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ON PRIMARY IDEALS OF POINTFREE FUNCTION RINGS

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Abstract. We study primary ideals of the ring $\mathcal{R}L$ of real-valued continuous functions on a completely regular frame L. We observe that prime ideals and primary ideals coincide in a P-frame. It is shown that every primary ideal in $\mathcal{R}L$ is contained in a unique maximal ideal, and an ideal Q in $\mathcal{R}L$ is primary if and only if $Q \cap \mathcal{R}^*L$ is a primary ideal in $\mathcal{R}^*L$. We show that every pseudo-prime (primary) ideal in $\mathcal{R}L$ is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal. Finally, we prove that if L is a connected frame, then the zero ideal in $\mathcal{R}L$ is decomposable if and only if $L = \mathbf{2}$.

1. Introduction

Throughout, all our rings are commutative with identity, and by the term "ideal" we mean a proper ideal. Recall from [23] that for an ideal I in a ring R, the *radical ideal* $\sqrt{I}$ is defined to be

$$\sqrt{I} = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

As defined in [23], we say that an ideal Q in a ring R is *primary ideal* if $a, b \in R$ and $ab \in Q$ imply $a \in Q$ or $b \in \sqrt{Q}$. Let Q be a primary ideal of a ring R. Then $P = \sqrt{Q}$ is a prime ideal of R, and we say that Q is P-primary. Let $C(X)$ be the ring of all real-valued continuous functions on a completely regular Hausdorff space X. Primary ideals in $C(X)$ have been studied by several authors (see [4, 16, 21, 24]). In

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[1], we begin the study of primary ideals in the ring $\mathcal{R}L$, and there, we investigate the relation between these ideals and z-ideals (d-ideals). For example, it is shown that the sum of a primary ideal and a z-ideal (a d-ideal) in $\mathcal{R}L$ which are not in a chain is a prime z-ideal (a prime d-ideal)

Denote by $\mathrm{Max}(R)$ the set of all maximal ideals of a ring R. Recall from [22] that an ideal I of a ring R is a *z-ideal* if whenever $\mathfrak{M}(x) = \mathfrak{M}(y)$ and $x \in I$, then $y \in I$. We not that for $a \in R$, $\mathfrak{M}(a) = \{M \in \mathrm{Max}(R) : a \in M\}$. In Proposition 3.4, we observe every z-ideal I of a ring R is prime if and only if it is primary. This enables us to see that if L is a P-frame, then every ideal in $\mathcal{R}L$ is prime if and only if it is primary (Corollary 3.5). And also, the socle of $\mathcal{R}L$ is a primary ideal if and only if $L = \mathbf{2}$ (Corollary 3.7).

In [9, Proposition 5.4.], Dube has shown that every prime ideal in the ring $\mathcal{R}L$ is contained in a unique maximal ideal. We show the same fact for primary ideals and then it turns out that if Q is a primary ideal of $\mathcal{R}L$, then there is a unique point $I \in \beta L$ such that $\mathbf{O}^I \subseteq Q \subseteq \mathbf{M}^I$.

In [17], Gillman and Kohls have defined that an ideal I in a ring R is *pseudo-prime* if for $a, b \in R$, $ab = 0$ implies, $a \in I$ or $b \in I$. They show that an ideal I in the ring $C(X)$ is pseudo-prime if and only if $\sqrt{I}$ is prime. In Proposition 4.1, we show that this fact is true in the ring $\mathcal{R}L$ and then it turns out that every primary ideal in $\mathcal{R}L$ is pseudo-prime because it is a well-known fact that the radical of a primary ideal is prime.

Azarpanah [3] has shown that every pseudo-prime ideal in $C(X)$ is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal. In the paper [13], Dube has proved that this fact is true for prime ideals in $\mathcal{R}L$. In Theorem 4.5, we show that this fact is true for pseudo-prime ideals in $\mathcal{R}L$, and also, it is true for primary ideals since they are pseudo-prime.

In the last section, we show that if L is a connected frame, then $|L| > 2$ if and only if the zero ideal in $\mathcal{R}L$ is not decomposable (see Theorem 5.1).

2. Preliminaries

2.1. **Frames.** Our reference for frames is [20]. The bottom element and the top element of a frame L is denoted by $\bot$ and $\top$, respectively. We denote the *completely below* relations by $\prec\!\!\prec$, and recall that a frame L is completely regular if each of its elements is the join of the elements completely below it. All frames considered in this paper are assumed to be completely regular. As usual, we denote by βL the Stone-Čech compactification of a completely regular frame L. We write r_L for the right adjoint of the join map $\beta L \to L$ given by

$$r_L(x) = \{a \in L : a \prec\!\!\prec x\}.$$

Recall that r_L preserve $\prec\!\!\prec$, and $\bigvee r_L(x) = x$ for any $x \in L$.

The *pseudocomplement* of an element $a \in L$ is denoted by a^* and we have

$$a^* = \bigvee\{x \in L : a \wedge x = \bot\}.$$

An element $a \in L$ is called *dense* if $a^* = \bot$, and *complemented* if $a \vee a^* = \top$. We note that $a \prec\!\!\prec a$ if and only if a is complemented, and if $a \prec\!\!\prec b$, then $b^* \prec\!\!\prec a^*$.

An element $\top \neq p \in L$ is a *prime* (or a *point*) element if for every $a, b \in L$, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We write $Pt(L)$ for the set of all points of a frame L.

2.2. **The ring $\mathcal{R}L$ and some of its ideals.** For undefined terms and notations see [18] on $C(X)$. Good references for results about $\mathcal{R}L$ are [5, 7]. We use the notation of [7]. A *cozero element* of L is an element of the form $\operatorname{coz}\varphi$ for some $\varphi \in \mathcal{R}L$. The cozero part of a frame L is denoted by $\operatorname{Coz} L = \{\operatorname{coz}\varphi : \varphi \in \mathcal{R}L\}$, and it is a regular sub-σ-frame of L. See [5, 6, 7] for details.

Recall from [9] that associated with each $I \in \beta L$ are two ideals, $\mathbf{M}^I$ and $\mathbf{O}^I$, of $\mathcal{R}L$ defined by $\mathbf{M}^I = \{\varphi \in \mathcal{R}L : r_L(\operatorname{coz}\varphi) \subseteq I\}$ and

$$\mathbf{O}^I = \{\varphi \in \mathcal{R}L : r_L(\operatorname{coz}\varphi) \prec\!\!\prec I\} = \{\varphi \in \mathcal{R}L : \operatorname{coz}\varphi \in I\}.$$

(1). Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\mathbf{M}^I$, for $I \in Pt(\beta L)$.

(2). If P is a prime ideal, there is a unique point $I \in \beta L$ such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$.

3. Primary ideals in $\mathcal{R}L$

Let Q be an ideal of the ring R such that $\sqrt{Q} = M$, a maximal ideal of R. Then Q is an M-primary ideal of R. Consequently, all positive powers M^n $(n \in \mathbb{N})$ of the maximal ideal M are M-primary. But, in general, any positive power of a prime ideal P need not be P-primary (see [23]). We aim to show that every positive power of a prime ideal in $\mathcal{R}L$ is primary.

In the ring $C(X)$, every function has an n^{th} root for every odd integer $n \geq 1$, and every positive function has an n^{th} root for every integer $n \geq 1$. Ighedo [19, Ch. 7], by these facts and this fact that the bounded part of $\mathcal{R}L$, denoted by $\mathcal{R}^*L$, is a $C(X)$, showed that:

(i) Every element of $\mathcal{R}L$ has an n^{th} root, for any odd $n \in \mathbb{N}$.

(ii) Every positive element of $\mathcal{R}L$ has an n^{th} root, for any $n \in \mathbb{N}$.

However, it is instructive to construct a direct proof. Let $n \in \mathbb{N}$ be an odd number. Suppose that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt[n]{x}$ for all $x \in \mathbb{R}$. Then for every $p, q \in \mathbb{Q}$,

$$f^{-1}\Big(\{x \in \mathbb{R} : p < x < q\}\Big) = \{x \in \mathbb{R} : p^n < x < q^n\}.$$

So, by definition of the frame of the reals $\mathcal{L}(\mathbb{R})$ there is a frame map $\rho : \mathcal{L}(\mathbb{R}) \to \mathcal{L}(\mathbb{R})$ such that for every $p, q \in \mathbb{Q}$, $\rho(p, q) = (p^n, q^n)$. Now, let $\alpha \in \mathcal{R}L$. We define the frame map $\sqrt[n]{\alpha} : \mathcal{L}(\mathbb{R}) \to L$ given by $\sqrt[n]{\alpha} = \alpha \circ \rho$. By the following proposition, $\sqrt[n]{\alpha}$ is an n^{th} root of α. We note that if $\alpha, \beta \in \mathcal{R}L$ and for every $r, s \in \mathbb{Q}$, $\alpha(r, s) \leq \beta(r, s)$, then $\alpha = \beta$.

Proposition 3.1. *Let $\alpha \in \mathcal{R}L$ and let $n \in \mathbb{N}$ be an odd number. Then $(\sqrt[n]{\alpha})^n = \alpha$.*

Proof. Let $p, q \in \mathbb{Q}$. Suppose that $(r, s) = (r_1, s_1) \wedge \cdots \wedge (r_n, s_n)$ such that $p < r_1 r_2 \cdots r_n < s_1 s_2 \cdots s_n < q$. It follows that $r = r_1 \vee \cdots \vee r_n$ and $s = s_1 \wedge \cdots \wedge s_n$; so $p < r_1 r_2 \cdots r_n \leq r^n < s^n \leq s_1 s_2 \cdots s_n < q$. In consequence, $\rho(r, s) \leq (p, q)$. Therefore,

$$\begin{aligned}
(\sqrt[n]{\alpha})^n(p, q) &= (\alpha \circ \rho)^n(p, q) \\
&= \bigvee \Big\{ \textstyle\bigwedge_{k=1}^{n} \alpha \circ \rho(r_k s_k) : p < r_1 r_2 \cdots r_n < s_1 s_2 \cdots s_n < q \Big\} \\
&= \alpha\Big(\bigvee \big\{ \big((\textstyle\bigvee_{i=1}^{n} r_i)^n, (\bigwedge_{i=1}^{n} s_i)^n\big) : p < r_1 r_2 \cdots r_n < s_1 s_2 \cdots \\
&\qquad s_n < q \big\}\Big) \\
&\leq \alpha\Big(\bigvee \big\{ (r^n, s^n) : p < r^n < s^n < q \big\}\Big) \\
&\leq \alpha(p, q).
\end{aligned}$$

This implies that $(\sqrt[n]{\alpha})^n = \alpha$, and the proof is complete. □

Proposition 3.2. *If P and Q are prime ideals of $\mathcal{R}L$, then $P \cap Q = PQ$.*

Proof. Let $\alpha \in P \cap Q$. Since $\alpha^{\frac{2}{3}} \alpha^{\frac{1}{3}} = \alpha \in P \cap Q$, we conclude that $\alpha^{\frac{1}{3}} \in P \cap Q$. Therefore $\alpha = \alpha^{\frac{2}{3}} \alpha^{\frac{1}{3}} \in PQ$, that is, $P \cap Q \subseteq PQ$. Now, the proof is complete since the reverse inclusion is always true. □

As a result, we get the upcoming corollary.

Corollary 3.3. *Every positive power of a prime ideal in $\mathcal{R}L$ is a primary ideal.*

It is known that every prime ideal in a ring is primary but the converse is not true. In what follows, we are going to show that the concepts of prime and primary coincide in the P-frames.

Proposition 3.4. *Every z-ideal I of a ring R is prime if and only if it is primary.*

Proof. To prove the nontrivial part of the proposition, let $x, y \in R$ such that $xy \in I$ but $x \notin I$. Since I is primary and $x \notin I$, we must have $y^n \in I$ for some $n \in \mathbb{N}$. On the other hand, $\mathfrak{M}(y) = \mathfrak{M}(y^n)$ implies that $y \in I$ because I is a z-ideal. Therefore I is a prime ideal. □

This proposition has some important corollaries, given in the following.

A frame L is called P-*frame* if $c \vee c^* = \top$ for each $c \in \operatorname{Coz} L$. Dube [11, Proposition 2.1] has shown that a frame L is P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal. Therefore, by the foregoing proposition, the next corollary is obvious.

Corollary 3.5. *If L is a P-frame, then an ideal in $\mathcal{R}L$ is prime if and only if it is primary.*

The combination of [2, Lemma 2.8] with the foregoing proposition allows us to obtain the following result.

Corollary 3.6. *For any z-ideal I of $\mathcal{R}L$, the following statements are equivalent.*

(1) *I is a primary ideal.*
(2) *I contains a primary ideal.*
(3) *For all $\alpha, \beta \in \mathcal{R}L$, if $\alpha\beta = \mathbf{0}$, then $\alpha \in I$ or $\beta \in I$.*
(4) *For every $\varphi \in \mathcal{R}L$, there is a cozero element $c \in \operatorname{Coz}[I] = \{\operatorname{coz}\varphi : \varphi \in I\}$ such that $\varphi(0, -) \leq c$ or $\varphi(-, 0) \leq c$.*

Dube [13, Proposition 4.2] has shown that the socle of $\mathcal{R}L$ is a prime ideal if and only if $L = \mathbf{2}$. Since the socle of a ring $\mathcal{R}L$ is a z-ideal, by Proposition 3.4, we can conclude the next corollary.

Corollary 3.7. *The socle of $\mathcal{R}L$ is a primary ideal if and only if $L = \mathbf{2}$.*

It is well known that every prime ideal in $\mathcal{R}L$ is contained in a unique maximal ideal (see [9, Proposition 5.4]). We aim to show that this fact is true for every primary ideal in $\mathcal{R}L$. We begin with the following lemma.

Lemma 3.8. *If I, J are two z-ideals in a ring R and $I \cap J$ is primary, then I and J are in a chain.*

Proof. Suppose $I \not\subseteq J$. Then there exists an $x \in I \setminus J$. Let $y \in J$. Then $xy \in I \cap J$. Since $x \notin I \cap J$ and $I \cap J$ is a primary ideal, there exists $n \in \mathbb{N}$ such that $y^n \in I \cap J$. Since $\mathfrak{M}(y) = \mathfrak{M}(y^n)$ and since $I \cap J$ is a z-ideal, it follows that $y \in I \cap J$. Thus, $J \subseteq I$, hence I and J are in a chain. □

Theorem 3.9. *Every primary ideal in $\mathcal{R}L$ is contained in a unique maximal ideal.*

Proof. We know that every ideal is contained in at least one maximal ideal. Let Q be a primary ideal in $\mathcal{R}L$. Suppose that M and M_1 are maximal ideals such that $Q \subseteq M$ and $Q \subseteq M_1$. Since $M \cap M_1$ is a z-ideal and $Q \subseteq M \cap M_1$, we can conclude from Corollary 3.6 that $M \cap M_1$ is a primary ideal. Now, the preceding lemma shows that $M \subseteq M_1$ or $M_1 \subseteq M$. Therefore $M = M_1$. □

Let P be a prime ideal of $\mathcal{R}L$. It turned out that there is a unique point $I \in \beta L$ such that $\mathbf{O}^I \subseteq P \subseteq \mathbf{M}^I$ (for details see [9]). We show that this fact is true for every primary ideal in $\mathcal{R}L$. In [1], we prove that if Q is an ideal of $\mathcal{R}L$ and S is a z-ideal such that $S \subseteq \sqrt{Q}$, then $S \subseteq Q$. We will use this result in our proof.

Proposition 3.10. *Let Q be primary ideal of $\mathcal{R}L$. Then there is a unique point $I \in \beta L$ such that $\mathbf{O}^I \subseteq Q \subseteq \mathbf{M}^I$.*

Proof. By Theorem 3.9, there is a unique maximal ideal M such that $Q \subseteq M$. Take $I \in Pt(\beta L)$ with $M = M^I$. It is enough to show that $\mathbf{O}^I \subseteq Q$. Let $\delta \in \mathbf{O}^I$. By Lemma 5.3 in [9] there exists $\gamma \notin \mathbf{M}^I$ such that $\delta\gamma = \mathbf{0}$. Thus $\delta \in \sqrt{Q}$ since Q is primary and $\gamma \notin Q$. In consequence, $\mathbf{O}^I \subseteq \sqrt{Q}$. This shows that $\mathbf{O}^I \subseteq Q$ since $\mathbf{O}^I$ is a z-ideal. □

We now wish to show that an ideal of $\mathcal{R}L$ is primary precisely when its contraction to the subring $\mathcal{R}^*L$ is a primary ideal. The left-to-right implication is true for any ring and any of its subrings. In fact, we have the following easy observation.

Observation: If $\phi : A \to B$ is a ring homomorphism and $Q \subseteq B$ is a primary ideal, then $\phi^{-1}[Q]$ is a primary ideal of A.

Proof. Let $xy \in \phi^{-1}[Q]$ with $x \notin \phi^{-1}[Q]$. Then $\phi(x) \notin Q$, and so there is an $n \in \mathbb{N}$ such that $\phi(y^n) = \phi(y)^n \in Q$. Thus, $y^n \in \phi^{-1}[Q]$. □

Now recall that an f-ring A is said to have *bounded inversion* if every $a \geq 1$ is a unit in A. Recall also that the *bounded part* of A is the subring

$$A^* = \{a \in A : |a| \leq n.1 \text{ for some } n \in \mathbb{N}\}.$$

It is well known that $\mathcal{R}L$ has bounded inversion. We now have the following result.

Proposition 3.11. *Let A be an f-ring with bounded inversion. Then an ideal Q of A is primary (resp. prime) if and only if $A^* \cap Q$ is a primary (resp. prime) ideal of A^*.*

Proof. We prove the result only for the primary case, as the other one can be proved similarly. The left-to-right implication follows from the observation above.

Conversely, assume that $A \cap Q$ is a primary ideal of A^*. Consider $a, b \in A$ with $ab \in Q$ and $a \notin Q$. Then $\frac{a}{1+|a|} \cdot \frac{b}{1+|b|} \in A^* \cap Q$ and $\frac{a}{1+|a|} \notin A^* \cap Q$. There is therefore an $n \in \mathbb{N}$ such that $\left(\frac{b}{1+|b|}\right)^n \in A^* \cap Q$, which clearly implies $b^n \in Q$, whence we deduce that Q is a primary ideal of A. □

Corollary 3.12. *An ideal of $\mathcal{R}L$ is primary (resp. prime) if and only if its contraction to $\mathcal{R}^*L$ is primary (resp. prime).*

4. PSEUDO-PRIME IDEALS

Let A be a partially ordered commutative ring. An ideal I in A is *convex* if $0 \leq x \leq y$ and $y \in I$ implies $x \in I$. Convexity is the necessary and sufficient condition that A/I be partially ordered, under the definition: $x + I \geq 0$ provided that $x + I = a + I$ for some $a \geq 0$. We note that arbitrary intersections of convex ideals are convex (see [18, Ch. 5]).

Proposition 4.1. *The following statements are equivalent for any ideal I in $\mathcal{R}L$.*

(1) *I contains a prime ideal.*
(2) *I is pseudo-prime.*
(3) *For every convex ideal $J \supseteq I$, $\mathcal{R}L/J$ is a totally ordered ring.*
(4) *$\mathcal{R}L/c(I)$ is a totally ordered ring, where $c(I)$ is the smallest convex ideal containing I.*
(5) *The convex ideals containing I form a chain.*
(6) *The prime ideals containing I form a chain.*
(7) *$\sqrt{I}$ is prime.*
(8) *$\mathcal{R}L/\sqrt{I}$ is a totally ordered ring.*

Proof. (1) ⇒ (2). It is obvious because any ideal containing a prime ideal is pseudo-prime.

(2) ⇒ (3). It is clear that for every $\varphi \in \mathcal{R}L$, $(\varphi + |\varphi|)(\varphi - |\varphi|) = \mathbf{0}$. Thus, [17, 3.5] shows that (2) implies (3)

(3) ⇒ (4). Obvious.

(4) ⇒ (5). The convex ideals in $\mathcal{R}L/c(I)$ form a chain. By [18, 14.3(b)], the convex ideals containing $c(I)$ form a chain. But these latter are the same as the convex ideals containing I.

(5) ⇒ (6). It is evident since, by [10, Lemma 3.5], we can infer that every prime ideal in $\mathcal{R}L$ is convex.

(6) $\Rightarrow$ (7). It is obvious because the intersection of any chain of prime ideals is prime.

(7) $\Rightarrow$ (1) Let P be a minimal prime ideal contained in the prime ideal $\sqrt{I}$. Then, by the corollary after Theorem 1.1 in [22], P is a prime z-ideal. We show that $P \subseteq I$. Suppose $\alpha \in P$. Put

$$\varphi = \sum_{n=1}^{\infty} \mathbf{2}^{-n}\alpha^{2/3n}(\mathbf{1} + \alpha^{2/3n})^{-1}.$$

Then, by [8, Section 6], we have $\varphi \in \mathcal{R}L$. Obviously, $\operatorname{coz}\varphi = \operatorname{coz}\alpha$. So φ belongs to the z-ideal P, and hence $\varphi \in \sqrt{I}$. This means that $\varphi^n \in I$ for some $n \in \mathbb{N}$. But, $\mathbf{2}^{-n}\frac{\alpha^{\frac{2}{3n}}}{\mathbf{1}+\alpha^{\frac{2}{3n}}} \leq \varphi$, and so $|\alpha| \leq |\varphi^n \mathbf{2}^{n^2}(\mathbf{1} + \alpha^{\frac{2}{3n}})^n|^{\frac{3}{2}}$. Now, by [19, Lemma 7.2.1], α is a multiple of $\varphi^n \mathbf{2}^{n^2}(\mathbf{1} + \alpha^{\frac{2}{3n}})^n$, which implies that α is a multiple of φ^n. Therefore $\alpha \in I$.

This establishes the equivalence of statements $(1) - (7)$. The fact that radical ideals of $\mathcal{R}L$ are convex is well-known (see [10, Lemma 3.5]). Thus, $c(\sqrt{I}) = \sqrt{I}$. Now, the equivalence of (4) with (7) yields their equivalence with (8). □

In what follows, we give a number of results that may be deduced immediately from Proposition 4.1.

(1) A pseudo-prime ideal in $\mathcal{R}L$ is prime if and only if it coincides with its radical.

(2) If I is an intersection of prime ideals in $\mathcal{R}L$, then I is prime if and only if $\mathcal{R}L/I$ is a totally ordered ring.

(3) A convex ideal I in $\mathcal{R}L$ is pseudo-prime if and only if $\mathcal{R}L/I$ is a totally ordered ring.

(4) If I is a convex ideal in $\mathcal{R}L$, and $\mathcal{R}L/I$ is a totally ordered ring, then $\mathcal{R}L/I$ has a prime radical.

Recall that an ideal of a ring is said to be *essential* if it intersects every nonzero ideal of the ring non-trivially. Let us recall the following lemmas from [9] and [10], respectively.

Lemma 4.2. *An ideal I of $\mathcal{R}L$ is essential if and only if $\bigvee \operatorname{Coz}[I]$ is dense.*

Lemma 4.3. *For any $I \in \beta L$, $\bigvee \operatorname{Coz}[\mathbf{O}^I] = \bigvee \operatorname{Coz}[M^I] = \bigvee I$.*

In [13, Lemma 4.5], Dube showed that each prime ideal of $\mathcal{R}L$ is either essential or simultaneously maximal and generated by an idempotent. In the latter case, it is also a minimal prime ideal. We show that this fact is true for pseudo-prime (primary) ideals. We begin with the next lemma.

Lemma 4.4. *The following statements hold for a frame L.*

1. *If $p \in Pt(L)$, then either $p^* = \bot$ or $p \vee p^* = \top$.*
2. *If $p \in Pt(L)$ and $p \vee p^* = \top$, then $r_L(p) \vee \big(r_L(p)\big)^* = \top_{\beta L}$.*
3. *If I is a complemented point of βL, then $\mathbf{O}^I = \mathbf{M}^I$.*

Proof. (1). Since $p \leq p \vee p^*$ and every prime element in the regular frames is maximal, we have $p = p \vee p^*$ or $p \vee p^* = \top$. The former case implies that $p^* \leq p$ which means that $p^* \wedge p^* = \bot$, that is, $p^* = \bot$. Therefore, the proof is complete.

(2). Since $\big(r_L(p)\big)^* = r_L(p^*)$, $p \in r_L(p)$, and $p^* \in r_L(p^*)$, we can conclude that $r_L(p) \vee \big(r_L(p)\big)^* = \top_{\beta L}$.

(3). We need to show $\mathbf{M}^I \subseteq \mathbf{O}^I$. Suppose $\varphi \in \mathbf{M}^I$. Then $r_L(\operatorname{coz}\varphi) \subseteq I$. Since $I \lll I$, we have $r_L(\operatorname{coz}\varphi) \lll I$. It follows that $\varphi \in \mathbf{O}^I$, that is, $\mathbf{M}^I \subseteq \mathbf{O}^I$. □

An ideal Q of $\mathcal{R}L$ is fixed if $\bigvee \operatorname{Coz}[Q] < \top$. Recall from [14, Proposition 3.3] that the fixed maximal ideals of $\mathcal{R}L$ are exactly the ideals M_p for $p \in Pt(L)$.

Theorem 4.5. *The non-essential pseudo-prime ideals of $\mathcal{R}L$ are precisely the ideals M^I, for I a complemented point of βL. Each is therefore principal, generated by an idempotent. Furthermore, each is minimal prime.*

Proof. Let Q be a non-essential pseudo-prime ideal in $\mathcal{R}L$. In view of the foregoing proposition, we can choose a prime ideal P such that $P \subseteq Q$. Take $I \in \beta L$ such that $\mathbf{O}^I \subseteq P \subseteq Q \subseteq \mathbf{M}^I$. Lemma 4.3 shows that

$$\bigvee \operatorname{Coz}[Q] = \bigvee \operatorname{Coz}[\mathbf{O}^I] = \bigvee \operatorname{Coz}[M^I] = \bigvee I.$$

Since Q is non-essential, we have $\bigvee I < \top$ which means that M^I is a fixed maximal ideal. Thus, $I = r_L(p)$ for some point p of L, and hence $p^* \neq \bot$ because $\bigvee \operatorname{Coz}[Q] = \bigvee I = \bigvee r_L(p) = p$. In consequence, Lemma 4.4 implies that $p \vee p^* = \top$, that is, p is complemented, it follows that I is a complemented point of βL. This implies that $\mathbf{O}^I = \mathbf{M}^I$, and so $\mathbf{O}^I = Q = \mathbf{M}^I$. Now, for each $p, q \in \mathbb{Q}$, define

$$\eta(p,q) = \begin{cases} \bot & \text{if } p < q \leq 0 \text{ or } 1 \leq p < q \\ p^* & \text{if } p < 0 < q \leq 1 \\ p & \text{if } 0 \leq p < 1 < q \\ \top & \text{if } p < 0 < 1 < q. \end{cases}$$

By [5, 8.4], $\eta \in \mathcal{R}L$ such that $\operatorname{coz}\eta = p$ and $\operatorname{coz}(\mathbf{1} - \eta) = p^*$. It is clear that $\eta^2 = \eta$ and M^I is equal to the principal generated by η, that is, $M^I = \langle \eta \rangle$. Finally, to show that Q is a minimal prime, let S be a prime ideal such that $S \subseteq Q$. Then $\eta(\mathbf{1}-\eta) = \mathbf{0}$ implies that $\eta \in S$ since $\mathbf{1} - \eta \notin S$. This shows that $Q = S$. □

The fact that the radical of a primary ideal is prime is well-known. Thus, by Proposition 4.1, every primary ideal in $\mathcal{R}L$ is pseudo-prime. Now, an immediate consequence of the previous theorem, is the following corollary.

Corollary 4.6. *The non-essential primary ideals of $\mathcal{R}L$ are precisely the ideals M^I, for I a complemented point of βL. Each is therefore principal, generated by an idempotent. Furthermore, each is minimal prime.*

5. Decomposable ideals in $\mathcal{R}L$

Recall from [23] that an ideal I in a ring R is called *decomposable* if $I = \bigcap_{i=1}^{n} Q_i$, where Q_i is P_i-primary for all $i = 1, \dots, n$. Before we discuss decomposability of the zero ideal in $\mathcal{R}L$, we need some background.

For any $x \in L$, we denote $\mathbf{M}^{r_L(x)}$ as $\mathbf{M}_x$, and $\mathbf{O}^{r_L(x)}$ as $\mathbf{O}_x$. Then we have:

$$\mathbf{M}_x = \{\varphi \in \mathcal{R}L : \operatorname{coz}\varphi \leq x\} \quad \text{ and } \quad \mathbf{O}_x = \{\varphi \in \mathcal{R}L : \operatorname{coz}\varphi \lll x\}.$$

Let us remind the reader that a frame L is called *connected* if $a \vee b = \top$ and $a \wedge b = \bot$ implies $a = \top$ or $b = \top$, for any $a, b \in L$. A ring R is called *reduced* if it has no nonzero nilpotent element. It is easy to see that for every frame L, $\mathcal{R}L$ is reduced.

Theorem 5.1. *If L is a connected frame, then $|L| > 2$ if and only if the zero ideal in $\mathcal{R}L$ is not decomposable.*

Proof. Necessity. Let L be a connected frame and $|L| > 2$. We suppose, by way of contradiction, that the ideal 0 has a minimal primary decomposition, and look for a contradiction. Let P be an associated prime of 0, that is, $P \in ass(0)$. Then, by [23, Theorem 4.17], there is $\varphi \in \mathcal{R}L$ such that $\sqrt{(0:\varphi)} = P$. We claim that $P = (0:\varphi) = \mathbf{M}_{(\operatorname{coz}\varphi)^*}$. Clearly, $(0:\varphi) \subseteq P$. For the reverse inclusion, let $\alpha \in P = \sqrt{(0:\varphi)}$. Then there is $n \in \mathbb{N}$ such that $\alpha^n \in (0:\varphi)$, that is, $\alpha^n\varphi = \mathbf{0}$, which implies that $\alpha\varphi = \mathbf{0}$ since $\mathcal{R}L$ is a reduced ring. It follows that $\alpha \in (0:\varphi)$, and so $P \subseteq (0:\varphi)$. Thus $P = (0:\varphi)$. On the other hand, by Lemma 3.1 in [12], $(0:\varphi) = \mathbf{M}_{(\operatorname{coz}\varphi)^*}$, and we have the claimed equality. Since $P = \mathbf{M}_{(\operatorname{coz}\varphi)^*}$ is a prime ideal, there is $I \in P(\beta L)$ such that

$\mathbf{O}^I \subseteq \mathbf{M}_{(\operatorname{coz}\alpha)^*} \subseteq \mathbf{M}^I$, implying that

$$\bigvee I = \bigvee \operatorname{Coz}[\mathbf{O}^I] = \bigvee \operatorname{Coz}[\mathbf{M}^I]$$
$$= \bigvee \operatorname{Coz}[\mathbf{M}_{(\operatorname{coz}\varphi)^*}] \qquad \dagger$$
$$= \bigvee r_L\big((\operatorname{coz}\varphi)^*\big) = (\operatorname{coz}\varphi)^*.$$

But $(\operatorname{coz}\varphi)^* \neq \top$, else $P = \mathbf{M}_\top = \mathcal{R}L$ which is a contradiction. Thus, $\bigvee \operatorname{Coz}[\mathbf{M}^I] < \top$, that is, $\mathbf{M}^I$ is a fixed maximal ideal of $\mathcal{R}L$. It follows that $I = r_L(p)$ for some prime element p of L, and so, by †, we have $p = \bigvee r_L(p) = \bigvee I = (\operatorname{coz}\varphi)^*$, which means that $(\operatorname{coz}\varphi)^* \in Pt(L)$. Therefore, in view of the first part of Lemma 4.4, either $(\operatorname{coz}\varphi)^{**} = \bot$ or $(\operatorname{coz}\varphi)^* \vee (\operatorname{coz}\varphi)^{**} = \top$. The former case implies that $(\operatorname{coz}\varphi)^* = \top$, which is a contradiction. In consequence, $(\operatorname{coz}\varphi)^* \vee (\operatorname{coz}\varphi)^{**} = \top$. Now, since L is a connected frame and $(\operatorname{coz}\varphi)^* \neq \top$, we must have $(\operatorname{coz}\varphi)^{**} = \top$, that is, $(\operatorname{coz}\varphi)^* = \bot$. This shows that $0 = \mathbf{M}_\top = \mathbf{M}_{(\operatorname{coz}\varphi)^*}$ is a maximal ideal of $\mathcal{R}L$; that is to say that $\mathcal{R}L$ is a field. Now, it is easy to show $L = \{\bot, \top\}$. With this contradiction, the proof is complete.

Sufficiency. It is obvious due to this fact $\mathcal{R}\mathbf{2} \cong \mathbb{R}$. □

A direct consequence of the above theorem is the following result.

Corollary 5.2. *If L is a connected frame, then the zero ideal in $\mathcal{R}L$ is decomposable if and only if $L = \mathbf{2}$.*

Before proving the last proposition, let us notice the following about reduced rings. If R is a reduced ring and e is an idempotent of R, then the principal ideal generated by e is a minimal ideal if and only if the principal ideal generated by $1 - e$ is a maximal ideal. We first discuss the two lemmas.

Lemma 5.3. *If $\varphi \in \mathcal{R}L$ such that $\operatorname{coz}\varphi$ is complemented and $(\operatorname{coz}\varphi)^* = \bigvee_{i=1}^n \operatorname{coz}\alpha_i$, $\alpha_1, \alpha_2 \cdots \alpha_n \in \mathcal{R}L$, then $\langle\varphi\rangle = \bigcap_{i=1}^n \mathbf{M}_{(\operatorname{coz}\alpha_i)^*}$.*

Proof. Let $\delta \in \langle\varphi\rangle$. Then $\operatorname{coz}\delta \le \operatorname{coz}\varphi$, and hence $\bigvee_{i=1}^n \operatorname{coz}\alpha_i = (\operatorname{coz}\varphi)^* \le (\operatorname{coz}\delta)^*$. It follows that for each $i = 1, 2, \dots n$, $\operatorname{coz}\alpha_i \le (\operatorname{coz}\delta)^*$, implying that $\operatorname{coz}\delta \le (\operatorname{coz}\delta)^{**} \le (\operatorname{coz}\alpha_i)^*$, that is, $\delta \in \mathbf{M}_{(\operatorname{coz}\alpha_i)^*}$. In consequence, $\langle\varphi\rangle \subseteq \bigcap_{i=1}^n \mathbf{M}_{(\operatorname{coz}\alpha_i)^*}$. To establish the reverse inclusion, let $\delta \in \bigcap_{i=1}^n \mathbf{M}_{(\operatorname{coz}\alpha_i)^*}$. This shows that for each i, $\operatorname{coz}\delta \le (\operatorname{coz}\alpha_i)^*$, which implies that $\operatorname{coz}\alpha_i \le (\operatorname{coz}\alpha_i)^{**} \le (\operatorname{coz}\delta)^*$. Thus, $(\operatorname{coz}\varphi)^* = \bigvee_{i=1}^n \operatorname{coz}\alpha_i \le (\operatorname{coz}\delta)^*$, showing that $\operatorname{coz}\delta \le (\operatorname{coz}\delta)^{**} \le \operatorname{coz}\varphi \prec\!\!\prec \operatorname{coz}\varphi$. Now, Lemma 3.3 in [11] shows that δ a multiple of φ, that is, $\alpha \in \langle\varphi\rangle$, and we have proved that $\bigcap_{i=1}^n \mathbf{M}_{\operatorname{coz}\beta_i} \subseteq \langle\varphi\rangle$. □

Lemma 5.4. *If $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\alpha$ is an atom, then there is an idempotent $\eta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha = \operatorname{coz}\eta$ and $\langle\alpha\rangle = \langle\eta\rangle = \mathbf{M}_{\operatorname{coz}\eta}$ is a minimal ideal.*

Proof. For each $p, q \in \mathbb{Q}$, define

$$\eta(p,q) = \begin{cases} \bot & \text{if } p < q \leq 0 \text{ or } 1 \leq p < q \\ (\operatorname{coz}\alpha)^* & \text{if } p < 0 < q \leq 1 \\ \operatorname{coz}\alpha & \text{if } 0 \leq p < 1 < q \\ \top & \text{if } p < 0 < 1 < q. \end{cases}$$

By [5, 8.4], $\eta \in \mathcal{R}L$ such that $\operatorname{coz}\eta = \operatorname{coz}\alpha$ and $\eta^2 = \eta$. In view of [11, Lemma 3.3], it is easy to see that $\langle\alpha\rangle = \langle\eta\rangle = \mathbf{M}_{\operatorname{coz}\eta}$. Now, Lemma 3.4 in [13] implies that $\langle\alpha\rangle = \langle\eta\rangle = \mathbf{M}_{\operatorname{coz}\eta}$ is minimal since $\operatorname{coz}\eta$ is an atom. □

We are now ready to prove the following result.

Proposition 5.5. *If $\varphi \in \mathcal{R}L$ such that $\operatorname{coz}\varphi$ is complemented and $(\operatorname{coz}\varphi)^*$ is a join of finitely many atoms, then the principal ideal $\langle\varphi\rangle$ is decomposable and there are idempotent elements $\eta_1, \eta_2 \ldots \eta_n$ such that $\langle\varphi\rangle = \bigcap_{i=1}^{n} \mathbf{M}_{\operatorname{coz}(\mathbf{1}-\eta_i)}$.*

Proof. Suppose $(\operatorname{coz}\varphi)^* = c_1 \vee c_2 \vee \ldots \vee c_n$ where each c_i is an atom. For each i, by the foregoing lemma, we can choose an idempotent $\eta_i \in \mathcal{R}L$ such that $\operatorname{coz}\eta_i = c_i$ and $\langle\eta_i\rangle = \mathbf{M}_{\operatorname{coz}\eta_i}$ is a minimal ideal of $\mathcal{R}L$. Clearly, $\mathbf{M}_{\operatorname{coz}(\mathbf{1}-\eta_i)} = \langle \mathbf{1} - \eta_i\rangle$. Thus, each $\mathbf{M}_{\operatorname{coz}(\mathbf{1}-\eta_i)}$ is a maximal ideal. Since for each i, $(\operatorname{coz}\eta_i)^* = \operatorname{coz}(\mathbf{1} - \eta_i)$, Lemma 3.5 shows that $\langle\varphi\rangle = \bigcap_{i=1}^{n} \mathbf{M}_{\operatorname{coz}(\mathbf{1}-\eta_i)}$. This means that the principal ideal $\langle\varphi\rangle$ is decomposable and the proof is complete. □

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ON PRIMARY IDEALS OF POINTFREE FUNCTION RINGS

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ایده‌آل‌های ابتدایی حلقه‌های توابع بدون نقطه

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ایده‌آل‌های ابتدایی حلقه‌ی $\mathcal{R}L$ از توابع پیوسته حقیقی-مقدار روی یک قاب کاملاً منظم L را مطالعه می‌کنیم. مشاهده می‌کنیم که ایده‌آل‌های اول و ابتدایی در یک P- قاب برهم منطبق هستند. نشان داده شده است که هر ایده‌آل ابتدایی در $\mathcal{R}L$ مشمول در یک ایده‌آل ماکسیمال منحصربفرد است و یک ایده‌آل Q در $\mathcal{R}L$ ابتدایی است اگر و تنها اگر $Q \cap \mathcal{R}^*L$ ایده‌آلی ابتدایی در $\mathcal{R}^*L$ است. نشان می‌دهیم که هر ایده‌آل شبه-اول (ابتدایی) در $\mathcal{R}L$ یک ایده‌آل اساسی است و یا ایده‌آلی ماکسیمال است که همزمان یک ایده‌آل اول مینیمال است. سرانجام، نشان می‌دهیم که اگر L یک قاب همبند باشد، آن‌گاه ایده‌آل صفر در $\mathcal{R}L$ تجزیه‌پذیر است اگر و تنها اگر $L = \mathbf{2}$.

کلمات کلیدی: قاب، ایده‌آل ابتدایی، ایده‌آل شبه-اول، حلقه توابع پیوسته حقیقی-مقدار روی یک قاب، ایده‌آل تجزیه‌پذیر.