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A CHARACTERIZATION FOR METRIC TWO-DIMENSIONAL GRAPHS AND THEIR ENUMERATION

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Abstract. The *metric dimension* of a connected graph G is the minimum number of vertices in a subset B of G such that all other vertices are uniquely determined by their distances to the vertices in B. In this case, B is called a *metric basis* for G. The *basic distance* of a metric two-dimensional graph G is the distance between the elements of B. Giving a characterization for those graphs whose metric dimensions are two, we enumerate the number of n vertex metric two-dimensional graphs with the basic distance 1.

1. Introduction

Let $G = (V, E)$ be a connected simple graph. For two vertices u and v of G, the distance $d_G(x, y)$ or $d(x, y)$ of x and y is the length of a minimum path connecting x to y. For a subset $R = \{r_1, \dots, r_k\}$ of V and a vertex v, the representation of v with respect to R is the k-tuple $\langle v|R\rangle = (d(v, r_1), \dots, d(v, r_k))$. The subset R is called a resolving set for G if any vertex has a unique representation with respect to R. A resolving set B of V is called a metric basis for G if it has the minimum possible number of elements for a resolving set. The metric dimension G, denoted by $\dim_{\mathrm{M}}(G)$ is then equal to this minimum number. For a study about these notions, we refer the reader to [4] and [8].

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As a simple known fact, $\dim_{\mathrm{M}}(G) = 1$ if and only if G is a path. The metric dimension of an n vertex graph G is $n-1$ if and only if G is the complete graph K_n; see [3].

The concept of a resolving set has various applications in different areas including network discovery and verification [1], problems of pattern recognition and image processing [6], robot navigation [5], mastermind game [2], and combinatorial search and optimization [7].

2. A Characterization for $\dim_{\mathrm{M}}(G) = 2$

In this section, we aim to characterize all two metric dimensional graphs, but prior to this we need to extend the notion of a path.

Definition 2.1. Let x and d be two positive integers with $x \geqslant d$ and let y be a nonnegative integer. An *extended path* $\mathcal{P}(x, y, d)$ *of the length* x*, width* y*, and height* $2d+1$ is a simple graph with the following properties:

i. $V(\mathcal{P}) = \cup_{i=0}^{x} P_i$, where $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \ldots, v_{i,i+d}\}$ for $0 \leqslant i \leqslant y$ and $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \ldots, v_{i,i+d-1}\}$ for $y+1 \leqslant i \leqslant x$;
ii. neighbors of $v_{i,j}$ are $v_{k,\ell}$ with $|i-k| \leqslant 1$ and $|j-\ell| \leqslant 1$.

Example 2.2. As an example, the generalized path $\mathcal{P}(7, 4, 1)$ has vertices of the form

$$
\begin{array}{cccccccc}
 & & & & & & & \mathbf{7}7 \\
 & & & & & & 66 & \mathbf{7}6 \\
 & & & & \mathbf{4}5 & 55 & 65 & \\
 & & & 34 & 44 & 54 & & \\
 & & 23 & 33 & 43 & & & \\
 & 12 & 22 & 32 & & & & \\
01 & 11 & 21 & & & & & \\
 & \mathbf{1}0 & & & & & &
\end{array}
$$

and there is an edge between any two vertices, which are horizontally, vertically, or diagonally adjacent. Whence any horizontal, vertical, or diagonal line is a path. Here, P_i's are vertical lines numbered from left to right by $P_0, P_1, \ldots, P_7$. The length of the first diagonal path from top is $y = \mathbf{4}$, the left coordinate of any vertex in the last path is $x = \mathbf{7}$, and the left coordinate of the only vertex in the first horizontal path from down is $d = \mathbf{1}$.

As another example, the generalized path $\mathcal{P}(6, 3, 4)$ has vertices of the form

						69
					58	**6**8
			37	47	57	**6**7
		26	36	46	56	**6**6
	15	25	35	45	55	**6**5
04	14	24	34	44	54	**6**4
	13	23	33	43	53	**6**3
		22	32	42	52	**6**2
			31	41	51	
				40		

and there is an edge between any two vertices which are horizontally, vertically, or diagonally adjacent.

Definition 2.3. Let G be a metric two-dimensional graph with the metric basis $B = \{a, b\}$. Then $d(a, b)$ is called the *basic distance* of G with respect to B and is denoted by $BD_B(G)$.

Proposition 2.4. *Let x and d be two positive integers with $x \geqslant d$ and let y be a nonnegative integer. If $(x, y, d) \neq (1, 0, 1)$, then the generalized path $\mathcal{P}(x, y, d)$ is a metric two-dimensional graph with the metric basis $B = \{v_{0,d}, v_{d,0}\}$ and the basic distance d with respect to B. Moreover, $\langle v_{i,j}|B\rangle = (i, j)$ for each $v_{i,j} \in \mathcal{P}$.*

Proof. At first we note that if $(x, y, d) \neq (1, 0, 1)$, then $\mathcal{P}(x, y, d)$ is not a path. We can therefore deduce that $\dim_{\mathrm{M}}(\mathcal{P}(x, y, d)) \geqslant 2$. We show that $B = \{a := v_{0,d}, b := v_{d,0}\}$ is a metric basis for $\mathcal{P}(x, y, d)$. In fact, we use induction on $i+j$ to show that $\langle v_{i,j}|B\rangle = (i, j)$ for each $v_{i,j} \in \mathcal{P}$.

The minimum possible value for $i + j$ is d. There are $d + 1$ vertices

$$v_{i,j} = v_{0,d}, v_{1,d-1}, \dots, v_{d-1,1}, v_{d,0}$$

with $i + j = d$. Consider the shortest path

$$a = v_{0,d}, v_{1,d-1}, \dots, v_{d-1,1}, v_{d,0} = b$$

to see that $\langle v_{i,j}|B\rangle = (i, j)$ for these vertices. In particular note that $d(a, b) = d$. Thus $BD_B(\mathcal{P}(x, y, d)) = d$.

Now let $\langle v_{i,j}|B\rangle = (i, j)$ for each vertex $v_{i,j}$ with $i + j < N$. Let $v_{k,\ell}$ be a vertex with $k + \ell = N$. Any path from $v_{i,j}$ to a should pass from one of the vertices $v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}$. The distance between each of these vertices to a is $i - 1$, by the induction hypothesis. Thus $d(v_{i,j}, a) = i$. A similar argument shows that $d(v_{i,j}, b) = j$. □

Lemma 2.5. *Let x and d be two positive integers with $x \geqslant d$ and let y be a nonnegative integer. Then $\mathcal{P}(x, y, d) = \mathcal{P}(x, y, 1) \cup \mathcal{P}(x, x, d - 1)$ and $\mathcal{P}(x, y, 1) \cap \mathcal{P}(x, x, d - 1)$ is a path.*

Proof. Let $V(\mathcal{P}) = \cup_{i=0}^{x} P_i$, where $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \dots, v_{i,i+d}\}$ for $0 \leqslant i \leqslant y$ and $P_i = \{v_{i,|i-d|}, v_{i,|i-d|+1}, \dots, v_{i,i+d-1}\}$ for $y+1 \leqslant i \leqslant x$.

Put

$$\begin{aligned} P_i' &= \{v'_{i,j-(d-1)} : v_{i,j} \in P_i \text{ and } j \geqslant i+d-2\}, \\ P_i'' &= \{v''_{i-1,j} : v_{i,j} \in P_i \text{ and } j \leqslant i+d-2\}. \end{aligned}$$

Now if $\mathcal{P}'$ is the subgraph of $\mathcal{P}$ induced by $\cup_{i=0}^{x} P_i'$ and $\mathcal{P}''$ is the subgraph of $\mathcal{P}$ induced by $\cup_{i=1}^{x} P_i''$, then $\mathcal{P}' = \mathcal{P}(x, y, 1), \mathcal{P}'' = \mathcal{P}(x, x, d-1), \mathcal{P}(x, y, d) = \mathcal{P}' \cup \mathcal{P}''$ and $\mathcal{P}' \cap \mathcal{P}''$ is the path $\{v_{1,d-1}, v_{2,d}, \dots, v_{x,x+d-2}\}$. □

Theorem 2.6. *A simple graph G is a metric two-dimensional graph with the basic distance d if and only if it is a subgraph of a generalized path $\mathcal{P}(x, y, d)$ with $(x, y, d) \neq (1, 0, 1)$ satisfying the following properties:*

i. $v_{0,d}, v_{d,0} \in G$;
ii. $N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}\} \neq \emptyset$ *and* $N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i,j-1}, v_{i+1,j-1}\} \neq \emptyset$ *for each* $v_{ij} \in G$.

Proof. An inductive argument proves that any subgraph of $\mathcal{P}(x, y, d)$ with $(x, y, d) \neq (1, 0, 1)$ possessing the properties (i) and (ii) is a metric two-dimensional graph with the basis $B = \{a := v_{0,d}, b := v_{d,0}\}$ and the basic distance d.

Conversely, suppose that G is a metric two-dimensional graph with the basis $B = \{a, b\}$ and the basic distance d. Let

$$x := \max\{d(v, a) : v \in G\},$$

and

$$y := \max\{i : (i, i+d) = \langle v|B\rangle, \text{ for some } v \in G\}.$$

Define $\varphi : G \to \mathcal{P}(x, y, d)$ by $\varphi(v) = v_{i,j}$, where $(i, j) = \langle v|B\rangle$. We show that

$$\begin{aligned} &|i-d| \leqslant j \leqslant i+d, && \text{for } i = 0, \dots, y, \\ &|i-d| \leqslant j \leqslant i+d-1, && \text{for } i = y+1, \dots, x. \end{aligned}$$

We have $d(v, a) = i$ and $d(v, b) = j$, since $(i, j) = \langle v|B\rangle$. The triangle inequality implies that $d = d(a, b) \leqslant d(a, v) + d(v, b) = i + j$. Moreover, $j = d(v, b) \leqslant d(v, a) + d(a, b) = i + d$ and $i = d(v, a) \leqslant d(v, b) + d(b, a) = j + d$. Thus $|i - d| \leqslant j \leqslant i + d$ for each $0 \leqslant i \leqslant x$.

If $i \geqslant y + 1$, then j cannot be $i + d$, since otherwise we should have $(i, i + d) = (i, j) = \langle v|B\rangle$ which contradicts the definition of y. Hence $j \leqslant i + d - 1$ for $i \geqslant y + 1$.

We therefore have $\varphi(V(G)) \subseteq V(\mathcal{P}(x,y,d))$. Now let $e = uv$ be an edge in $V(G)$. If $\varphi(u) = v_{i,j}$ and $\varphi(v) = v_{k\ell}$, then

$$i = d(u,a) \leqslant d(u,v) + d(v,a) = 1 + k,$$

and

$$k = d(v,a) \leqslant d(v,u) + d(u,a) = 1 + i.$$

Thus $|i-k| \leqslant 1$. By the same argument, $|j-\ell| \leqslant 1$. This shows that $k = i-1, i$ or $i+1$ and $\ell = j-1, j$ or $j+1$. Whence $\varphi(e)$ is an edge in $\mathcal{P}(x,y,d)$ and so G is a subgraph of $\mathcal{P}(x,y,d)$.

Clearly, $v_{0,d} = a, v_{d,0} = b \in G$. To show that (ii) does also hold, note that if, for example, $N(v_{i,j}) \cap \{v_{i-1,j-1}, v_{i-1,j}, v_{i-1,j+1}\} = \emptyset$, then there is no path with the length i from $v_{i,j}$ to a. □

3. Enumerating of Metric two-dimensional Graphs with the Basic Distance 1

Lemma 2.5 shows that any generalized path $\mathcal{P}(x,y,d)$ can be regarded as a larger path $\mathcal{P}(x',y',d')$. Thus the generalized path mentioned in Theorem 2.6 is not unique. A simple argument based on the property (ii) of Theorem 2.6 implies that if $x = \max\{d(v,a) : v \in G\}, y = \max\{i : (i,i+d) = \langle v|B\rangle, \text{ for some } v \in G\}$ and $d = d(a,b)$, then the *boundary* $\partial\mathcal{P}(x,y,d)$

$$v_{0,d}, v_{1,d-1}, v_{2,d-2}, \ldots, v_{d,0}, v_{d+1,1} v_{d+2,2}, \ldots, v_{x,x-d}, v_{1,d+1}, v_{2,d+2}, \ldots, v_{y,y+d}$$

of $\mathcal{P}(x,y,d)$ are vertices of G. Whence this x, y and d are the least possible values such that G is a subgraph of $\mathcal{P}(x,y,d)$.

Definition 3.1. Let G be a simple metric two-dimensional graph. We say that G *is fitted in* $\mathcal{P}(x,y,d)$, denoted by $G \sqsubseteq \mathcal{P}(x,y,d)$, if

$$x = \max\{d(v,a) : v \in G\},$$
$$y = \max\{i : (i,i+d) = \langle v|B\rangle, \text{ for some } v \in G\},$$
$$d = d(a,b),$$

or equivalently G contains the *boundary* $\partial\mathcal{P}(x,y,d)$

$$v_{0,d}, v_{1,d-1}, v_{2,d-2}, \ldots, v_{d,0}, v_{d+1,1} v_{d+2,2}, \ldots, v_{x,x-d}, v_{1,d+1}, v_{2,d+2}, \ldots, v_{y,y+d}$$

of $\mathcal{P}(x,y,d)$. The parameters x and y are called the length and width of G and are denoted by $\ell(G)$ and $w(G)$, respectively.

We now want to enumerate the number of n vertex metric two-dimensional graph with the basic distance 1. Prior to this, we enumerate the number of n vertex metric two-dimensional graph with the length x, width y, and the basic distance 1. We denote the latter number by $\nu(n;x,y)$.

Lemma 3.2. *$\nu(n;x,y) \geqslant 1$ if and only if $x+y+2 \leqslant n \leqslant 2x+y+1$.*

Proof. Suppose that there is an n vertex metric two-dimensional graph G with the length x, width y and the basic distance 1. Using Theorem 2.6, we fit it in $\mathcal{P}(x,y,1)$. Since the boundary of $\mathcal{P}(x,y,1)$ has $x+y+1$ elements, we should have $n \geqslant x+y+1$. If $n = x+y+1$, then $G = \partial\mathcal{P}(x,y,1)$ which is a path and has metrics dimension 1. Thus $n \geqslant x+y+2$. Moreover, $n = |V(G)| \leqslant |V(\mathcal{P}(x,y,1))| = 2x+y+1$.

On the other hand, if $x+y+2 \leqslant n \leqslant 2x+y+1$, then we can write $n = x+y+1+r$, where $1 \leqslant r \leqslant x$. Now consider the subgraph of $\mathcal{P}(x,y,1)$ induced by $\partial\mathcal{P}(x,y,1) \cup \{v_{1,1},\ldots,v_{r,r}\}$. This is an n vertex subgraph of $\mathcal{P}(x,y,1)$ satisfying (i) and (ii) of Theorem 2.6. □

Based on Lemma 3.2, for simplicity, we denote $\nu(n;x,y)$ by $\mu(m;x,y)$. We note that $\mu(m;x,y) \geqslant 1$ if and only if $1 \leqslant m \leqslant x$.

Lemma 3.3. *$\mu(x;x,y) = 4 \times 20^{y-1} \times 10^{x-y}$ for each $x \geqslant y \geqslant 1$ and $\mu(x;x,0) = 2 \times 10^{x-1}$ for each $x \geqslant 1$.*

Proof. Let G be an n vertex metric two-dimensional graph G with the length x, width y and the basic distance 1, where $n = 2x+y+1$. Thus $G \sqsubseteq \mathcal{P}(x,y,1)$ and the induced subgraph $\partial\mathcal{P}(x,y,1)$ of $\mathcal{P}(x,y,1)$ should be a subgraph of G. For other vertices

$$\{v_{1,1},\ldots,v_{y,y},v_{y+1,y+1},\ldots,v_{x,x}\},$$

we should put the edges in such a way that (ii) of Theorem 2.6 is satisfied. For $v_{1,1}$ putting edges $v_{1,1}v_{0,1}$ and $v_{1,1}v_{1,0}$ is compulsory, and we have 4 choices for 'to put' or 'not to put' the edges $v_{1,1}v_{1,2}$ and $v_{1,1}v_{2,1}$.

If $1 < i \leqslant y$, then for $v_{i,i}$ putting one of the 5 sets of edges,

$$\begin{aligned}&\{v_{i,i}v_{i-1,i-1}\},\{v_{i,i}v_{i-1,i},v_{i,i}v_{i,i-1}\},\{v_{i,i}v_{i-1,i-1},v_{i,i}v_{i-1,i}\},\\&\{v_{i,i}v_{i-1,i-1},v_{i,i}v_{i,i-1}\},\{v_{i,i}v_{i-1,i-1},v_{i,i}v_{i-1,i},v_{i,i}v_{i,i-1}\}\end{aligned}$$

is compulsory and we have 4 choices for 'to put' or 'not to put' the edges $v_{i,i}v_{i,i+1}$ and $v_{i,i}v_{i+1,i}$.

If $y+1 \leqslant i \leqslant x$, then the 4 choices decreases into 2 choices, since we do not have $v_{i,i+1}$.

Finally, if $y = 0$, then we have 2 choices for $v_{1,1}$ and 10 choices for $v_{i,i}$ when $1 < i \leqslant x$. □

Though we know that $\mu(0;x,y) = 0$, but for the following recursive relation, we need to assume, as a convenient, that $\mu(0;x,y) = 1$.

Furthermore, for $y \geqslant 1$, we assume that

$$\omega(j) = \begin{cases} 4, & j = 1, \\ 20, & 2 \leqslant j \leqslant y, \\ 10, & y+1 \leqslant j \leqslant x, \end{cases}$$

and for $y = 0$ we assume that

$$\omega(j) = \begin{cases} 2, & j = 1, \\ 10, & 2 \leqslant j \leqslant x. \end{cases}$$

Theorem 3.4. *Let x be a positive integer, let y be a nonnegative integer, and let $1 \leqslant m < x$. Then $\mu(m; x, y)$ satisfies the recursive relation*

$$\mu(m; x, y) = \sum_{i=1}^{m+1} (\prod_{j=1}^{i-1} \omega(j)) \cdot \mu(m-(i-1); x-i, \max\{y-i, 0\}),$$

with the boundary values

$$\mu(0; x, y) = 1, \quad \mu(x; x, y) = \prod_{j=1}^{x} \omega(j).$$

Proof. To determine $\mu(m; x, y)$, we in fact need to enumerate the number of $n = x + y + 1 + m$ vertex metric two-dimensional subgraphs G of $\mathcal{P}(x, y, 1)$ with the basic distance 1. Let $m < x$. Then there is a vertex $v_{i,i} \in \mathcal{P}(x, y, 1) \setminus G$. Let i be the first index such that $v_{i,i} \in \mathcal{P}(x, y, 1) \setminus G$. Then $1 \leqslant i \leqslant m + 1$. Since $v_{1,1}, \ldots, v_{i-1,i-1} \in G$, we have $\prod_{j=1}^{i-1} \omega(j)$ choices for selecting appropriate edges. Then we have $\mu(m-(i-1); x-i, \max\{y-i, 0\})$ choices for selecting other edges for other vertices of G. □

Corollary 3.5. *Let x be a positive integer and let $1 \leqslant m < x$. Then*

$$\mu(m; x, 0) = \mu(m; x-1, 0) + \sum_{i=2}^{m+1} 2 \times 10^{i-2} \cdot \mu(m-(i-1); x-i, 0).$$

Example 3.6. We evaluate $\mu(m; x, 0)$ for $m = 1, 2, 3$ and $x > m$.

A simple verification shows that $\mu(1; x, 0) = 2x$. For $m = 2 < x$ we have

$$\begin{aligned} \mu(2; x, 0) &= \mu(2; x-1, 0) + 2\mu(1; x-2, 0) + 20\mu(0; x-3, 0) \\ &= \mu(2; x-1, 0) + 2 \cdot 2(x-2) + 20 \\ &= \mu(2; x-1, 0) + 4(x+3). \end{aligned}$$

Iterating the above equation, we have

$$\begin{aligned}
\mu(2;x,0) &= \mu(2;x-1,0)+4(x+3)\\
&= \mu(2;x-2,0)+4(x+2)+4(x+3)\\
&= \mu(2;x-3,0)+4(x+1)+4(x+2)+4(x+3)\\
&= \dots\\
&= \mu(2;2,0)+4(2+4)+\dots 4(x+3)\\
&= 20+4\left(\frac{(x+3)(x+4)}{2}-15\right)\\
&= 2(x-1)(x+8).
\end{aligned}$$

Finally, for $m=3<x$, we have

$$\begin{aligned}
\mu(3;x,0) &= \mu(3;x-1,0)+2\mu(2;x-2,0)+20\mu(1;x-3,0)\\
&+ 200\mu(0;x-4,0)\\
&= \mu(3;x-1,0)+2\cdot 2(x-3)(x+6)+20\cdot 2(x-3)+200\\
&= \mu(3;x-1,0)+4(x^2+13x+2).
\end{aligned}$$

A similar method gives

$$\mu(3;x,0)=\frac{4}{3}x^3+28x^2+\frac{104}{3}x-192.$$

Corollary 3.7. *Let x be a positive integer and let $1\leqslant m<x$. Then $\mu(m;x,0)$ is a polynomial of x of degree m.*

Proof. Using induction on $m+x$, we can assume that the right hand side of Corollary 3.5 is a polynomial of x of degree m. Whence the left hand side is also a polynomial of x of degree m. □

We now can simply evaluate $\nu(n)$; the number of all n vertex labeled metric two-dimensional graph with the basis $B=\{a,b\}$ and the basic distance 1.

Theorem 3.8. *The number of all n vertex labeled metric two-dimensional graph G with the basis $B=\{a,b\}$ and the basic distance 1, is*

$$\nu(n)=\sum_{y=0}^{\lfloor\frac{n-1}{3}\rfloor}\sum_{x=\lceil\frac{n-y-1}{2}\rceil}^{n-y-2}\mu(n-x-y-1;x,y).$$

Proof. Each G can be fitted in a $\mathcal{P}(x,y,1)$ where, by Lemma 3.2, we should have $x+y+2\leqslant n\leqslant 2x+y+1$. Thus the valid values of x and y are $0\leqslant y\leqslant\lfloor\frac{n-1}{3}\rfloor$ and $\lceil\frac{n-y-1}{2}\rceil\leqslant x\leqslant n-y-2$. We know that the number of metric two-dimensional subgraph of $\mathcal{P}(x,y,1)$ is $\mu(n-x-y-1;x,y)$. □

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مشخص‌سازی گراف‌های با بعد متریک دو و شمارش آن‌ها

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بعد متریک گراف G عبارت است از حداقل تعداد راس‌های لازم در زیرمجموعه‌ی B از رئوس گراف، به‌طوری‌که تمام راس‌های دیگر به واسطهٔ فاصله‌ی آنها تا راس‌های B، به‌طور منحصربه‌فرد تعیین شوند. در این حالت B را پایه متریک گراف G می‌نامیم. فاصله پایه یک گراف دو بعدی G، به‌صورت فاصله بین دو عنصر B تعریف می‌شود. در این مقاله، ابتدا گراف‌های با بعد متریک دو مشخص‌سازی می‌شود و سپس تعداد رئوس گراف‌های با بعد متریک دو، که فاصله پایه آن‌ها یک باشد را خواهیم شمرد.

کلمات کلیدی: بعد متریک گراف‌ها، مجموعه کاشف، پایه متریک گراف، فاصله پایه‌ای، شمارش گراف‌ها.