is a prime number, we obtain that any minimal subgroup of \, G\, is of order \, p. As \, G\, is the subgroup of \, G\, generated by all its minimal subgroups, it follows that each element \, g \in G\, with \, g \neq e\, is of order \, p. We know from [10, Example 2.5, p.146] that there exist cyclic subgroups \, A_1, A_2, \ldots, A_n\, of \, G\, satisfying the following properties: \, o(A_i) = p\, for each \, i \in \{1,2,\ldots,n\}\, and \, G\, is the internal direct product of \, A_1, A_2, \ldots, A_n. This shows that \, G\, is the internal direct product of cyclic subgroups \, A_1, A_2, \ldots, A_n\, with \, o(A_i) = p\, for each \, i \in \{1,2,\ldots,n\}.\\

(ii) \Rightarrow (i) Assume that there exist cyclic subgroups \, A_1, A_2, \ldots, A_n\, with \, o(A_i) = p\, for each \, i \in \{1,2,\ldots,n\}\, and \, G\, is the internal direct product of \, A_1, A_2, \ldots, A_n.\\

Suppose that \, n = 2.\, Then we know from the proof of (ii) \Rightarrow (i) of Lemma 2.5 that \, (\Gamma(G))^c\, is \, K_{p+1}.\, Therefore, \, diam((\Gamma(G))^c) = 1.\\

Let us next suppose that \, n \geq 3.\, Let \, H_1, H_2\, be two distinct nontrivial subgroups of \, G\, with \, H_1 \neq H_2.\, We show that there exists a path of length at most two between \, H_1\, and \, H_2\, in \, (\Gamma(G))^c.\, We can assume that \, H_1\, and \, H_2\, are not adjacent in \, (\Gamma(G))^c.\, If \, H_1, H_2\, are not comparable under the inclusion relation, then it is clear that \, H_1 \cup H_2\, is not a subgroup of \, G\, and therefore, \, H_1 \cup H_2 \neq G.\, Let \, g \in G\, be such that \, g \not\in H_1 \cup H_2.\, Let \, K = \langle g \rangle.\, Note that \, o(K) = p\, and \, H_i \cap K = \{e\}\, for each \, i \in \{1,2\}.\, Hence, \, H_1 - K - H_2\, is a path of length two between \, H_1\, and \, H_2\, in \, (\Gamma(G))^c.\, Suppose that \, H_1\, and \, H_2\, are comparable under the inclusion relation. We can assume without loss of generality that \, H_1 \subset H_2.\, Since \, H_2 \neq G,\, it follows that \, A_i \not\subset H_2.
for some \( i \in \{1, 2, \ldots, n\} \). As \( o(A_i) = p \), it follows that \( H_2 \cap A_i = \{e\} \) and so, \( H_1 \cap A_i = \{e\} \). Hence, \( H_1 - A_i - H_2 \) is a path of length two between \( H_1 \) and \( H_2 \) in \((\Gamma(G))^c\). This shows that \((\Gamma(G))^c\) is connected and \( diam((\Gamma(G))^c) \leq 2 \). We next verify that \( e(S) \geq 2 \) in \((\Gamma(G))^c\) for any nontrivial subgroup \( S \) of \( G \). Note that \( o(S) = p^i \) for some \( i \) with \( 1 \leq i < n \). Observe that there exists a subgroup \( W \) of \( S \) with \( o(W) = p \). If \( i > 1 \), then \( W \neq S \) and \( S \) and \( W \) are not adjacent in \((\Gamma(G))^c\) and so, \( d(S,W) \geq 2 \) in \((\Gamma(G))^c\). Suppose that \( i = 1 \). Now, \( A_k \not\subseteq S \) for some \( k \in \{1, 2, \ldots, n\} \). Hence, \( A_k \cap S = \{e\} \). Observe that \( SA_k \) is a subgroup of \( G \) and it follows from [6, Theorem 2.5.1, p.45] that \( o(SA_k) = p^2 \). As \( o(G) = p^n \) with \( n \geq 3 \), it is clear that \( SA_k \) is a nontrivial subgroup of \( G \). Since \( S \cap SA_k \neq \{e\} \), we get that \( S \) and \( SA_k \) are not adjacent in \((\Gamma(G))^c\). Therefore, \( d(S,SA_k) \geq 2 \) in \((\Gamma(G))^c\). This proves that \( e(S) \geq 2 \) in \((\Gamma(G))^c\) for each nontrivial subgroup \( S \) of \( G \). This proves that \( diam((\Gamma(G))^c) = r((\Gamma(G))^c) = 2 \).

The proof of the moreover part is contained in the proof of \((ii) \Rightarrow (i)\) of this Proposition. \(\square\)

Let \( G \) be a finite abelian group with \( o(G) = \prod_{i=1}^{k} p_i^{n_i} \), where \( k \geq 2 \) and \( p_1, p_2, \ldots, p_k \) are distinct prime numbers and \( n_i \geq 1 \) for each \( i \in \{1, 2, \ldots, k\} \). We next proceed to characterize \( G \) such that \((\Gamma(G))^c\) is connected and determine its diameter when it is connected.

**Proposition 2.9.** Let \( G \) be a finite abelian group such that \( o(G) = \prod_{i=1}^{k} p_i^{n_i} \), where \( k \geq 2 \) and \( p_1, p_2, \ldots, p_k \) are distinct prime numbers and \( n_i \geq 1 \) for each \( i \in \{1, 2, \ldots, k\} \). For each \( i \in \{1, 2, \ldots, k\} \), let \( P_i \) denote the unique \( p_i \)-Sylow subgroup of \( G \). Then the following statements are equivalent:

(i) \( (\Gamma(G))^c \) is connected.

(ii) Given \( i \in \{1, 2, \ldots, k\} \), either \( o(P_i) = p_i \) or \((\Gamma(P_i))^c\) is connected.

**Proof.** \((i) \Rightarrow (ii)\) Assume that \((\Gamma(G))^c \) is connected. Since \( k \geq 2 \), \( G \) has at least two nontrivial subgroups. Indeed, \( P_i \) is a nontrivial subgroup of \( G \) for each \( i \in \{1, 2, \ldots, k\} \) and \( o(P_i) = p_i^{n_i} \) for each \( i \in \{1, 2, \ldots, k\} \). It is well-known that \( G \) is the internal direct product of \( P_1, P_2, \ldots, P_k \). As \((\Gamma(G))^c \) is connected, we obtain from \((i) \Rightarrow (ii)\) of Proposition 2.1 that \( N_G = G \). Let \( g \in G \), \( g \neq e \). It follows from \( N_G = G \) that \( o(g) = \prod_{j \in A} p_j \) for some nonempty subset \( A \) of \( \{1, 2, \ldots, k\} \). Let \( i \in \{1, 2, \ldots, k\} \). Suppose that \( o(P_i) \neq p_i \). Hence, \( n_i \geq 2 \). As any element \( x \) of \( P_i \) with \( x \neq e \) is of order \( p_i \), it follows from [10, Example 2.5, p.146] that there exist cyclic subgroups \( A_{i_1}, A_{i_2}, \ldots, A_{i_{n_i}} \) of \( P_i \) such that \( o(A_{i_j}) = p_i \) for each \( j \in \{1, 2, \ldots, n_i\} \) and \( P_i \) is the
internal direct product of $A_{i_1}, A_{i_2}, \ldots, A_{i_n}$. Now, it follows from $(ii) \Rightarrow (i)$ of Proposition 2.8 that $(\Gamma(P_i))^c$ is connected.

$(ii) \Rightarrow (i)$ It is well-known that $G$ is the internal direct product of $P_1, P_2, \ldots, P_k$. Let $g \in G$, $g \neq e$. Now, there exist unique elements $x_1, x_2, \ldots, x_k$ with $x_i \in P_i$ for each $i \in \{1, 2, \ldots, k\}$ such that $g = \prod_{i=1}^k x_i$. As $g \neq e$, it follows that $x_i \neq e$ for at least one $i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots, k\}$ be such that $x_i \neq e$. By assumption, either $o(P_i) = p_i$ or $(\Gamma(P_i))^c$ is connected. If $o(P_i) = p_i$, then $o(x_i) = p_i$. Suppose that $(\Gamma(P_i))^c$ is connected. Then it follows from $(i) \Rightarrow (ii)$ of Proposition 2.8 that $o(x_i) = p_i$. Hence, in any case $o(x_i) = p_i$. Now, it follows from $g = \prod_{i=1}^k x_i$ that $g \in N_G$ and so, $N_G = G$. Therefore, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(G))^c$ is connected.\[\square\]

Let $G$ be a finite abelian group and let $o(G)$ be as in the statement of Proposition 2.9. Suppose that $(\Gamma(G))^c$ is connected. In Proposition 2.11, we determine $\text{diam}((\Gamma(G))^c)$. We use Lemma 2.10 in the proof of Proposition 2.11.

**Lemma 2.10.** Let $G$ be a finite abelian group such that $G$ has at least two nontrivial subgroups. Suppose that $(\Gamma(G))^c$ is connected. Then the following hold:

(i) $\text{diam}((\Gamma(G))^c) = 2$ if and only if $G$ admits a nontrivial subgroup which is not a minimal subgroup of $G$ and if $H_1, H_2$ are distinct maximal subgroups of $G$ with $H_1 \cap H_2 \neq \{e\}$, then $H_1$ and $H_2$ are isomorphic.

(ii) $\text{diam}((\Gamma(G))^c) = 3$ if and only if there exist nonisomorphic maximal subgroups $H_1, H_2$ of $G$ such that $H_1 \cap H_2 \neq \{e\}$.

**Proof.** Since $(\Gamma(G))^c$ is connected, we know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $\text{diam}((\Gamma(G))^c) \leq 3$.

(i) Assume that $\text{diam}((\Gamma(G))^c) = 2$. We know from Lemma 2.3 that $G$ admits at least one nontrivial subgroup which is not a minimal subgroup of $G$. Let $H_1, H_2$ be distinct maximal subgroups of $G$ such that $H_1 \cap H_2 \neq \{e\}$. Note that $H_1$ and $H_2$ are not adjacent in $(\Gamma(G))^c$. As $\text{diam}((\Gamma(G))^c) = 2$, there exists a nontrivial subgroup $K$ of $G$ such that $H_1 - K - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(G))^c$. Hence, $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. As $H_i$ is a maximal subgroup of $G$, we obtain that $H_i K = G$. Therefore, we obtain from the second isomorphism theorem of groups [3, Theorem 2.3, p.98] that $\frac{G}{K} = \frac{H_i K}{K} \simeq \frac{H_i}{H_i \cap K} = H_i$ for each $i \in \{1, 2\}$.

Conversely, assume that $G$ admits at least one nontrivial subgroup which is not a minimal subgroup and any two distinct maximal subgroups of $G$ which are not adjacent in $(\Gamma(G))^c$ are isomorphic. As
there exists at least one nontrivial subgroup of \( G \) which is not a minimal subgroup of \( G \), it follows from Lemma 2.3 that \( \text{diam}(\Gamma(G)^c) \geq 2 \). Let \( W_1, W_2 \) be nontrivial subgroups of \( G \). We prove that there exists a path of length at most two between \( W_1 \) and \( W_2 \) in \((\Gamma(G))^c\). We can assume that \( W_1 \) and \( W_2 \) are not adjacent in \((\Gamma(G))^c\). That is, \( W_1 \cap W_2 \neq \{e\} \). Let \( H_i \) be a maximal subgroup of \( G \) such that \( W_i \subseteq H_i \) for each \( i \in \{1, 2\} \). Observe that \( H_1 \cap H_2 \neq \{e\} \). It can happen that \( H_1 = H_2 \). And in the case, \( H_1 \neq H_2 \), we know from the assumption that \( H_1 \) and \( H_2 \) are isomorphic. Thus in any case, \( o(H_1) = o(H_2) \). Hence, we obtain that \( o(G_{H_1}) = o(G_{H_2}) \). Since \( G_{H_i} \) is an abelian simple group, we get that \( \frac{G}{H_i} \) is a cyclic group for each \( i \in \{1, 2\} \) with \( o(G_{H_i}) = o(G_{H_i}) = p \), where \( p \) is a prime number. As \((\Gamma(G))^c\) is connected, we know from \((i) \Rightarrow (ii)\) of Proposition 2.1 that \( N_G = G \). Let \( i \in \{1, 2\} \). As \( H_i \neq G \), there exists a minimal subgroup \( M_i \) of \( G \) such that \( M_i \nsubseteq H_i \). We know from Remark 2.4 that \( o(M_i) \) is a prime number. Since \( H_i \) is a maximal subgroup of \( G \), we obtain that \( H_i M_i = G \). It follows from \( H_i \cap M_i = \{e\} \) and [6, Theorem 2.5.1, p.45] that \( o(G) = o(H_i)o(M_i) \). Therefore, we obtain that \( o(M_i) = \frac{o(G)}{o(H_i)} = o(G_{H_i}) = p \). If \( M_2 \nsubseteq H_1 \), then it follows from \( H_i \cap M_2 = H_2 \cap M_2 = \{e\} \) that \( W_i \cap M_2 = \{e\} \) for each \( i \in \{1, 2\} \) and so, \( W_1 - M_2 - W_2 \) is a path of length two between \( W_1 \) and \( W_2 \) in \((\Gamma(G))^c\). Similarly, if \( H_2 \cap M_1 = \{e\} \), then it follows that \( W_1 - M_1 - W_2 \) is a path of length two between \( W_1 \) and \( W_2 \) in \((\Gamma(G))^c\). Suppose that \( M_2 \subseteq H_1 \) and \( M_1 \subseteq H_2 \). Note that \( M_i \) is a cyclic group with \( o(M_i) = p \) for each \( i \in \{1, 2\} \). Let \( g_1 \in M_1 \setminus M_2 \) and let \( g_2 \in M_2 \setminus M_1 \). Observe that \( o(g_1g_2) = p \) and let us denote \( g_1g_2 > M \). It is clear that \( M \) is a minimal subgroup of \( G \) and \( M \nsubseteq H_i \) for each \( i \in \{1, 2\} \). Therefore, \( W_i \cap M \subseteq H \cap M = \{e\} \) for each \( i \in \{1, 2\} \). Hence, we obtain that \( W_1 - M - W_2 \) is a path of length two between \( W_1 \) and \( W_2 \) in \((\Gamma(G))^c\). Therefore, we get that \( \text{diam}(\Gamma(G)^c) = 2 \).

(ii) Assume that \( \text{diam}(\Gamma(G)^c) = 3 \). Let \( W_1, W_2 \) be distinct nontrivial subgroups of \( G \) such that \( d(W_1, W_2) = 3 \) in \((\Gamma(G))^c\). Let \( i \in \{1, 2\} \). Let \( H_i \) be a maximal subgroup of \( G \) such that \( W_i \subseteq H_i \). From \( W_1 \cap W_2 \neq \{e\} \), it follows that \( H_1 \cap H_2 \neq \{e\} \). If \( H_1 \cong H_2 \) as groups, then it follows from the proof of the if part of \((i)\) that \( d(W_1, W_2) = 2 \) in \((\Gamma(G))^c\). This is in contradiction to the assumption that \( d(W_1, W_2) = 3 \) in \((\Gamma(G))^c\). Therefore, \( H_1 \) and \( H_2 \) are nonisomorphic. This proves that there exist nonisomorphic maximal subgroups \( H_1, H_2 \) of \( G \) such that \( H_1 \cap H_2 \neq \{e\} \).

Conversely, assume that there exist nonisomorphic maximal subgroups \( H_1, H_2 \) of \( G \) such that \( H_1 \cap H_2 \neq \{e\} \). It follows from the
Let that

2.1

2.10

H

We can assume that

P

G

W

P

trivial subgroups. If

H

S

Proposition

2.12

Remark

H

G

clear that

Suppose that

P

H

G

nontrivial subgroups of

P

the internal direct product of

P

let

P

is connected. Then the following hold.

If

k = 2,

then

diam(\((\Gamma(G))^c\)) = 1 if and only if

n_i = n_2 = 1.

If

n_i \geq 2

for some

i \in \{1, 2\},

then

diam(\((\Gamma(G))^c\)) = 3.

If

k \geq 3,

then

diam(\((\Gamma(G))^c\)) = 3.

Proof. Suppose that

\((\Gamma(G))^c\)

is connected. For each

i \in \{1, 2, \ldots, k\},

let

P_i

denote the unique

p_i-Sylow subgroup of

G. We know that

G

is the internal direct product of

P_1, P_2, \ldots, P_k.

Suppose that

k = 2. If

n_1 = n_2 = 1,

then

P_1,

P_2

are the only nontrivial subgroups of

G

and

\((\Gamma(G))^c\)

is

K_2

and so,

\(diam(\((\Gamma(G))^c\)) = 1.

Suppose that

n_i \geq 2

for some

i \in \{1, 2\}. Without loss of generality, we can assume that

n_1 \geq 2.

Let

W_i

be a subgroup of

P_i

with

o(W_i) = p_i^{n_i-1}

for each

i \in \{1, 2\}. Let

H_1

be the internal direct product of

W_1

and

P_2

and

H_2

be the internal direct product of

P_1

and

W_2.

Observe that

\(o(H_1) = p_1^{n_1-1}p_2^{n_2}

and

\(o(H_2) = p_1^{n_1}p_2^{n_2-1}.

It is clear that

H_1

and

H_2

are nonisomorphic maximal subgroups of

G

with

H_1 \cap H_2 \neq \{e\}. Hence, it follows from Lemma 2.10(ii) that

\(diam(\((\Gamma(G))^c\)) = 3.

Suppose that

k \geq 3.

Let

W_1

be the internal direct product of

P_1, P_2, \ldots, P_{k-1}.

Let

W_2

be the internal direct product of

P_2, \ldots, P_k.

Let

U

be a subgroup of

P_1

with

o(U) = p_1^{n_1-1}

and let

W

be a subgroup of

P_k

with

o(W) = p_k^{n_k-1}.

Let

H_1

be the internal direct product of

W_1

and

W

and let

H_2

be the internal direct product of

W_2

and

U.

It is clear that

\(o(H_1) = (\prod_{i=1}^{k-1} p_i^{n_i}) p_k^{n_k-1},

\(o(H_2) = p_1^{n_1-1}(\prod_{j=2}^{k} p_j^{n_i}),

H_1

and

H_2

are nonisomorphic maximal subgroups of

G

with

H_1 \cap H_2 \neq \{e\}. Therefore, we obtain from Lemma 2.10(ii) that

\(diam(\((\Gamma(G))^c\)) = 3.

Remark 2.12. Let

G

be a finite group which admits at least two nontrivial subgroups. If

\((\Gamma(G))^c\)

is connected, then

\(e(H) \leq 2\)

in

\((\Gamma(G))^c\)

for any minimal subgroup

H

of

G.

Proof. Let

H

be a minimal subgroup of

G. Let

W

be any nontrivial subgroup of

G

with

W \neq H.

We claim that

\(d(H, W) \leq 2\)

in

\((\Gamma(G))^c\).

We can assume that

H

and

W

are not adjacent in

\((\Gamma(G))^c\).

Hence,

\(H \cap W \neq \{e\}.\) As

H

is a minimal subgroup of

G,

it follows that

\(H \subset W.\) Since

\((\Gamma(G))^c\)

is connected, we know from

(i) \(\Rightarrow\) (ii)

of Proposition 2.1 that

\(N_G = G.\) It follows from

W \neq G

that there exists a minimal subgroup

S

of

G

such that

\(S \subset W.\) Observe that

\(H \cap S = W \cap S = \{e\}.\) Therefore,

\(H - S - W\)

is a path of length
two between $H$ and $W$ in $(\Gamma(G))^c$. This proves that $d(H,W) \leq 2$ in $(\Gamma(G))^c$ for any nontrivial subgroup $W$ of $G$ and so, $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup $H$ of $G$. \hfill \Box$

Remark 2.13. Let $G$ be a finite abelian group and let $o(G) = \prod_{i=1}^{k} p_i^{n_i}$, where $k \geq 2$ and $p_1,p_2,\ldots,p_k$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1,2,\ldots,k\}$. Suppose that $(\Gamma(G))^c$ is connected and in the case $k = 2$, either $n_1 > 1$ or $n_2 > 1$. Then $r((\Gamma(G))^c) = 2$.

Proof. Let $H$ be any minimal subgroup of $G$. We know from Remark 2.12 that $e(H) \leq 2$ in $(\Gamma(G))^c$.

In the case $k \geq 3$, it is clear that if $H$ is a minimal subgroup of $G$, then there exists at least one nontrivial subgroup $W$ of $G$ such that $H \subset W$ and so, $H$ and $W$ are not adjacent in $(\Gamma(G))^c$. In the case $k = 2$, we are assuming that either $n_1 > 1$ or $n_2 > 1$. Hence, in this case also, given a minimal subgroup $H$ of $G$, there exists a nontrivial subgroup $W$ of $G$ such that $H \subset W$ and so, $H$ and $W$ are not adjacent in $(\Gamma(G))^c$. Therefore, $d(H,W) \geq 2$ in $(\Gamma(G))^c$. It is already shown that $e(H) \leq 2$ in $(\Gamma(G))^c$ for any minimal subgroup $H$ of $G$. This proves that $e(H) = 2$ in $(\Gamma(G))^c$ for any minimal subgroup $H$ of $G$. As for a given nontrivial subgroup $W$ of $G$, there exists a minimal subgroup $H$ of $G$ such that $H \subseteq W$, it follows that $e(W) \geq 2$ in $(\Gamma(G))^c$. Therefore, we obtain that $r((\Gamma(G))^c) = 2$. \hfill \Box$

Let $n \geq 3$. Let $S_n$ denote the symmetric group of degree $n$. We know from [6, Lemma 2.10.2, p.78] that any $\sigma \in S_n$ is a product of transpositions. If $\tau = (i,j)$ is any transposition, then $o(\tau) = 2$ in $S_n$. Therefore, $N_{S_n} = S_n$ and so, we obtain from $(ii) \Rightarrow (i)$ of Proposition 2.1 that $(\Gamma(S_n))^c$ is connected. In Proposition 2.14, we determine $\text{diam}((\Gamma(S_n))^c)$.

Proposition 2.14. Let $n \geq 3$. Then $(\Gamma(S_n))^c$ is connected and $\text{diam}((\Gamma(S_n))^c) = 1$, whereas $\text{diam}((\Gamma(S_n))^c) = 2$ for all $n \geq 4$.

Proof. It is already noted above that $(\Gamma(S_n))^c$ is connected. Observe that $o(S_3) = 6 = 2 \times 3$ and $S_3$ is not abelian. We know from Remark 2.7 that $(\Gamma(S_3))^c$ is a complete graph on four vertices. Therefore, we obtain that $\text{diam}((\Gamma(S_3))^c) = 1$. Let $n \geq 4$. Let $\sigma = (1,2,3,4)$. Let $H = < \sigma >$ and let $K = < \sigma^2 >$. Observe that $o(H) = 4$ and $o(K) = 2$ and $H \cap K = K$ is nontrivial. Hence, $H$ and $K$ are not adjacent in $(\Gamma(S_n))^c$. Therefore, $\text{diam}((\Gamma(S_n))^c) \geq 2$. We next verify that $\text{diam}((\Gamma(S_n))^c) \leq 2$. Let $H_1,H_2$ be any nontrivial subgroups of $S_n$ with $H_1 \neq H_2$. We claim that there exists a path of length at most two between $H_1$ and $H_2$ in $(\Gamma(S_n))^c$. We can assume that $H_1$ and $H_2$...
Let that $6$ adjacent in $(\Gamma(S_n))^c$. It is well-known that $S_n$ is generated by the set of 2-cycles $\{(1, i) : i \in \{2, 3, \ldots, n\}\}$. Since $H_1 \neq S_n$, it follows that $(1, i) \notin H_1$ for some $i \in \{2, 3, \ldots, n\}$. If $(1, i) \notin H_2$, then with $H = \langle (1, i) \rangle$, we get that $H_1 \cap H = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - H - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(S_n))^c$. Suppose that $(1, i) \in H_2$. As $H_2 \neq S_n$, we obtain that there exists $j \in \{2, 3, \ldots, n\}$ such that $(1, j) \notin H_2$. It is clear that $i \neq j$. If $(1, j) \notin H_1$, then $H_1 - (1, j) > -H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(S_n))^c$. Suppose that $(1, j) \in H_1$. Thus $(1, j) \in H_1 \setminus H_2$ and $(1, i) \in H_2 \setminus H_1$. Let $\rho = (1, i)(1, j)$. Note that $\rho = (1, j, i)$ is a cycle of length 3 and let $H_3 = \langle (1, j, i) \rangle$. It is clear that $H_3 = \{e, \rho, \rho^2\}$ and $\rho \notin H_1 \cup H_2$. Hence, we get that $H_1 \cap H_3 = \{e\}$ for each $i \in \{1, 2\}$. Therefore, $H_1 - H_3 - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(S_n))^c$. From the above discussion, it is clear that $\text{diam}((\Gamma(S_n))^c) \leq 2$ and so, $\text{diam}((\Gamma(S_n))^c) = 2$. $\square$

Remark 2.15. Let $n \geq 4$. Then $r((\Gamma(S_n))^c) = 2$.

Proof. Let $n \geq 4$. We know from Proposition 2.14 that $(\Gamma(S_n))^c$ is connected and $\text{diam}((\Gamma(S_n))^c) = 2$. Therefore, $e(H) \leq 2$ in $(\Gamma(S_n))^c$ for each nontrivial subgroup $H$ of $S_n$. Hence, to prove this remark, it is enough to show that $e(H) \geq 2$ in $(\Gamma(S_n))^c$ for any nontrivial subgroup $H$ of $S_n$. Let $H$ be any nontrivial subgroup of $S_n$. If $H$ is not a minimal subgroup of $S_n$, then it is clear that $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Hence, we can assume that $H$ is a minimal subgroup of $S_n$. Note that either $H \subseteq A_n$ or $H \not\subseteq A_n$, where $A_n$ is the alternating group of degree $n$.

It is known that $o(A_n) = \frac{o(S_n)}{2}$ [6, Lemma 2.10.3, p.80] Thus, $A_n$ is a maximal subgroup of $S_n$ and is a normal subgroup of $S_n$. If $H \subseteq A_n$, then as $A_n$ is not a minimal subgroup of $S_n$, it follows that $H \neq A_n$. Therefore, it follows from $H \cap A_n = H \neq \{e\}$ that $H$ and $A_n$ are not adjacent in $(\Gamma(S_n))^c$. Hence, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that $H \not\subseteq A_n$. Then $A_n H = S_n$ and $H \cap A_n = \{e\}$. We know from [6, Theorem 2.5.1, p.45] that $o(S_n) = o(A_n)o(H)$ and so, $o(H) = 2$. Let $\sigma \in S_n$ be such that $H = \{e, \sigma\}$. Note that $o(\sigma) = 2$. It follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $\tau_1, \ldots, \tau_k$ such that $\sigma = \prod_{i=1}^{k} \tau_i$. As $\sigma$ is an odd permutation, $k$ must be odd. Suppose that $k = 1$. Let $\sigma = \tau_1 = (i_1, i_2)$. As $n \geq 4$, there exist distinct symbols $i_3, i_4$ such that $i_3, i_4 \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2\}$. Let $K$ be the subgroup of $S_n$ generated by $\{\sigma, (i_1, i_3)\}$. It is clear that $(i_1, i_3) \notin K$ and so, $K \neq S_n$. Since $(i_1, i_3) \notin H$, it follows that $H \subset K$. Hence, $H$ and $K$ are not adjacent in $(\Gamma(S_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(S_n))^c$. Suppose that $k$ is odd and $k \geq 3$. Let $\tau_1 = (i_1, i_2), \tau_2 = (i_3, i_4), \ldots, \tau_k = (i_{2k-1}, i_{2k})$. Let $K$ be the subgroup of $S_n$ generated by $\{\sigma, (i_1, i_2)\}$. It is clear that
Let \((i_1, i_3) \notin K\) and so, \(K \neq S_n\). Since \((i_1, i_2) \in K \setminus H\), it follows that \(H \subset K\). Hence, \(H\) and \(K\) are not adjacent in \((\Gamma(S_n))^c\). Therefore, we get that \(e(H) \geq 2\) in \((\Gamma(S_n))^c\). This proves that \(r((\Gamma(S_n))^c) = 2\). □

**Remark 2.16.** Let \(n \geq 4\). It is well-known that \(A_n\) is generated by the set of 3-cycles \\{(1, 2, i) : i \in \{3, 4, \ldots, n\}\} [10, Proposition 4.5.1, p.55]. If \(\sigma \in S_n\) is any 3-cycle, then \(o(\sigma) = 3\) and so, \(<\sigma> = \{e, \sigma, \sigma^2\}\).

It follows from the above given arguments that \(N_{A_n} = A_n\) and so, we obtain from \((ii) \Rightarrow (i)\) of Proposition 2.17 that \((\Gamma(A_n))^c\) is connected. We prove in Proposition 2.17 that \(\text{diam}((\Gamma(A_n))^c) = 2\).

**Proposition 2.17.** Let \(n \geq 4\). Then \((\Gamma(A_n))^c\) is connected and \(\text{diam}((\Gamma(A_n))^c) = 2\).

**Proof.** It is noted in Remark 2.16 that \((\Gamma(A_n))^c\) is connected. Let \(H = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}\) and \(K = \{e, (1, 2)(3, 4)\}\). It is clear that \(H, K\) are subgroups of \(A_n\) and \(H \cap K = K\) is non-trivial. Hence, \(H\) and \(K\) are not adjacent in \((\Gamma(A_n))^c\). Therefore, \(d(H, K) \geq 2\) in \((\Gamma(A_n))^c\) and so, \(\text{diam}((\Gamma(A_n))^c) \geq 2\). We next verify that \(\text{diam}((\Gamma(A_n))^c) \leq 2\). Let \(H_1, H_2\) be nontrivial subgroups of \(A_n\) with \(H_1 \neq H_2\). We show that there exists a path of length at least two between \(H_1\) and \(H_2\) in \((\Gamma(A_n))^c\). We can assume that \(H_1\) and \(H_2\) are not adjacent in \((\Gamma(A_n))^c\). Since \(H_1 \neq A_n\), \((1, 2, i) \notin H_1\) for some \(i \in \{3, 4, \ldots, n\}\). If \((1, 2, i) \notin H_2\), then \(H_1 - <(1, 2, i)> - H_2\) is a path of length two between \(H_1\) and \(H_2\) in \((\Gamma(A_n))^c\). Suppose that \((1, 2, i) \in H_2\). As \(H_2 \neq A_n\), there exists \(j \in \{3, 4, \ldots, n\}\) such that \((1, 2, j) \notin H_2\). If \((1, 2, j) \notin H_1\), then \(H_1 - <(1, 2, j)> - H_2\) is a path of length two between \(H_1\) and \(H_2\) in \((\Gamma(A_n))^c\). Suppose that \((1, 2, j) \in H_1\). Now, \((1, 2, j) \in H_1 \setminus H_2\) and \((1, 2, i) \in H_2 \setminus H_1\). Let \(\rho = (1, 2, i)(1, 2, j)\). Observe that \(\rho \notin H_1 \cup H_2\) and \(\rho = (1, i)(2, j)\). Let \(H_3 = <\rho>\). As \(H_3 = \{e, \rho\}\), we obtain that \(H_i \cap H_3 = \{e\}\) for each \(i \in \{1, 2\}\) and so, \(H_1 - H_3 - H_2\) is a path of length two between \(H_1\) and \(H_2\) in \((\Gamma(A_n))^c\). This proves that \(\text{diam}((\Gamma(A_n))^c) \leq 2\) and so, we obtain that \(\text{diam}((\Gamma(A_n))^c) = 2\). □

**Proposition 2.18.** Let \(n \geq 4\). Then the following hold.

(i) \(r((\Gamma(A_4))^c) = 1\).

(ii) Let \(n \geq 5\). Let \(H\) be a minimal subgroup of \(A_n\). If \(o(H) \in \{2, 3\}\) or \(o(H) \equiv 1(\text{mod} 4)\), then there exists a nontrivial subgroup \(W\) of \(A_n\) such that \(H \subset W\).

**Proof.** Let \(n \geq 4\). It is proved in Proposition 2.17 that \((\Gamma(A_n))^c\) is connected and \(\text{diam}((\Gamma(A_n))^c) = 2\).

(i) We verify that \(r((\Gamma(A_4))^c) = 1\). Let \(\sigma = (1, 2, 3)\) and let \(H = <\sigma>\). Observe that \(o(H) = 3\). We claim that \(e(H) = 1\) in \((\Gamma(A_4))^c\). Let
$K$ be any nontrivial subgroup of $A_4$ with $K \neq H$. We assert that $H \cap K = \{e\}$. Suppose that $H \cap K \neq \{e\}$. Then as $H$ is a minimal subgroup of $A_4$, it follows that $H \subset K$. It follows from Lagrange’s theorem [6, Theorem 2.4.1, p.41] that $o(K) = 3t$ for some $t \in \mathbb{N}$ with $t \geq 2$. As $o(K)$ is a divisor of $o(A_4) = 12$ and $K \neq A_4$, it follows that $o(K) = 6$. This is impossible since it is well-known that $A_4$ has no subgroup of order 6 [10, Example 3.3.6, p.75] Therefore, $H \cap K = \{e\}$ and so, $H$ and $K$ are adjacent in $(\Gamma(A_4))^c$. This proves that $e(H) = 1$ in $(\Gamma(A_4))^c$ and therefore, $r((\Gamma(A_4))^c) = 1$.

(ii) Let $n \geq 5$. Let $H$ be a minimal subgroup of $A_n$. Note that $o(H) = p$, where $p$ is a prime number and $H = \langle \sigma \rangle$ for any $\sigma \in H \setminus \{e\}$. We discuss two cases.


Let $\sigma \in H \setminus \{e\}$. As $\sigma \in A_n$ and $o(\sigma) = 2$, it follows from [6, Lemma 2.10.1, p.78] that there exist disjoint transpositions $(i_1, i_2), (i_3, i_4), \ldots, (i_{2k-1}, i_{2k})$ with $k \geq 2$ is even and is such that $\sigma = \prod_{s=1}^k (i_{2s-1}, i_{2s})$. If $k = 2$, then $\sigma = (i_1, i_2)(i_3, i_4)$. Observe that $W = \{e, \sigma, (i_1, i_2)(i_3, i_4), (i_1, i_4)(i_2, i_3)\}$ is a nontrivial subgroup of $A_n$ such that $o(W) = 4$ and $H \subset W$. As $H$ and $W$ are not adjacent in $(\Gamma(A_n))^c$, it follows that $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $k \geq 4$. Let $\sigma_1 = (i_1, i_2)(i_3, i_4)$ and let $\sigma_2 = \prod_{s=3}^k (i_{2s-1}, i_{2s})$. Note that $\sigma_1, \sigma_2 \in A_n$, $o(\sigma_i) = 2$ for each $i \in \{1, 2\}$ and $\sigma_1\sigma_2 = \sigma = \sigma_2\sigma_1$ and $W_1 = \{e, \sigma_1, \sigma_2, \sigma\}$ is a nontrivial subgroup of $A_n$ with $o(W_1) = 4$ and $H \subset W_1$. Hence, $H$ and $W_1$ are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Case (2): $p$ is odd.

Let $\sigma \in H \setminus \{e\}$. Note that $\sigma \in A_n$ and $o(\sigma) = p$. Hence, it follows from [6, Lemma 2.10.1, p.78] that there exists $t \in \mathbb{N}$ and disjoint cycles $C_1, \ldots, C_t$ such that $C_i$ is of length $p$ for each $i \in \{1, \ldots, t\}$ and $\sigma = \prod_{i=1}^t C_i$. Suppose that $t = 1$. Then $\sigma = C_1 = (i_1, i_2, \ldots, i_p)$. Observe that either $p = 3$ or $p \geq 5$. Assume that $p = 3$. Let $i_1 \in \{1, 2, 3, \ldots, n\} \setminus \{i_1, i_2, i_3\}$. Let $W = \{e, (i_1, i_2, i_3), (i_1, i_3, i_2), (i_1, i_2, i_4), (i_1, i_4, i_2), (i_1, i_3, i_4), (i_1, i_4, i_3), (i_2, i_3, i_4), (i_2, i_4, i_3), (i_1, i_3)(i_2, i_4), (i_1, i_4)(i_2, i_3)\}$. Note that $W$ is a nontrivial subgroup of $A_n$ with $o(W) = 12$, $W \cong A_4$ as groups, and $H \cap W = H$. Therefore, $H$ and $W$ are not adjacent in $(\Gamma(A_n))^c$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. Suppose that $p \geq 5$. Note that $\sigma = (i_1, i_2, i_3, \ldots, i_p)$. Suppose that $p \equiv 1 \pmod{4}$. Let $\tau \in S_n$ be given by $\tau(i_1) = i_1, \tau(i_j) = i_{p-j+2}$ for each $j \in \{2, 3, \ldots, p\}$. Observe that $\tau = \prod_{j=2}^{p+1}(i_j, i_{p-j+2})$ is the product of $\frac{p-1}{2}$ disjoint transpositions. As $\frac{p-1}{2}$ is even, we obtain that $\tau \in A_n$. Observe
that $\sigma^p = e, \tau^2 = e, \sigma^{p-1} = \tau \sigma$. Let $W$ be the subgroup of $A_n$ generated by $\{\sigma, \tau\}$. Note that $W = \{e, \sigma, \sigma^2, \ldots, \sigma^{p-1}, \tau, \sigma \tau, \sigma^2 \tau, \ldots, \sigma^{p-1} \tau = \tau \sigma\}$ and $W \cong D_p$ as groups, where $D_p$ is the dihedral group of degree $p$. It is clear that $W$ is a nontrivial subgroup of $A_n$ and $H \subset W$. Hence, $H \cap W = H \neq \{e\}$ and so, $H$ and $W$ are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Suppose that $p$ is an odd prime number and $\sigma$ is the product of $t$ $(t \geq 2)$ disjoint cycles $C_1, C_2, \ldots, C_t$ such that $C_i$ is of length $p$ for each $i \in \{1, 2, \ldots, t\}$. Let $\sigma_1 = C_1$ and let $\sigma_2 = \prod_{j=2}^{t} C_j$. Note that $\sigma_i \in A_n$ for each $i \in \{1, 2\}$, $o(\sigma_1) = o(\sigma_2) = p$, and $\sigma = \sigma_1\sigma_2 = \sigma_2\sigma_1$. Let $W$ be the subgroup of $A_n$ generated by $\{\sigma_1, \sigma_2\}$. Let $H_1 = < \sigma_1 >$ and let $H_2 = < \sigma_2 >$. It is clear that $o(H_1) = o(H_2) = p$, $H_1 \cap H_2 = \{e\}$, and $W = H_1H_2$. It follows from [6, Theorem 2.5.1, p.45] that $o(W) = o(H_1)o(H_2) = p^2$. Observe that $H = < \sigma > \subset W$ and so, $H \cap W = H \neq \{e\}$. Hence, $H$ and $W$ are not adjacent in $(\Gamma(A_n))^c$. Therefore, $e(H) \geq 2$ in $(\Gamma(A_n))^c$.

Thus for any $n \geq 5$, it is shown that if $H$ is any minimal subgroup of $A_n$ with $o(H) \in \{2, 3\}$ or $o(H) \equiv 1(mod4)$, then there exists a nontrivial subgroup $W$ of $A_n$ such that $H \subset W$ and so, $e(H) \geq 2$ in $(\Gamma(A_n))^c$. \hfill \square

Remark 2.19. Let $n \geq 3$. Recall from [3, Theorem 5.2, p.87 and p.88] that the dihedral group of degree $n$ denoted by $D_n$ is the subgroup of $S_n$ generated by $\sigma$ and $\tau$, where $\sigma$ is the cycle given by $\sigma = (1, 2, 3, \ldots, n)$ and $\tau$ is given by $\tau(1) = 1, \tau(i) = n - i + 2$ for each $i \in \{2, 3, \ldots, n\}$. Note that $o(\sigma) = n$, $o(\tau) = 2$, $\sigma^{n-1} = \tau \sigma$, $o(D_n) = 2n$ and indeed, $D_n = \{e, \sigma, \sigma^2, \ldots, \sigma^{n-1}, \tau, \sigma \tau, \sigma^2 \tau, \ldots, \sigma^{n-1} \tau = \tau \sigma\}$. Observe that $D_3 = S_3$ and it is already shown in Proposition 2.14 that $(\Gamma(S_3))^c$ is connected and $diam((\Gamma(S_3))^c) = 1$. Hence, in discussing the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$. Suppose that $n$ is a prime number. Then $n$ is odd and $o(D_n) = 2n$ is the product of two distinct prime numbers. As $D_n$ is not abelian, it follows from Remark 2.7 that $(\Gamma(D_n))^c$ is a complete graph on $n + 1$ vertices. Therefore, in our discussion regarding the connectedness of $(\Gamma(D_n))^c$, we can assume that $n \geq 4$ and $n$ is not a prime number. We prove in Proposition 2.20 that $(\Gamma(D_n))^c$ is connected and moreover, we determine $diam((\Gamma(D_n))^c)$.

Proposition 2.20. Let $n \geq 4$ and suppose that $n$ is not a prime number. Then $(\Gamma(D_n))^c$ is connected. Moreover, the following hold.

(i) $diam((\Gamma(D_n))^c) = 2$ if either $n$ is odd or $n = 2m$, where $m \geq 3$ is odd.

(ii) $diam((\Gamma(D_n))^c) = 3$ if $n = 2^kt$, where $k \geq 2$ and $t \geq 1$ is odd.
Proof. We know that $D_n$ is the subgroup of $S_n$ generated by $\sigma$ and $\tau$, where $\sigma$ and $\tau$ are mentioned as above in Remark 2.19. Note that $o(\sigma) = n, o(\tau) = o(\sigma^i\tau) = 2$ for each $i \in \{1, 2, \ldots, n-1\}$. It is clear that $D_n$ has at least two nontrivial subgroups and as $D_n$ is generated by $\sigma \tau$ and $\tau$, it follows that $N_{D_n} = D_n$. Therefore, we obtain from $(ii) \implies (i)$ of Proposition 2.1 that $(\Gamma(D_n))^c$ is connected. Moreover, we know from the proof of $(ii) \implies (i)$ of Proposition 2.1 that $\text{diam}((\Gamma(D_n))^c) \leq 3$.

Let $n \geq 4$ and suppose that $n$ is not a prime number. Then $\langle \sigma \rangle$ is not a minimal subgroup of $D_n$. Let $H$ be a nontrivial subgroup of $\langle \sigma \rangle$ such that $H < \langle \sigma \rangle$. Observe that $H$ and $\langle \sigma \rangle$ are not adjacent in $(\Gamma(D_n))^c$. Hence, $d(H, \langle \sigma \rangle) \geq 2$ in $(\Gamma(D_n))^c$ and so, we obtain that $\text{diam}((\Gamma(D_n))^c) \geq 2$. Let $H_1, H_2$ be nontrivial subgroups of $D_n$ with $H_1 \neq H_2$. Suppose that $\tau \notin H_1 \cup H_2$. Note that $K = \langle \tau \rangle$ is a subgroup of $D_n$ with $o(K) = 2$ and $H_1 \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(D_n))^c$. As $o(\sigma \tau) = 2$, it follows that if $\sigma \tau \notin H_1 \cup H_2$, then $H_1 - \langle \sigma \tau \rangle - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(D_n))^c$. Hence, in finding $d(H_1, H_2)$ in $(\Gamma(D_n))^c$, we can assume that $\tau, \sigma \tau \in H_1 \cup H_2$. Since $D_n$ is generated by $\sigma \tau$ and $\tau$, both $\tau$ and $\sigma \tau$ cannot be in $H_i$ for each $i \in \{1, 2\}$. Without loss of generality, we can assume that $\tau \in H_1 \setminus H_2$ and $\sigma \tau \in H_2 \setminus H_1$. Since $D_n$ is generated by $\tau$ and $\sigma \tau$ and as $\tau \in H_1$, it follows that $\tau \sigma \notin H_1$.

(i) Suppose that $n \geq 4$ and $n$ is odd. We claim that $\tau \sigma \notin H_2$. For, if $\tau \sigma \in H_2$, then $(\sigma \tau)(\tau \sigma) = \sigma^2 \in H_2$. As $n$ is odd, $o(\sigma^2) = o(\sigma) = n$. This implies that $\sigma \in H_2$ and so, $H_2 = D_n$. This is a contradiction. Therefore, $\tau \sigma \notin H_2$. It is now clear that $H_1 - \langle \tau \sigma \rangle - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(D_n))^c$. This proves that for any nontrivial subgroups $H_1, H_2$ of $D_n$ with $H_1 \neq H_2$, $d(H_1, H_2) \leq 2$ in $(\Gamma(D_n))^c$. Therefore, we get that $\text{diam}((\Gamma(D_n))^c) \leq 2$ and so, $\text{diam}((\Gamma(D_n))^c) = 2$.

Suppose that $n = 2m$, where $m \geq 3$ and $m$ is odd. If $\tau \sigma \notin H_2$, then $H_1 - \langle \tau \sigma \rangle - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(D_n))^c$. Suppose that $\tau \sigma \in H_2$. Then it follows that $(\sigma \tau)(\tau \sigma) = \sigma^2 \in H_2$. Thus $\sigma^2, \sigma \tau \in H_2$ and so, $\langle \sigma^2, \sigma \tau \rangle \subseteq H_2$. As $\sigma \notin H_2$, $\sigma^2 \notin H_2$, and $m$ is odd we obtain that $\sigma^m \notin H_2$. Suppose that $\sigma^m \notin H_1$. Let $K = \langle \sigma^m \rangle$. Note that $o(K) = 2$ and $H_i \cap K = \{e\}$ for each $i \in \{1, 2\}$. Hence, $H_1 - K - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(D_n))^c$. Suppose that $\sigma^m \in H_1$. As $\sigma \notin H_1$, it follows that $\sigma^2 \notin H_1$. Since $\tau \in H_1$, we obtain that $\sigma^2 \tau \notin H_1$. As $\sigma^2 \in H_2$ and $\tau \notin H_2$, we obtain that $\sigma^2 \tau \notin H_2$. Thus $(\sigma \tau)^2 \notin H_1 \cup H_2$. As $o(\langle \sigma^2 \tau \rangle) = 2$, we obtain that $H_i \cap \langle \sigma^2 \tau \rangle = \{e\}$ for each $i \in \{1, 2\}$. Therefore, $H_1 - \langle \sigma^2 \tau \rangle - H_2$ is a path of length two between $H_1$
and \( H_2 \) in \((\Gamma(D_n))^c\). It follows from the above given arguments that 
\[
diam((\Gamma(D_n))^c) \leq 2 \quad \text{and so,} \quad diam((\Gamma(D_n))^c) = 2.
\]

(ii) Suppose that \( n = 2^kt \), where \( k \geq 2 \) and \( t \geq 1 \) is odd. Let \( H_1 \) be the subgroup of \( D_n \) generated by \( \sigma^2 \) and \( \tau \) and let \( H_2 \) be the subgroup of \( D_n \) generated by \( \sigma^2 \) and \( \sigma \tau \). Observe that \( < \sigma^2 > \) is a characteristic subgroup of \( < \sigma > \). Since \([D_n : < \sigma >] = 2\), it follows that \( < \sigma > \) is a normal subgroup of \( D_n \). Therefore, we obtain from \([6, \text{Problem 9, p.70}]\) that \( < \sigma^2 > \) is a normal subgroup of \( D_n \). Therefore, \( H_1 = < \sigma^2 > < \tau > \) and \( H_2 = < \sigma^2 > < \sigma \tau > \). Note that \( o(< \sigma^2 >) = 2^{k-1}t \) and \( o(< \tau >) = o(\sigma \tau >) = 2 \) and \( < \sigma^2 > \cap < \tau > = < \sigma^2 > \cap < \sigma \tau > = \{e\} \). Therefore, we obtain from \([6, \text{Theorem 2.5.1, p.45}]\) that \( o(H_1) = o(H_2) = (2^{k-1}t)(2) = 2^k t \). Hence, \( H_1 \) and \( H_2 \) are maximal subgroups of \( D_n \) and they are also normal subgroups of \( D_n \). Since \( \sigma^2 \in H_1 \cap H_2 \), it follows that \( H_1 \) and \( H_2 \) are not adjacent in \((\Gamma(D_n))^c\). We claim that there exists no path of length two between \( H_1 \) and \( H_2 \) in \((\Gamma(D_n))^c\). Suppose that there exists a path of length two between \( H_1 \) and \( H_2 \) in \((\Gamma(D_n))^c\). Let \( H_3 \) be a nontrivial subgroup of \( D_n \) such that \( H_1 - H_3 - H_2 \) is a path of length two in \((\Gamma(D_n))^c\). Then \( H_i \cap H_3 = \{e\} \) for each \( i \in \{1, 2\} \). Note that \( H_3 \not\subseteq H_1 \) and \( H_1 \) is a maximal and a normal subgroup of \( D_n \). Therefore, we obtain that \( H_1 H_3 = D_n \). Hence, \( o(H_1)o(H_3) = o(D_n) \) and so, \( o(H_3) = 2 \). Observe that \( S = \{\sigma^{2k-2} \tau, \sigma \tau, \sigma^2 \tau, \ldots, \sigma^{n-2} \tau\} \) is the set of all elements of order 2 in \( D_n \). Hence, \( H_3 = < s > \) for some \( s \in S \). Note that \( \{\sigma^{2k-2} \tau, \sigma \tau, \sigma^2 \tau, \ldots, \sigma^{n-2} \tau\} \subseteq H_1 \) and \( \{\sigma \tau, \sigma^3 \tau, \ldots, \sigma^{n-1} \tau\} \subseteq H_2 \). This implies that \( S \subseteq H_1 \cup H_2 \) and so, either \( H_3 \subseteq H_1 \) or \( H_3 \subseteq H_2 \). This is a contradiction. Therefore, there exists no path of length two between \( H_1 \) and \( H_2 \) in \((\Gamma(D_n))^c\). Hence, we obtain that \( diam((\Gamma(D_n))^c) \geq 3 \) and as \( diam((\Gamma(D_n))^c) \leq 3 \), it follows that \( diam((\Gamma(D_n))^c) = 3 \). \(\square\)

Remark 2.21. Let \( n \geq 4 \) be such that \( n \) is not a prime number. Then \( r((\Gamma(D_n))^c) = 2 \).

Proof. It is already noted in Remark 2.19 that \((\Gamma(D_n))^c\) is connected and \( diam((\Gamma(D_n))^c) \) is determined in Proposition 2.20. Let \( \sigma, \tau \) be as mentioned in Remark 2.19. Let \( H \) be any minimal subgroup of \( D_n \). We know from Remark 2.12 that \( e(H) \leq 2 \) in \((\Gamma(D_n))^c\). We next verify that \( e(H) \geq 2 \) in \((\Gamma(D_n))^c\). We consider the following cases.

Case(1): \( H \subseteq < \sigma > \).

Note that \( H = < \sigma^p > \) for some prime number \( p \) such that \( p \) is a divisor of \( n \). Observe that \( o(H) = p \). Since \( H \) is a characteristic subgroup of \( < \sigma > \) and \( < \sigma > \) is a normal subgroup of \( D_n \), we obtain from \([6, \text{Problem 9, p.70}]\) that \( H \) is a normal subgroup of \( D_n \). Let \( K \) be the subgroup of \( D_n \) generated by \( \sigma^p \) and \( \tau \). Observe that \( K = H < \)
\( \tau > \). As \( o(\tau) = 2 \) and \( H \cap \tau = \{ e \} \), it follows from [6, Theorem 2.5.1, p.45] that \( o(K) = 2p < 2n = o(D_n) \). Hence, \( K \) is a nontrivial subgroup of \( D_n \). Since \( H \cap K = H \neq \{ e \} \), we get that \( H \) and \( K \) are not adjacent in \( (\Gamma(D_n))^c \). Therefore, \( d(H, K) \geq 2 \) in \( (\Gamma(D_n))^c \) and so, \( e(H) \geq 2 \) in \( (\Gamma(D_n))^c \).

Case (2): \( H \not\subset \sigma > \).

In this case \( H = \langle \sigma^i \tau \rangle \) for some \( i \in \{0, 1, \ldots, n-1\} \). Note that \( o(H) = 2 \). Let \( p \) be a prime number such that \( p \) is a divisor of \( n \). Let \( K \) be the subgroup of \( D_n \) generated by \( \sigma^2 \) and \( \sigma^i \). Using the same arguments as in Case (1), we obtain that \( o(K) = 2p < 2n = o(D_n) \).

Hence, \( K \) is a nontrivial subgroup of \( D_n \). It is clear that \( H \) and \( K \) are not adjacent in \( (\Gamma(D_n))^c \). Therefore, \( d(H, K) \geq 2 \) in \( (\Gamma(D_n))^c \). This proves that \( e(H) \geq 2 \) in \( (\Gamma(D_n))^c \).

Therefore, \( e(H) = 2 \) in \( (\Gamma(D_n))^c \) for any minimal subgroup \( H \) of \( D_n \). It is clear that if \( K \) is any nontrivial subgroup of \( D_n \) which is not minimal, then \( e(K) \geq 2 \) in \( (\Gamma(D_n))^c \). Therefore, we obtain that \( r((\Gamma(D_n))^c) = 2 \). \( \square \)

**Proposition 2.22.** Let \( G, \overline{G} \) be finite groups such that both of them admit at least two nontrivial subgroups. Let \( \phi : G \to \overline{G} \) be a surjective homomorphism of groups. If \( (\Gamma(G))^c \) is connected, then \( (\Gamma(\overline{G}))^c \) is also connected. Moreover, if \( \text{diam}((\Gamma(G))^c) \leq 2 \), then \( \text{diam}((\Gamma(\overline{G}))^c) \leq 2 \).

**Proof.** Let \( e \) denote the identity element of \( G \) and let us denote the identity element of \( \overline{G} \) by \( \overline{e} \). Let us denote \( \text{Ker}\phi \) by \( N \). It is clear that \( N \neq G \). If \( N = \{ e \} \), then \( G \cong \overline{G} \) as groups. Hence, the graphs \( (\Gamma(G))^c \) and \( (\Gamma(\overline{G}))^c \) are isomorphic. Therefore, there is nothing to prove in this case. So, we can assume that \( N \neq \{ e \} \). Let \( y \in \overline{G}, \ y \neq \overline{e} \). Since \( \phi \) is a surjective homomorphism from \( G \) onto \( \overline{G} \), there exists \( x \in G \setminus \{ e \} \) such that \( y = \phi(x) \). We are assuming that \( (\Gamma(G))^c \) is connected. Therefore, we obtain from (i) \( \Rightarrow \) (ii) of Proposition 2.1 that \( N_G = G \). Note that there exist \( k \geq 1 \) and elements \( g_1, \ldots, g_k \in G \) such that \( o(g_i) \) is a prime number for each \( i \in \{1, \ldots, k\} \) and \( x = \prod_{i=1}^{k} g_i \). Hence, \( y = \phi(x) = \prod_{i=1}^{k} \phi(g_i) \). Since \( y \neq \overline{e} \), it follows that \( \phi(g_i) \neq \overline{e} \) for at least one \( i \in \{1, \ldots, k\} \) and for such an \( i \), \( o(\phi(g_i)) = o(g_i) \) is a prime number. The above discussion implies that \( N_{\overline{G}} = \overline{G} \). Therefore, we obtain from (ii) \( \Rightarrow \) (i) of Proposition 2.1 that \( (\Gamma(\overline{G}))^c \) is connected.

We next prove the moreover part. Suppose that \( \text{diam}((\Gamma(G))^c) \leq 2 \). We show that \( \text{diam}((\Gamma(\overline{G}))^c) \leq 2 \). Let \( W_1, W_2 \) be nontrivial subgroups of \( \overline{G} \) with \( W_1 \neq W_2 \). We now show that there exists a path of length at most two between \( W_1 \) and \( W_2 \) in \( (\Gamma(\overline{G}))^c \). We can assume that \( W_1 \) and \( W_2 \) are not adjacent in \( (\Gamma(\overline{G}))^c \). We know from [6, Lemma 2.7.5, p.63]
that there exist nontrivial subgroups $H_1, H_2$ of $G$ with $N \subset H_i$ for each $i \in \{1, 2\}$ and $W_i = \phi(H_i)$ for each $i \in \{1, 2\}$. It is clear that $H_1 \neq H_2$ and as $H_1 \cap H_2 \neq \{e\}$, we obtain that $H_1$ and $H_2$ are not adjacent in $(\Gamma(G))^c$. We are assuming that $\text{diam}((\Gamma(G))^c) \leq 2$. Hence, there exists a nontrivial subgroup $K$ of $G$ such that $H_1 - K - H_2$ is a path of length two between $H_1$ and $H_2$ in $(\Gamma(G))^c$. We assert that $W_i \cap \phi(K) = \{\overline{e}\}$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. Let $z \in W_i \cap \phi(K)$. Then $z = \phi(h_i) = \phi(k)$ for some $h_i \in H_i$ and $k \in K$. Hence, $kh_i^{-1} \in N \subset H_i$ and so, $k \in H_i \cap K = \{e\}$. Therefore, $z = \phi(k) = \phi(e) = \overline{e}$. This shows that $W_i \cap \phi(K) = \{\overline{e}\}$ for each $i \in \{1, 2\}$. From $H_1 \cap K = \{e\}$ and $N \subset H_1$, it follows that $\phi(K) \neq \{\overline{e}\}$. Hence, $W_1 - \phi(K) - W_2$ is a path of length two between $W_1$ and $W_2$ in $(\Gamma(G))^c$. This proves that $\text{diam}((\Gamma(G))^c) \leq 2$. □

**Remark 2.23.** Let $G, \overline{G}$ be finite groups such that both $G$ and $\overline{G}$ admit at least two nontrivial subgroups. Let $\phi : G \to \overline{G}$ be a surjective homomorphism of groups. Suppose that $(\Gamma(G))^c$ is connected. Then $(\Gamma(\overline{G}))^c$ is connected. If $\text{diam}((\Gamma(\overline{G}))^c) = 3$, then $\text{diam}((\Gamma(G))^c) = 3$.

**Proof.** We know from Proposition 2.22 that $(\Gamma(\overline{G}))^c$ is connected. If $\text{diam}((\Gamma(\overline{G}))^c) = 3$, then it follows from Proposition 2.22 that $\text{diam}((\Gamma(G))^c) \geq 3$. We know from the proof of $(ii) \Rightarrow (i)$ of Proposition 2.1 that $\text{diam}((\Gamma(G))^c) \leq 3$ and so, we get that $\text{diam}((\Gamma(G))^c) = 3$. □

### 3. Some More Results

Let $G$ be a finite group which admits at least one nontrivial subgroup. The aim of this section is to determine $\omega((\Gamma(G))^c)$ and $\text{girth}((\Gamma(G))^c)$.

**Proposition 3.1.** Let $G$ be a finite group. Then $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$, where $k$ is the number of minimal subgroups of $G$.

**Proof.** Since $G$ is a finite group with at least one nontrivial subgroup, $G$ has at least one minimal subgroup and $G$ has only a finite number of minimal subgroups. Let $k$ be the number of minimal subgroups of $G$. Let $\{W_1, \ldots, W_k\}$ be the set of all minimal subgroups of $G$. Since $W_i \cap W_j = \{e\}$ for all distinct $i, j \in \{1, 2, \ldots, k\}$, it follows that the subgraph of $(\Gamma(G))^c$ induced on $\{W_1, \ldots, W_k\}$ is a clique on $k$ vertices. Therefore, we get that $\omega((\Gamma(G))^c) \geq k$. We next verify that the vertices of $(\Gamma(G))^c$ can be properly colored using a set of $k$ distinct colors. Let $\{c_1, \ldots, c_k\}$ be a set of $k$ distinct colors. Now, color $W_i$ with $c_i$ for each $i \in \{1, \ldots, k\}$. Let $H$ be any nontrivial subgroup of $G$. It is clear that $H$ contains a minimal subgroup of $G$. Let $i \in \{1, \ldots, k\}$ be least with
Let us know from Proposition 3.1 that $H \supseteq W_i$. Then color $H$ using $c_i$. We claim that the above assignment of colors is a proper vertex coloring of $(\Gamma(G))^c$. Let $H_1, H_2$ be nontrivial subgroups of $G$ such that $H_1$ and $H_2$ are adjacent in $(\Gamma(G))^c$. Hence, $H_1 \cap H_2 = \{e\}$. Let $i \in \{1, \ldots, k\}$ be least with the property that $H_1 \supseteq W_i$ and let $j \in \{1, \ldots, k\}$ be least with the property that $H_2 \supseteq W_j$. Note that $H_1$ receives color $c_i$ and $H_2$ receives color $c_j$. As $H_1 \cap H_2 = \{e\}$, it is clear that $i \neq j$ and so, $c_i \neq c_j$. This shows that $(\Gamma(G))^c$ can be properly colored using a set of $k$ distinct colors. Therefore, we obtain that $\chi((\Gamma(G))^c) \leq k \leq \omega((\Gamma(G))^c) \leq \chi((\Gamma(G))^c)$.

This proves that $\omega((\Gamma(G))^c) = \chi((\Gamma(G))^c) = k$. □

**Proposition 3.2.** Let $G$ be a finite group. Then $girth((\Gamma(G))^c) = 3$ if and only if $G$ has at least three minimal subgroups.

**Proof.** Assume that $girth((\Gamma(G))^c) = 3$. Then there exist nontrivial subgroups $H_1, H_2, H_3$ such that $H_1 - H_2 - H_3 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Note that $H_1 \cap H_2 = H_2 \cap H_3 = H_3 \cap H_1 = \{e\}$. Let $i \in \{1, 2, 3\}$. Let $W_i$ be a minimal subgroup of $G$ such that $W_i \subseteq H_i$ for each $i \in \{1, 2, 3\}$. Observe that $W_1 \cap W_2 = W_2 \cap W_3 = W_3 \cap W_1 = \{e\}$. Hence, $W_i \neq W_j$ for all distinct $i, j \in \{1, 2, 3\}$. Therefore, $G$ has at least three minimal subgroups.

Conversely, assume that $G$ has at least three minimal subgroups. We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where $k$ is the number of minimal subgroups of $G$. As $k \geq 3$, it follows that $girth((\Gamma(G))^c) = 3$. □

**Proposition 3.3.** Let $G$ be a finite group. Let $o(G) = \prod_{i=1}^{t} p_i^{n_i}$ be the factorization of $o(G)$ into product of prime numbers (here, $p_1, \ldots, p_t$ are distinct prime numbers and $n_i \geq 1$ for each $i \in \{1, \ldots, t\}$ and in the case $t = 1$, $n_1 > 1$). Then $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, $G$ has only one subgroup $W_i$ with $o(W_i) = p_i$. Moreover, if $G$ is abelian, then $\omega((\Gamma(G))^c) = t$ if and only if $G$ is cyclic.

**Proof.** We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where $k$ is the number of minimal subgroups of $G$. Therefore, $\omega((\Gamma(G))^c) = t$ if and only if $G$ has exactly $t$ minimal subgroups. Let $i \in \{1, \ldots, t\}$. Since $p_i$ is a divisor of $o(G)$, we know from Cauchy’s theorem [6, Theorem 2.11.3, p.87] that there exists a subgroup $W_i$ of $G$ with $o(W_i) = p_i$. It is clear that $W_i$ is a minimal subgroup of $G$ for each $i \in \{1, \ldots, t\}$. Observe that if $W$ is any minimal subgroup of $G$, then $o(W) = p_i$ for some $i \in \{1, \ldots, t\}$. Hence, $\omega((\Gamma(G))^c) = t$ if and only if $\{W_1, \ldots, W_t\}$ is the set of all minimal subgroups of $G$. Therefore, we obtain that $\omega((\Gamma(G))^c) = t$ if and only if for each $i \in \{1, \ldots, t\}$, there exists only one subgroup $W_i$ of $G$ with $o(W_i) = p_i$. 

We next verify the moreover part of this Proposition. If $G$ is cyclic, then for each divisor $d$ of $o(G)$, there exists a unique subgroup $H$ of $G$ with $o(H) = d$. Hence, for each $i \in \{1, \ldots, t\}$, $W_i$ is the only subgroup of $G$ with $o(W_i) = p_i$. Therefore, $\omega((\Gamma(G))^c) = t$. Conversely, assume that $G$ is abelian and $\omega((\Gamma(G))^c) = t$. For each $i \in \{1, \ldots, t\}$, let $P_i$ be the unique $p_i$-Sylow subgroup of $G$. Note that $o(P_i) = p_i^{n_i}$ for each $i \in \{1, \ldots, t\}$ and $G$ is the internal direct product of $P_1, \ldots, P_t$. It is clear that $W_i$ is the only subgroup of $P_i$ with $o(W_i) = p_i$. We assert $P_i$ is cyclic for each $i \in \{1, \ldots, t\}$. Suppose that $P_i$ is not cyclic for some $i \in \{1, \ldots, t\}$. Then $n_i > 1$ and we know from the proof of the fundamental theorem of finite abelian groups [6, Theorem 2.14.1, p.109] that there exist $s \geq 2$ and cyclic subgroups $A_1, A_2, \ldots, A_s$ of $P_i$ such that $o(A_1) = p_i^{n_{i1}}, o(A_2) = p_i^{n_{i2}}, \ldots, o(A_s) = p_i^{n_{is}}$ with $n_{i1} \geq n_{i2} \cdots \geq n_{is} \geq 1$ and $P_i$ is the internal direct product of $A_1, A_2, \ldots, A_s$. We know from [3, Problem 6, p.154] that the number of minimal subgroups of $P_i$ equals $p_i^s - 1 = 1 + p_i + \cdots + p_i^{s-1} \geq 2$, since $s \geq 2$. This is impossible as $W_i$ is the only minimal subgroup of $P_i$. This proves that $P_i$ is cyclic for each $i \in \{1, \ldots, t\}$. As $o(P_i), o(P_j) = 1$ for all distinct $i, j \in \{1, \ldots, t\}$, it follows from [6, Problem 6, p.108] that $G$ is cyclic.

Remark 3.4. Let $G$ be a finite group such that $o(G)$ is divisible by at least three distinct prime numbers $p_1, p_2,$ and $p_3$. We know from Cauchy’s theorem [6, Theorem 2.11.3, p.87] that for each $i \in \{1, 2, 3\}$, there exists a subgroup $W_i$ of $G$ such that $o(W_i) = p_i$. It is clear that $W_i$ is a minimal subgroup of $G$ for each $i \in \{1, 2, 3\}$ and hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

Proposition 3.5. Let $G$ be a finite group such that $o(G) = p_1p_2$, where $p_1$ and $p_2$ are distinct prime numbers. Then $girth((\Gamma(G))^c) \in \{3, \infty\}$.

Proof. We can assume without loss of generality that $p_1 < p_2$. It is already noted in Remark 2.7 that $(\Gamma(G))^c$ is either $K_2$ or $K_{p_2+1}$. Therefore, we obtain that $girth((\Gamma(G))^c) \in \{3, \infty\}$.

Lemma 3.6. Let $G$ be a finite group such that $o(G) = p_1^{n_1}p_2^{n_2}$, where $p_1$ and $p_2$ are distinct prime numbers and $n_i > 1$ for each $i \in \{1, 2\}$. Then $girth((\Gamma(G))^c) \leq 4$.

Proof. Let $i \in \{1, 2\}$. Let $k \in \mathbb{N}$ be such that $k \leq n_i$. We know from [6, Theorem 2.12.1, p.92] that there exists a subgroup $H$ of $G$ such that $o(H) = p_i^k$. Let $V_i$ denote the set of all subgroups $H$ of $G$ such that $o(H) = p_i^k$ for some $k \in \mathbb{N}$ with $k \leq n_i$ for each $i \in \{1, 2\}$. It is clear that each member of $V_i$ is a nontrivial subgroup of $G$ and $V_i$ contains at least $n_i$ elements for each $i \in \{1, 2\}$. As $n_i \geq 2$, it follows that $V_i$
contains at least two elements for each $i \in \{1, 2\}$. Since $(p_1, p_2) = 1$, it follows from Lagrange’s theorem that $H \cap W = \{e\}$ for any $H \in V_1$ and $W \in V_2$. If there exist $H_1, H_2 \in V_1$ such that $H_1 \cap H_2 = \{e\}$, then for any $W \in V_2$, we obtain that $H_1 - W - H_2 - H_1$ is a cycle of length three in $(\Gamma(G))^c$. Similarly, if there exist $W_1, W_2 \in V_2$ such that $W_1 \cap W_2 = \{e\}$, then for any $H \in V_1$, we get that $W_1 - H - W_2 - W_1$ is a cycle of length three in $(\Gamma(G))^c$. Hence, we can assume that no two distinct members of $V_i$ are adjacent in $(\Gamma(G))^c$ for each $i \in \{1, 2\}$. Let $H_1, H_2 \in V_1$ with $H_1 \neq H_2$ and let $W_1, W_2 \in V_2$ with $W_1 \neq W_2$. Note that $H_1 - W_1 - H_2 - W_2 - H_1$ is a cycle of length four in $(\Gamma(G))^c$. This proves that $\text{girth}((\Gamma(G))^c) \leq 4$.

**Proposition 3.7.** Let $G$ be a finite cyclic group with $o(G) = p_1^{n_1}p_2^{n_2}$, where $p_1, p_2$ are distinct prime numbers and $n_i > 1$ for each $i \in \{1, 2\}$. Then $\text{girth}((\Gamma(G))^c) = 4$.

**Proof.** We know from Lemma 3.6 that $\text{girth}((\Gamma(G))^c) \leq 4$. Since $G$ is a cyclic group with $o(G) = p_1^{n_1}p_2^{n_2}$, it follows that $G$ has exactly two minimal subgroups. Hence, we obtain from Proposition 3.2 that $\text{girth}((\Gamma(G))^c) \neq 3$ and therefore, $\text{girth}((\Gamma(G))^c) = 4$. \hfill $\square$

**Proposition 3.8.** Let $G$ be a finite cyclic group with $o(G) = p_1^{n_1}p_2$, where $p_1$ and $p_2$ are distinct prime numbers and $n > 1$. Then $\text{girth}((\Gamma(G))^c) = \infty$.

**Proof.** Let $P_1$ be the subgroup of $G$ with $o(P_1) = p_1^n$ and let $P_2$ be the subgroup of $G$ with $o(P_2) = p_2$. Let $V_1$ denote the set of all subgroups $H$ of $P_1$ with $H \neq \{e\}$ and let $V_2 = \{P_2\}$. Since $P_1$ is cyclic, it is clear that $V_1$ contains exactly $n$ elements. As is noted in the proof of Lemma 3.6, $H \cap P_2 = \{e\}$ for any $H \in V_1$ and hence, $H$ and $P_2$ are adjacent in $(\Gamma(G))^c$. Let $W_1, W_2$ be any two distinct nontrivial subgroups of $G$ such that $W_i \notin V_1 \cup V_2$. Observe that $W_i = H_iP_2$ for some subgroup $H_i \in V_1$ such that $H_i \neq P_1$ for each $i \in \{1, 2\}$. It is clear that $W_i \cap H \neq \{e\}, W_i \cap P_2 \neq \{e\}$, $W_i \cap W_2 \neq \{e\}$ for each $i \in \{1, 2\}$ and for any subgroup $H \in V_1$. From the above discussion, we obtain that $V_1 \cup V_2$ is the set of all nonisolated vertices of $(\Gamma(G))^c$ and the subgraph of $(\Gamma(G))^c$ induced on $V_1 \cup V_2$ is a star graph. Indeed, it is $K_{1,n}$. Therefore, we get that $\text{girth}((\Gamma(G))^c) = \infty$. \hfill $\square$

**Proposition 3.9.** Let $G$ be a finite abelian group with $o(G) = p_1^{n_1}p_2^{n_2}$, where $p_1$ and $p_2$ are distinct prime numbers. Suppose that $G$ is not cyclic. Then $\text{girth}((\Gamma(G))^c) = 3$.

**Proof.** We know from Proposition 3.1 that $\omega((\Gamma(G))^c) = k$, where $k$ is the number of minimal subgroups of $G$. It is clear that $k \geq 2$. Thus, we have $\omega((\Gamma(G))^c) = 3$. Therefore, we get that $\text{girth}((\Gamma(G))^c) = 3$. \hfill $\square$
Since $G$ is abelian but not cyclic, we obtain from Proposition 3.3 that 
$\omega((\Gamma(G))^c) \geq 3$ and therefore, $girth((\Gamma(G))^c) = 3$. □

We mention an example in Example 3.10 to illustrate that the hypothesis that the group $G$ is abelian cannot be omitted in Proposition 3.9. For any $n \geq 2$, we denote the additive group of integers modulo $n$ by $\mathbb{Z}_n$.

**Example 3.10.** Let $Q_8$ be the quaternion group of order 8 given in [5, Exercise 44, p.187]. Let $G = Q_8 \times \mathbb{Z}_9$ be the external direct product of $Q_8$ and $\mathbb{Z}_9$. Observe that $o(G) = 2^33^2$. Note that $\{1, -1\} \times \{0\}$ and $\{1\} \times \{0, 3, 6\}$ are the only minimal subgroups of $G$. Hence, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) \neq 3$. We know from Lemma 3.6 that $girth((\Gamma(G))^c) \leq 4$ and therefore, $girth((\Gamma(G))^c) = 4$.

**Remark 3.11.** Let $G$ be a finite group with $o(G) = p^n$, where $p$ is a prime number and $n \geq 2$. If $G$ is cyclic, then $G$ has only one minimal subgroup and so, $girth((\Gamma(G))^c) = \infty$. If $G$ is abelian but not cyclic, then it is already noted in the proof of Proposition 3.3 that $G$ has at least three minimal subgroups and so, we obtain from Proposition 3.2 that $girth((\Gamma(G))^c) = 3$.

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SOME RESULTS ON THE COMPLEMENT OF THE INTERSECTION GRAPH OF SUBGROUPS OF A FINITE GROUP

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Some results on the complement of the intersection graph of subgroups of a finite group

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