A KIND OF F-INVERSE SPLIT MODULES

M. HOSSEINPOUR AND A. R. MONIRI HAMZEKOLAEE*

Abstract. Let $M$ be a right module over a ring $R$. In this manuscript, we shall study on a special case of $F$-inverse split modules where $F$ is a fully invariant submodule of $M$ introduced in [12]. We say $M$ is $\mathbb{Z}^2(M)$-inverse split provided $f^{-1}(\mathbb{Z}^2(M))$ is a direct summand of $M$ for each endomorphism $f$ of $M$. We prove that $M$ is $\mathbb{Z}^2(M)$-inverse split if and only if $M$ is a direct sum of $\mathbb{Z}^2(M)$ and a $\mathbb{Z}^2$-torsionfree Rickart submodule. It is shown under some assumptions that the class of right perfect rings $R$ for which every right $R$-module $M$ is $\mathbb{Z}^2(M)$-inverse split ($\mathbb{Z}(M)$-inverse split) is precisely that of right $GV$-rings.

1. Introduction

Throughout this paper $R$ denotes a ring with identity, modules are unital right $R$-modules and $S = \text{End}_R(M)$ denotes the ring of all right $R$-module endomorphisms of a module $M$ unless otherwise stated. Also $N \leq M$ states that $N$ is a submodule of a module $M$.

The notions of Rickart and Baer rings have their roots in functional analysis with close links to $C^*$-algebras and von Neumann algebras. Kaplansky introduced the notion of Baer rings in 1955 [4] which was extended to quasi-Baer rings in 1967 [1]. A ring $R$ is called (quasi-)Baer if the right annihilator of any nonempty subset (two-sided ideal) of $R$ is generated by an idempotent as a right ideal. Closely related to the concept of Baer rings is the more general notion of right Rickart

Keywords: Rickart module, $\mathbb{Z}(M)$-inverse split module, $\mathbb{Z}^2(M)$-inverse split module.
Received: 30 June 2018, Accepted: 27 January 2019.
*Corresponding author.
rings. The concept of right (left) Rickart rings has been comprehensively studied in the literature. A ring $R$ is called right Rickart if the right annihilator of any single element of $R$ is generated by an idempotent of $R$ as a right ideal. Let $R$ be a ring, $M$ be a right $R$-module and $S = \text{End}_R(M)$. Following [5], $M$ is a Rickart module if the right annihilator in $M$ of any single element of $S$ is generated by an idempotent of $S$, equivalently, $r_M(f) = \text{Ker} f$ is a direct summand of $M$ for every $f \in S$. It is easy to see that for $M = R_R$, the notion of a Rickart module coincides with that of a right Rickart ring. Hence $R_R$ is a Rickart module if $R$ is a Baer ring, a von Neumann regular ring or a right hereditary ring (see [5]). In [5], the authors investigated Rickart modules and their properties and study the connections between a Rickart module and its endomorphism ring.

A submodule $N$ of a module $M$ is said to be small in $M$ if $N + K \neq M$ for any proper submodule $K$ of $M$. Also a module $L$ is said to be a small module, in case $L$ is a small submodule of a module $T$. Following [2], a module $M$ is called lifting if every submodule $N \leq M$ there exists a direct summand $D$ of $M$ such that $N/D \ll M/D$. A submodule $N$ of $M$ is called a supplement in $M$ if there is a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll N$. A module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. A module $M$ is called amply supplemented if $M = A + B$, then $A$ contains a supplement of $B$ in $M$. A lifting module is amply supplemented and hence supplemented. Let $R$ be a ring and $M$ a right $R$-module. In [10], Talebi and Vanaja defined $\mathcal{Z}(M)$ as a dual of singular submodule as follows: $\mathcal{Z}(M) = \cap \{\text{Ker} f \mid f : M \to U, U \in \mathcal{S}\}$ (here $\mathcal{S}$ denotes the class of all small right $R$-modules). They called $M$ a cosingular (noncosingular) module if $\mathcal{Z}(M) = 0$ ($\mathcal{Z}(M) = M$). Clearly every small module is cosingular.

In [10], $\mathcal{Z}^\alpha(M)$ is defined by $\mathcal{Z}^0(M) = M$, $\mathcal{Z}^{\alpha+1}(M) = \mathcal{Z}(\mathcal{Z}^\alpha(M))$ and $\mathcal{Z}^\alpha(M) = \bigcap_{\beta < \alpha} \mathcal{Z}^\beta(M)$ if $\alpha$ is a limit ordinal. Hence there is a descending chain $M = \mathcal{Z}^0(M) \supseteq \mathcal{Z}(M) \supseteq \mathcal{Z}^2(M) \supseteq \ldots$ of submodules of $M$.

Recall that a submodule $F$ of a module $M$ is called fully invariant if $h(F) \subseteq F$ for every $h \in \text{End}_R(M)$. Let $F$ be a fully invariant submodule of a module $M$. Then $M$ is said to be $F$-inverse split [12], if $h^{-1}(F)$ is a direct summand of $M$ for every $h \in \text{End}_R(M)$. In [13], the authors defined $\mathcal{Z}(M)$-inverse split modules and investigated their properties. A module $M$ is $\mathcal{Z}(M)$-inverse split provided $h^{-1}(\mathcal{Z}(M))$ is a direct summand of $M$ for each endomorphism $h$ of $M$. They proved that a module $M$ is $\mathcal{Z}(M)$-inverse split if and only if $M$ is decomposed to a noncosingular submodule and a cosingular Rickart submodule if
and only if $M = \mathbb{Z}(M) \oplus N$ where $N$ is cosingular Rickart. In this article, we define $\mathbb{Z}^2(M)$-inverse split modules and try to investigate their general properties. We say $M$ is $\mathbb{Z}^2(M)$-inverse split provided $h^{-1}(\mathbb{Z}^2(M))$ is a direct summand of $M$ for each endomorphism $h$ of $M$. We prove that a module $M$ is $\mathbb{Z}^2(M)$-inverse split if and only if $M$ is decomposed to $\mathbb{Z}^2(M)$ and a $\mathbb{Z}^2$-torsion free Rickart submodule $N$. We also present, under some assumptions, new characterizations of right perfect right $GV$-rings in terms of $\mathbb{Z}^2$-inverse split modules.

2. $\mathbb{Z}^2$-Inverse Split Modules

In this section, we are interested in studying on a special case of $F$-inverse split modules. There are many important fully invariant submodules of a module. Among all of them, the second cosingular submodule of a module $M$, namely $\mathbb{Z}^2(M)$, has a key role in determining of some important modules such as lifting modules and amply supplemented modules.

We start with introducing $\mathbb{Z}^2$-inverse split modules.

Definition 2.1. A module $M$ is called $\mathbb{Z}^2(M)$-inverse split whenever $f^{-1}(\mathbb{Z}^2(M))$ is a direct summand of $M$ for every $f \in S$.

Here are examples of some known $\mathbb{Z}^2$-inverse split modules.

Example 2.2. (1) The class of $\mathbb{Z}^2$-inverse split modules contains a large class of modules namely semisimple modules. In particular, every Artinian (Noetherian) module $M$ over a Boolean ring $R$ is $\mathbb{Z}^2(M)$-inverse split.

(2) It is clear that, every noncosingular module $M$ is $\mathbb{Z}^2(M)$-inverse split. So that, every right $R$-module $M$ over a right $V$-ring $R$, is $\mathbb{Z}^2(M)$-inverse split.

(3) Every injective right $R$-module $M$ over a right hereditary ring is noncosingular (see [10, Proposition 2.7]) and hence is $\mathbb{Z}^2(M)$-inverse split.

(4) For a cosingular module $M$, two concepts ”Rickart” and ”$\mathbb{Z}^2(M)$-inverse split” coincide. Since a projective $\mathbb{Z}$-module has the form $M = \mathbb{Z}(I)$ where $I$ is an arbitrary nonempty index set, $M$ is cosingular. From [5, Theorem 2.26], $M$ is Rickart. So that $M$ is $\mathbb{Z}^2(M)$-inverse split.

We exhibit a characterization of $\mathbb{Z}^2(M)$-inverse split modules which will be applied excessively throughout the paper.
Theorem 2.3. Let $M$ be a module. Then the following statements are equivalent:

1. $M$ is $\mathbb{Z}^2(M)$-inverse split;
2. $M = \mathbb{Z}^2(M) \oplus N$ where $N$ is a Rickart module with $\mathbb{Z}^2(N) = 0$;
3. $M = K \oplus L$ where $K$ is noncosingular and $L$ is a Rickart module with $\mathbb{Z}^2(L) = 0$.

Proof. (1) $\Rightarrow$ (2) Let $M$ be $\mathbb{Z}^2(M)$-inverse split. So, $id^{-1}_M(\mathbb{Z}^2(M)) = \mathbb{Z}^2(M)$ is a direct summand of $M$. Set $M = \mathbb{Z}^2(M) \oplus N$ for $N \leq M$. We shall prove $N$ is Rickart. To verify this assertion, suppose $g \in \text{End}(N)$. Then $f = j \circ \pi_N$ is an endomorphism of $M$ where $\pi_N : M \to N$ is the projection of $M$ on $N$ and $j : N \to M$ is the inclusion. Now, being $M$ a $\mathbb{Z}^2(M)$-inverse split module leads us that $f^{-1}(\mathbb{Z}^2(M))$ is a direct summand of $M$. By a normal verification, we conclude that $f^{-1}(\mathbb{Z}^2(M)) = \text{Ker}g \oplus \mathbb{Z}^2(M)$. Hence $\text{Ker}g$ is a direct summand of $M$. As $\text{Ker}g$ is contained in $N$, we have $\text{Ker}g$ is a direct summand of $N$, showing $N$ is Rickart.

(2) $\Rightarrow$ (1) Let $M = \mathbb{Z}^2(M) \oplus N$ where $N$ is Rickart. Let $f \in \text{End}(M)$. Consider $h = \pi_N \circ f \circ j : N \to N$ which is an endomorphism of $N$ such that $\pi_N : M \to N$ is the projection of $M$ on $N$ and $j : N \to M$ is the inclusion. Being $N$ a Rickart module implies $\text{Ker}h$ is a direct summand of $N$. Set $\text{Ker}h \oplus L = N$. It is not hard to check that $f^{-1}(\mathbb{Z}^2(M)) = \text{Ker}h \oplus \mathbb{Z}^2(M)$. By the decomposition $M = (\text{Ker}h \oplus L) \oplus \mathbb{Z}^2(M)$ we come to a conclusion that $f^{-1}(\mathbb{Z}^2(M))$ is a direct summand.

(2) $\Rightarrow$ (3) Suppose that $M = \mathbb{Z}^2(M) \oplus N$. Then $\mathbb{Z}^4(M) = \mathbb{Z}^3(M) = \mathbb{Z}^2(M)$ showing that $\mathbb{Z}^2(M)$ is noncosingular.

(3) $\Rightarrow$ (2) If $M = K \oplus L$ where $K$ is noncosingular and $L$ is Rickart with $\mathbb{Z}^2(L) = 0$ then it is obvious that $K = \mathbb{Z}^2(M)$. \hfill $\square$

The following is an immediate consequence of Theorem 2.3.

Corollary 2.4. Every $\mathbb{Z}(M)$-inverse split module $M$ is $\mathbb{Z}^2(M)$-inverse split.

Example 2.5. Every Rickart module $M$ with $\mathbb{Z}^2(M)$ a direct summand of $M$, is a $\mathbb{Z}^2(M)$-inverse split module. Let now $M$ be a lifting Rickart module. Then by [10, Theorem 4.1], there is a decomposition $M = \mathbb{Z}^2(M) \oplus N$. It follows that $M$ is $\mathbb{Z}^2(M)$-inverse split. In particular, every nonsingular injective (extending) lifting module $M$ is $\mathbb{Z}^2(M)$-inverse split by [5, Example 2.3].
Definition 2.6. Let $M$ be a module. We call $M$ a $\mathbb{Z}^2$-torsionfree module provided $\mathbb{Z}^2(M) = 0$.

It is easy to see that every cosingular module is $\mathbb{Z}^2$-torsionfree. The class of $\mathbb{Z}^2$-torsionfree modules is closed under submodules, direct sums and direct products (see [10, Proposition 2.1]). It is also followed by [6, Theorem 4.41] and [10, Proposition 2.1 and Theorem 3.5] that for a perfect ring $R$, the class of $\mathbb{Z}^2$-torsionfree $R$-modules is also closed under factor modules.

Recall that a module $M$ satisfies $(D_0)$ in case $M = M_1 \oplus M_2$ implies $M_1$ and $M_2$ are relatively projective. We present a new characterization of right $GV$-rings in terms of $\mathbb{Z}^2$-inverse split modules.

We should note that the proofs for equivalences of (1), (2) and (3) of the following can be found distinctly in [8]. We state them here to make useful connections with $\mathbb{Z}^2$-inverse split modules.

Before presenting next result, it is worth to recall that a ring $R$ is a right $GV$-ring (generalized $V$-ring) provided every simple singular right $R$-module is injective. In [8], some characterizations of right $GV$-rings are given. Among them, it is proved that a ring $R$ is right $GV$ if and only if every simple cosingular right $R$-module is projective if and only if every small right $R$-module is projective ([8, Theorem 3.1 and Corollary 3.3]).

Theorem 2.7. Consider the following statements for a right perfect ring $R$:

1. $R$ is a right $GV$-ring;
2. Every $\mathbb{Z}^2$-torsionfree right $R$-module is projective;
3. Every right $R$-module is a direct sum of a noncosingular right $R$-module and a semisimple right $R$-module;
4. Every right $R$-module $M$ is $\mathbb{Z}^2(M)$-inverse split.

Then (1) $\iff$ (2) $\iff$ (3) $\Rightarrow$ (4). They are equivalent in case every $\mathbb{Z}^2$-torsionfree module satisfies $(D_0)$.

Proof. (1) $\Rightarrow$ (2) Assume that $R$ is right $GV$. Let $0 \neq M$ be a $\mathbb{Z}^2$-torsionfree $R$-module, $0 \neq x \in M$ and $K$ a maximal submodule of $xR$. Now the simple $R$-module $xR/K$ is either singular or projective (but not both). If $xR/K$ is singular, then it will be noncosingular by [7, Theorem 4.1]. Consider the natural epimorphism $\pi: xR \to xR/K$. Since $R$ is a right perfect ring, by [6, Theorem 4.41] $xR$ is amply supplemented. Therefore by [10, Theorem 3.5] we conclude that $0 = \pi(\mathbb{Z}^2(xR)) = \mathbb{Z}^2(xR/K) = \mathbb{Z}(xR/K) = xR/K$, which
is a contradiction. Then $xR/K$ is projective and so $K$ is a direct summand of $xR$. Hence $xR$ and as well as $xR$, the module $M$ is semisimple. Let $M = \bigoplus_{i \in I} M_i$ where each $M_i$ is simple. By mentioned argument in above, each $M_i$ is projective. Hence $M$ is projective.

(2) $\Rightarrow$ (3) Let $M$ be a right $R$-module. Since $R$ is a right perfect ring, $M$ is amply supplemented by [6, Theorem 4.41]. Now consider natural epimorphism $\pi : M \to M/\mathbb{Z}^2(M)$. Hence by [10, Theorem 3.5], we have $0 = \pi(\mathbb{Z}^2(M)) = \mathbb{Z}^2(M/\mathbb{Z}^2(M))$. Now, we conclude by assumption that $\mathbb{Z}^2(M)$ is a direct summand of $M$. Suppose $M = \mathbb{Z}^2(M) \oplus N$ for a submodule $N$ of $M$. It is clear that $\mathbb{Z}^2(M)$ is noncosingular. We shall show that $N$ is semisimple. To verify this assertion, let $0 \neq x \in N$. As $xR$ is finitely generated, it contains a maximal submodule say $K$. Consider the simple module $xR=K$ which must be small or injective, but not both. Assume that $xR=K$ is injective. Then $xR/K$ is noncosingular. Now, designate the natural epimorphism $\pi : xR \to xR/K$. Being $R$ a right perfect ring combining with [10, Theorem 3.5] implies that $0 = \pi(\mathbb{Z}^2(xR)) = \mathbb{Z}^2(xR/K) = \mathbb{Z}(xR/K) = xR/K$ which causes a contradiction. By the way, $xR/K$ is small and therefore by assumption $xR/K$ is projective concluding that $K$ is a direct summand of $xR$. Hence $xR$ is semisimple which implies $N$ is semisimple.

(3) $\Rightarrow$ (1) Let $M$ be a simple singular right $R$-module. Then $M$ is either small or injective. If $M$ is small, then it is projective which is a contradiction. It follows that $M$ is injective.

(3) $\Rightarrow$ (4) Let $M$ be a right $R$-module. By (3), there is a decomposition $M = N \oplus S$ where $N$ is noncosingular and $S$ is semisimple. Let us consider $S$ as $(\oplus_{\alpha \in A}(S_{\alpha})) \oplus (\oplus_{\beta \in B}(S'_{\beta}))$ while for each $\alpha \in A$, $S_{\alpha}$ is noncosingular and $S'_{\beta}$ for each $\beta \in B$ is small (note that a simple module is either injective (noncosingular) or small). Now, $M = [N \oplus (\oplus_{\alpha \in A}(S_{\alpha}))] \oplus [(\oplus_{\beta \in B}(S'_{\beta}))]$ is a direct sum of $\mathbb{Z}^2(M)$ and a semisimple (Rickart) module. Hence, $M$ is $\mathbb{Z}^2(M)$-inverse split by Theorem 2.3.

Now let every $\mathbb{Z}^2$-torsionfree module satisfy $(D_0)$.

(4) $\Rightarrow$ (3) Let $M$ be a module. Then by assumption, $M = \mathbb{Z}^2(M) \oplus N$ where $N$ is a Rickart module. We shall prove that $N$ is semisimple. To verify this assertion, take an arbitrary nonzero $x$ in $N$. Being $xR$ finitely generated implies $xR$ has at least a maximal submodule. Suppose $K$ is a maximal submodule of $xR$. As $\mathbb{Z}^2(xR \oplus xR/K) = 0$, it satisfies $(D_0)$ by assumption (note that $\mathbb{Z}^2(xR/K) = (\mathbb{Z}^2(xR) + K)/K = 0$ as $R$ is right perfect). It follows that $xR/K$ is $xR$-projective which
implies that \( K \) is a direct summand of \( xR \). Therefore, \( xR \) and hence \( N \) are semisimple.

\[ \square \]

Theorem 2.7 combining characterizations of \( GV \)-rings in [8], gives us the following:

**Corollary 2.8.** Let \( R \) be a right perfect ring. Consider the following:

1. \( R \) is a right \( GV \)-ring;
2. Every (simple) cosingular right \( R \)-module is projective;
3. Every right \( R \)-module is a direct sum of a noncosingular right \( R \)-module and a semisimple right \( R \)-module;
4. Every right \( R \)-module \( M \) is \( \mathbb{Z}^2(M) \)-inverse split.

Then (1) \( \iff \) (2) \( \iff \) (3) \( \implies \) (4). They are equivalent in case every \( \mathbb{Z}^2 \)-torsionfree module satisfies \((D_0)\).

**Proof.** It follows from Theorem 2.7, [8, Theorems 3.1 and 3.18].  \( \square \)

We shall prove under some assumptions that the class of right perfect rings \( R \) for which every right \( R \)-module \( M \) is \( \mathbb{Z}(M) \)-inverse split is precisely that of right \( GV \)-rings.

**Corollary 2.9.** Let \( R \) be a right perfect ring such that every \( \mathbb{Z}^2 \)-torsionfree module satisfies \((D_0)\). Then the following statements are equivalent:

1. \( R \) is a right \( GV \)-ring;
2. Every right \( R \)-module \( M \) is \( \mathbb{Z}(M) \)-inverse split;
3. Every right \( R \)-module \( M \) is \( \mathbb{Z}^2(M) \)-inverse split.

**Proof.** (1) \( \Rightarrow \) (2) Let \( M \) be a right \( R \)-module. It follows from Corollary 2.8 that \( M = N \oplus S \) where \( N \) is noncosingular and \( S \) is semisimple. Similar to argument mentioned in (3) \( \Rightarrow \) (4) of the proof of Theorem 2.7, we can conclude that \( M = \mathbb{Z}(M) \oplus H \) where \( H \) is semisimple. Hence by [13, Theorem 3.3], \( M \) is \( \mathbb{Z}(M) \)-inverse split.

(2) \( \Rightarrow \) (3) It is obvious.

(3) \( \Rightarrow \) (1) It follows from Theorem 2.7.  \( \square \)

**Corollary 2.10.** Let \( R \) be a right perfect ring such that every noncosingular submodule of a module is a direct summand of that module. If every \( \mathbb{Z}^2 \)-torsionfree right \( R \)-module satisfies \((D_0)\), then every right \( R \)-module \( M \) is \( \mathbb{Z}^2(M) \)-inverse split.

**Proof.** Let \( M \) be an arbitrary right \( R \)-module. By assumption \( \mathbb{Z}^2(M) \) is a direct summand of \( M \). Set \( M = \mathbb{Z}^2(M) \oplus L \) for some submodule \( L \) of
It is clear that $\mathbb{Z}^2(L) = 0$, so that by assumption $L$ satisfies $(D_0)$. By a similar argument stated in $(4) \Rightarrow (3)$ of the proof of Theorem 2.7, $L$ is semisimple. Therefore, $M$ is $\mathbb{Z}^2(M)$-inverse split by Theorem 2.3. □

**Corollary 2.11.** Let $R$ be a left and right Artinian serial ring with $J(R)^2 = 0$. If every injective module is noncosingular, then every left and right $R$-module $M$ is $\mathbb{Z}^2(M)$-inverse split. In particular, over a hereditary left and right Artinian serial ring $R$ with $J(R)^2 = 0$, every $R$-module $M$ is $\mathbb{Z}^2(M)$-inverse split.

**Proof.** By [2, 29.10], every $R$-module $M$ has a decomposition $M = E \oplus S$ where $E$ is an injective $R$-module and $S$ is a semisimple $R$-module. Now, by assumption $E$ is noncosingular. The result follows from Theorem 2.7. The last assertion follows from first part and [10, Proposition 2.7]. □

The following introduces a ring over which every module $M$ is $\mathbb{Z}^2(M)$-inverse split.

**Example 2.12.** ([8, Example 3.15]) Let $F$ be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ the ring of $2 \times 2$ upper triangular matrices over $F$. By [3, Example 13.6], every singular (left and right) $R$-module is injective. Hence $R$ is a left and right $GV$-ring. Since $J(R) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, $R$ can not be a (left and right) $V$-ring. Also $R$ is (left and right) hereditary Artinian serial from [3, Example 13.6]. It is easy to check that $J(R)^2 = 0$. Hence by Corollary 2.11, every right $R$-module $M$ is $\mathbb{Z}^2(M)$-inverse split.

There are $\mathbb{Z}^2(M)$-inverse split modules which are not Rickart. Now, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^I}$ for an arbitrary non-empty indexed set $I$. Then $M$ is not Rickart, since $\mathbb{Z}_{p^\infty}$ is not a Rickart $\mathbb{Z}$-module by [5, Example 2.17]. On the other hand, $M$ is noncosingular and so it is $\mathbb{Z}^2(M)$-inverse split. Generally, every non-Rickart noncosingular module provides an example of a $\mathbb{Z}^2(M)$-inverse split module that is not Rickart.

**Proposition 2.13.** Let $M$ be an indecomposable module. Then the following are equivalent:

1. $M$ is $\mathbb{Z}^2(M)$-inverse split;
2. $M$ is noncosingular or $M$ is Rickart with $\mathbb{Z}^2(M) = 0$. 


Proof. (1) \( \Rightarrow \) (2) Let \( M \) be \( \mathbb{Z}^2(M) \)-inverse split. Then by Theorem 2.3, \( M = \mathbb{Z}^2(M) \oplus N \) where \( N \) is Rickart. Being \( M \) indecomposable implies \( M = \mathbb{Z}^2(M) \) or \( M = N \). First case yields \( M \) is noncosingular and the second one implies \( M \) is Rickart with \( \mathbb{Z}^2(M) = 0 \).

(2) \( \Rightarrow \) (1) It is straightforward. \( \square \)

Example 2.14. Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z}_{p^n} \) where \( p \) is a prime and \( n \in \mathbb{N} \). Then \( M \) is an indecomposable cosingular \( \mathbb{Z} \)-module. Now, earmark the endomorphism \( h : M \to M \) by \( h(x) = px \). It is clear that \( 0 \neq Kerh < M \). Therefore, \( M \) is not a Rickart \( \mathbb{Z} \)-module. As \( M \) is cosingular indecomposable, \( M \) is not \( \mathbb{Z}^2(M) \)-inverse split by Proposition 2.13.

Recall from [9] that a module \( M \) has \( C^* \)-property provided that every submodule \( N \) of \( M \) contains a direct summand \( D \) of \( M \) such that \( N/D \) is cosingular. Let \( R \) be a ring. Then every right \( R \)-module satisfies \( C^* \) if and only if every right \( R \)-module is a direct sum of an injective module and a cosingular module (see [9, Theorem 2.9]). Recall also from [2] that a ring \( R \) is right Harada in case every injective right \( R \)-module is lifting. It follows from [2, 28.10] that \( R \) is right Harada if and only if every right \( R \)-module is decomposed to an injective right \( R \)-module and a small right \( R \)-module. So, over a right Harada ring every right \( R \)-module satisfies \( C^* \).

Proposition 2.15. Let \( R \) be a right perfect ring such that every right \( R \)-module has \( C^* \)-property. Then every Rickart \( R \)-module \( M \) is \( \mathbb{Z}^2(M) \)-inverse split. In particular, every Rickart module \( M \) over a Harada ring (quasi-Frobenius ring) is \( \mathbb{Z}^2(M) \)-inverse split.

Proof. Let \( M \) be a Rickart module. As \( R \) is right perfect, \( \mathbb{Z}^2(M) \) is a noncosingular submodule of \( M \). Now, from [9, Theorem 2.9], there is a decomposition \( \mathbb{Z}^2(M) = E \oplus C \) such that \( E \) is injective and \( C \) is cosingular. It follows that \( \mathbb{Z}^2(M) \) is injective and hence a direct summand of \( M \). Being \( M \) a Rickart module implies that \( M \) is \( \mathbb{Z}^2(M) \)-inverse split. \( \square \)

The following contains an example of a \( \mathbb{Z}^2(M) \)-inverse split module which is not \( \mathbb{Z}(M) \)-inverse split showing that the concept of \( \mathbb{Z}^2(M) \)-inverse split modules is a proper generalization of the \( \mathbb{Z}(M) \)-inverse split modules.
Example 2.16. Let $K$ be a field and

$$R = \begin{cases} \left( \begin{array}{ccc} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & c \\ d & 0 & 0 \\ c & 0 & 0 \end{array} \right) \mid a, b, c, d \in K \end{cases}.$$ 

Then $R$ is a subring of $M_{4\times 4}(K)$. Now consider $e = e_{11} + e_{22}$ and $f = e_{33} + e_{44}$ where $e_{ij}$ is an element of $R$ such that $(i, j)$-component is 1 and elsewhere is 0. Then $e$ and $f$ are two idempotents in $R$ and $R = eR \oplus fR$. The ring $R$ is a (4-dimensional) Frobenius algebra and $eR$ is an indecomposable projective module where $\text{Soc}(eR) = e_{23}R$ is the only non-trivial proper submodule of $eR$ (it can be easily checked). Therefore, $eR$ is a local right $R$-module with $\text{Soc}(eR) = \text{Rad}(eR) \ll eR$. Now by [11, Corollary 2.8], we have $\overline{Z}(eR) = \text{Soc}(eR) \ll eR$, so that $\overline{Z}^2(eR) = 0$. Note that the only proper submodule of $eR$ is $\text{Soc}(eR)$. Now, suppose that $\varphi : eR \rightarrow eR$ is an arbitrary nonzero endomorphism of $eR$. Then $\text{Ker}\varphi = 0$ or $\text{Ker}\varphi = \text{Soc}(eR)$. Since $eR/\text{Soc}(eR)$ is not isomorphic to a submodule of $eR$, we conclude that $\text{Ker}\varphi = 0$. In fact, $eR/\text{Soc}(eR)$ is isomorphic to a submodule of $fR$. Hence, $\text{Ker}\varphi$ is a direct summand of $eR$ implying that $eR$ is a Rickart right $R$-module. Now by Theorem 2.3, $eR$ is $\overline{Z}^2(eR)$-inverse split while $eR$ is not $\overline{Z}(eR)$-inverse split as $\overline{Z}(eR) \ll eR$.

A submodule $N$ of a $\overline{Z}^2(M)$-inverse split module $M$ need not be $\overline{Z}^2(N)$-inverse split. Now, consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}_2$. By Theorem 2.3 and [5, Example 2.5], $M$ is $\overline{Z}^2(M)$-inverse split. Set $N = \mathbb{Z} \oplus \mathbb{Z}_2$ which is a cosingular submodule of $M$. By [5, Example 2.5], $N$ is a Rickart $\mathbb{Z}$-module. Hence $N$ is not a $\overline{Z}^2(N)$-inverse split module. We next show that a direct summand of a $\overline{Z}^2$-inverse split module inherits the property.

Proposition 2.17. Let $M$ be a $\overline{Z}^2(M)$-inverse split module and $N$ a direct summand of $M$. Then $N$ is $\overline{Z}^2(N)$-inverse split.

Proof. Let $M = N \oplus K$ be a $\overline{Z}^2(M)$-inverse split module with $N$ and $K$ submodules of $M$. By Theorem 2.3, there is a decomposition $M = \overline{Z}^2(M) \oplus L$ where $L$ is Rickart. Since $\overline{Z}^2(M) = \overline{Z}^2(N) \oplus \overline{Z}^2(K)$, we conclude that $M = \overline{Z}^2(N) \oplus \overline{Z}^2(K) \oplus L$. Modular law implies $N = \overline{Z}^2(N) \oplus ([\overline{Z}^2(K) \oplus L] \cap N)$. Let $Y = (\overline{Z}^2(K) \oplus L) \cap N$ and $f \in \text{End}_R(Y)$. It just remains to prove that $Y$ is a Rickart module. It is easy to check that $h = j_0 f o \pi_Y$ is an endomorphism of $M$ where
A KIND OF $F$-INVERSE SPLIT MODULES

$j : Y \rightarrow M$ is the inclusion map and $\pi_Y : M \rightarrow Y$ is the projection of $M$ on $Y$. Being $M$ a $\mathbb{Z}^2(M)$-inverse split module conduces that $h^{-1}(\mathbb{Z}^2(M)) = \mathbb{Z}^2(N) \oplus \mathbb{Z}^2(K) \oplus \text{Ker} f$ is a direct summand of $M$. By modular law, $\text{Ker} f$ is a direct summand of $Y$ resulting that $Y$ is a Rickart module.

**Theorem 2.18.** The following are equivalent for a module $M$:

1. $M$ is $\mathbb{Z}^2(M)$-inverse split and $\text{Ker} f$ is a direct summand of $f^{-1}(\mathbb{Z}^2(M))$ for any $f \in S$;
2. $M$ is Rickart and $\mathbb{Z}^2(M)$ is a direct summand of $M$.

**Proof.** (1) $\Rightarrow$ (2) Let $M$ be a $\mathbb{Z}^2(M)$-inverse split module and $f \in S$. Then $f^{-1}(\mathbb{Z}^2(M))$ is a direct summand of $M$ and by hypothesis, $\text{Ker} f$ is a direct summand of $f^{-1}(\mathbb{Z}^2(M))$. It follows that $M$ is Rickart. In addition, by Theorem 2.3, $\mathbb{Z}^2(M)$ is a direct summand of $M$.

(2) $\Rightarrow$ (1) Let $M$ be a Rickart module and $M = \mathbb{Z}^2(M) \oplus N$ for some submodule $N$ of $M$. Then $N$ is Rickart and so $M$ is $\mathbb{Z}^2(M)$-inverse split by Theorem 2.3. Being $M$ Rickart leads us that $\text{Ker} f$ is a direct summand of $M$. The result follows from the fact that $\text{Ker} f$ is a submodule of $f^{-1}(\mathbb{Z}^2(M))$ for any $f \in S$. □

We shall state an analogue of [13, Theorem 3.12] for a $\mathbb{Z}^2(M)$-inverse split module.

**Proposition 2.19.** Let $f : M \rightarrow M'$ be an epimorphism of modules where $M$ is $\mathbb{Z}^2(M)$-inverse split. If $\text{Ker} f$ is noncosingular, then $M'$ is $\mathbb{Z}^2(M')$-inverse split.

**Proof.** Let $M$ be $\mathbb{Z}^2(M)$-inverse split. Then by Theorem 2.3, $M = \mathbb{Z}^2(M) \oplus N$ where $N$ is a Rickart module. It is easy to check that $\mathbb{Z}^2(N) = 0$. Taking image of $M$, we have $M' = f(\mathbb{Z}^2(M)) + f(N)$. As $\text{Ker} f$ is noncosingular, it is contained in $\mathbb{Z}^2(M)$. So by a similar argument given in the proof of [13, Theorem 3.12], we conclude that $M' = f(\mathbb{Z}^2(M)) \oplus f(N)$. Note that the condition $\text{Ker} f \subseteq \mathbb{Z}^2(M)$ implies there is an isomorphism between $N$ and $f(N)$. Existing such isomorphism implies $f(N)$ is a Rickart module and $\mathbb{Z}^2(f(N)) = 0$, as well as $\mathbb{Z}^2(N) = 0$. Hence $M' = \mathbb{Z}^2(M') \oplus f(N)$. The result then follows from Theorem 2.3. □
REFERENCES


Mehrab Hosseinpour
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran.
Email: m.hpour@umz.ac.ir

Ali Reza Moniri Hamzekolaee
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran.
Email: a.monirih@umz.ac.ir
A KIND OF $F$-INVERSE SPLIT MODULES

M. HOSSEINPOUR AND A. R. MONIRI HAMZEKOLAEE

یک نوع از مدول‌های $F$-وارون شکافته‌شدنی

مراح حسین‌پور و علیرضا منیری حمزرکایی

گروه ریاضی، دانشکده علوم ریاضی، دانشگاه مازندران، بابلسر، ایران