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ON SELBERG-TYPE SQUARE MATRICES INTEGRALS

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Abstract. In this paper we consider Selberg-type square matrices integrals with focus on Kummer-beta types I & II integrals. For generality of the results for real normed division algebras, the generalized matrix variate Kummer-beta types I & II are defined under the abstract algebra. Then Selberg-type integrals are calculated under orthogonal transformations.

1. Introduction and Some Preliminaries

Selberg-type gamma and beta integrals involving scalar functions of positive definite symmetric matrices as integrand are considered by several authors, for examples see [1] and [11]. Mathai [14], (p. 231, 4.1.2) lists Selberg-type gamma integrals containing a positive signature symmetric matrix. Recently Gupta and Kabe [10] presented some results on the Selberg-type gamma and beta integrals where the integrand is a scalar function of squared matrix. They also covered skew symmetric matrices. In this note, we extend the existing results in the literature for real normed division algebras to cover real, complex, quaternion and octonion spaces simultaneously. We mainly focus on Selberg-type square matrices Kummer-gamma and Kummer-beta integrals.

The hypercomplex multivariate analysis distribution theory developed by Kabe [12] in order to consider real, complex, quaternion and octonion spaces simultaneously.

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Afterward many researchers extended the results in many directions. Among them, the works of Prof. Jose A. Diaz-Garcia and his colleague should be acknowledged. The reader is referred to [7], [8], and [6].

A vector-space is always a finite-dimensional module over the field of real numbers. An algebra $\mathcal{F}$ is a vector space that is equipped by a bilinear map $m : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ termed multiplication and a non-zero element $1 \in \mathcal{F}$ termed the unit such that $m(1, a) = m(a, 1) = a$. As usual abbreviate $m(a, b) = ab$ as ab. Do not assume $\mathcal{F}$ is associative.

An algebra $\mathcal{F}$ is a division algebra if given $a, b \in \mathcal{F}$, then $ab = 0$ implies $a = 0$ or $b = 0$. Equivalently, $\mathcal{F}$ is a division algebra if the operation of left and right multiplications by any non-zero element is invertible. A normed division algebra is an algebra $\mathcal{F}$ that is also a normed vector space with $\|ab\| = \|a\|\|b\|$. This implies that $\mathcal{F}$ is a division algebra and $\|1\| = 1$.

There are exactly four normed division algebras (according to [2]):

(1) Real Numbers ($\mathbb{R}$),
(2) Complex Numbers ($\mathbb{C}$),
(3) Quaternions ($\mathbb{Q}$),
(4) Octonions ($\mathbb{O}$).

Moreover, they are the only alternative division algebras, and all division algebras have a real dimension of $1,\ 2,\ 4$ or 8, which is denoted by β, see [2], theorems 1, 2 and 3. The parameter $\alpha = 2/\beta$ is used, in other mathematical fields, see [5].

Let $\mathcal{L}^{\beta}_{p,n}$ be the linear space of all $n \times p$ matrices of rank $n \leq p$ over $\mathcal{F}$ with m distinct positive singular values, where $\mathcal{F}$ denotes a real finite-dimensional normed division algebra. Let $\mathcal{F}^{n\times p}$ be the set of all $n \times p$ matrices over $\mathcal{F}$. The dimension of $\mathcal{F}^{n\times p}$ over $\mathbb{R}$ is $np\beta$. Let $\boldsymbol{A} \in \mathcal{F}^{n\times p}$, then $\boldsymbol{A}^* = \bar{\boldsymbol{A}}^T$ denotes the usual conjugate transpose.

The set of matrices $\boldsymbol{H}_1 \in \mathcal{F}^{n\times p}$ such that $\boldsymbol{H}_1^*\boldsymbol{H}_1 = \boldsymbol{I}_p$ is a manifold denoted $\mathcal{V}^{\beta}_{p,n}$, is termed the Stiefel manifold ($\boldsymbol{H}_1$ is also known as semi-orthogonal ($\beta = 1$), semi-unitary ($\beta = 2$), semi-symplectic ($\beta = 4$) and semi-exceptional type ($\beta = 8$) matrices, see [4]. The dimension of $\mathcal{V}^{\beta}_{p,n}$ over $\mathbb{R}$ is $\left(np\beta - \frac{1}{2}p(p+1)\beta - p\right)$, is the maximal compact subgroup $\mathcal{U}^{\beta}(p)$ of $\mathcal{L}^{\beta}_{p,p}$ and consists of all matrices $\boldsymbol{H} \in \mathcal{F}^{p\times p}$ such that $\boldsymbol{H}^*\boldsymbol{H} = \boldsymbol{I}_p$. Therefore, $\mathcal{U}^{\beta}(p)$ is the real orthogonal group $\mathcal{O}(p)$ ($\beta = 1$), the unitary group $\mathcal{U}(p)$

($\beta = 2$), compact symplectic group $\mathcal{S}_p(p)$ ($\beta = 4$), or exceptional type matrices $\mathcal{O}_o(p)$ ($\beta = 8$), for $\mathcal{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$ or $\mathbb{O}$, respectively.

Denote by $\mathcal{C}_p^\beta$ the real vector space of all $\boldsymbol{S} \in \mathcal{F}^{p\times p}$ such that $\boldsymbol{S} = \boldsymbol{S}^*$. Let $\mathcal{B}_p^\beta$ be the cone of positive definite matrices $\boldsymbol{S} \in \mathcal{F}^{p\times p}$; then $\mathcal{B}_p^\beta$ is an open subset of $\mathcal{C}_p^\beta$. Over $\mathbb{R}$, $\mathcal{C}_p^\beta$ consists of symmetric matrices; over $\mathbb{C}$, Hermitian matrices; over $\mathbb{Q}$, quaternionic Hermitian matrices (also termed self-dual matrices) and over $\mathbb{O}$, octonionic Hermitian matrices. Generically, the elements of $\mathcal{C}_p^\beta$ are termed Hermitian matrices, irrespective of the nature of $\mathcal{F}$. The dimension of $\mathcal{C}_p^\beta$ over $\mathbb{R}$ is $\left(\frac{1}{2}[p(p+1)\beta + p]\right)$. Let $\mathcal{D}_p^\beta$ be the diagonal subgroup of $\mathcal{L}_{p,p}^\beta$ consisting of all $\boldsymbol{D} \in \mathcal{F}^{p\times p}$, $\boldsymbol{D} = \operatorname{diag}(d_1, \dots, d_p)$.
The surface area or volume of the Stiefel manifold $\mathcal{V}_{p,n}^\beta$ is given by

$$\operatorname{Vol}(\mathcal{V}_{p,n}^\beta) = \int_{\boldsymbol{H}_1 \in \mathcal{V}_{p,n}^\beta} (\boldsymbol{H}_1^* d\boldsymbol{H}_1) = \frac{2^p \pi^{\frac{np\beta}{2}}}{\Gamma_p^\beta\left(\frac{n\beta}{2}\right)},$$

and therefore

$$(d\boldsymbol{H}_1) = \frac{1}{\operatorname{Vol}(\mathcal{V}_{p,n}^\beta)} (\boldsymbol{H}_1^* d\boldsymbol{H}_1) = \frac{\Gamma_p^\beta\left(\frac{n\beta}{2}\right)}{2^p \pi^{\frac{np\beta}{2}}} (\boldsymbol{H}_1^* d\boldsymbol{H}_1)$$

is the normalized invariant measure on $\mathcal{V}_{p,n}^\beta$ and $(d\boldsymbol{H})$, i.e. with $(n = p)$, it defines the normalized Haar measure on $\mathcal{U}^\beta(p)$ and $\Gamma_p^\beta(a)$ denotes the multivariate gamma function for the space $\mathcal{C}_p^\beta$, defined by

$$\begin{aligned}\Gamma_p^\beta(a) &= \int_{\boldsymbol{A} \in \mathcal{B}_p^\beta} |\boldsymbol{A}|^{a - \frac{(p-1)\beta}{2} - 1} \operatorname{etr}(-\boldsymbol{A}) d\boldsymbol{A} \\ &= \pi^{\frac{p(p-1)\beta}{4}} \prod_{i=1}^{p} \Gamma\left(a - \frac{(i-1)\beta}{2}\right),\end{aligned}$$

where $\operatorname{Re}(a) > \frac{(p-1)\beta}{2}$, see [9].

A generalized form of multivariate gamma function is a function of weight κ for the space $\mathcal{C}_p^\beta$ with $\kappa = (k_1, k_2, \dots, k_p)$, $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$, $\operatorname{Re}(a) \geq (p-1)\beta/2 - k_p$, which is defined by: (see [9] and [7])

$$\Gamma_p^\beta(a, \kappa) = \int_{\boldsymbol{A} \in \mathcal{B}_p^\beta} \operatorname{etr}(-\boldsymbol{A}) |\boldsymbol{A}|^{a - \frac{(p-1)\beta}{2} - 1} q_\kappa(\boldsymbol{A}) d\boldsymbol{A} = (a)_\kappa^\beta \Gamma_p^\beta(a),$$

where for $\boldsymbol{A} \in \mathcal{C}_p^\beta$

$$q_\kappa(\boldsymbol{A}) = |\boldsymbol{A}_p|^{k_p} \prod_{i=1}^{p-1} |\boldsymbol{A}_i|^{k_i - k_{i+1}}$$

with $\boldsymbol{A}_m = (a_{rs})$, $r, s = 1, 2, \ldots, m$, $m = 1, 2, \ldots, p$ is termed the highest weight vector. Also $(a)_\kappa^\beta$ denotes the generalized Pochhammer symbol of weight κ, defined by

$$\begin{aligned}(a)_\kappa^\beta = \prod_{i=1}^{p}\left(a - \frac{(i-1)\beta}{2}\right)_{k_i} &= \frac{\pi^{\frac{p(p-1)\beta}{4}}\prod_{i=1}^{p}\Gamma\left(a + k_i - \frac{(i-1)\beta}{2}\right)}{\Gamma_p^\beta(a)} \\ &= \frac{\Gamma_p^\beta(a,\kappa)}{\Gamma_p^\beta(a)},\end{aligned}$$

where $\text{Re}(a) > (p-1)\beta/2 - k_p$ and

$$(a)_i = a(a+1)\ldots(a+i-1),$$

is the standard Pochhammer symbol. Thus $\Gamma_p^\beta(a, (0, 0, \ldots, 0)) = \Gamma_p^\beta(a)$
The multivariate beta function for the space $\mathcal{C}_p^\beta$ is defined as (see [7])

$$\begin{aligned}B_p^\beta(a,b) &= \int_{\boldsymbol{0}<\boldsymbol{X}<\boldsymbol{I}_p} |\boldsymbol{X}|^{a-\frac{(p-1)\beta}{2}-1}|\boldsymbol{I}_p - \boldsymbol{X}|^{b-\frac{(m-1)\beta}{2}-1}d\boldsymbol{X} \\ &= \int_{\boldsymbol{Y}\in\mathcal{B}_p^\beta} |\boldsymbol{Y}|^{a-\frac{(p-1)\beta}{2}-1}|\boldsymbol{I}_p + \boldsymbol{Y}|^{-(a+b)}d\boldsymbol{Y} \\ &= \frac{\Gamma_p^\beta(a)\Gamma_p^\beta(b)}{\Gamma_p^\beta(a+b)},\end{aligned}$$

where $\boldsymbol{Y} = (\boldsymbol{I}_p - \boldsymbol{X})^{-1} - \boldsymbol{I}_p$, $\text{Re}(a), \text{Re}(b) > (p-1)\beta/2$. From [13],

$$(\text{tr}(\boldsymbol{X}))^k = \sum_\kappa C_\kappa^\beta(\boldsymbol{X}),$$

where $C_\kappa^\beta(\boldsymbol{X})$ denotes the zonal polynomials.

Fix complex numbers $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$, and for all $1 \leq i \leq q$ and $1 \leq j \leq p$ do not allow $-b_i + (j-1)\beta/2$ to be a non-negative integer. Then the hypergeometric function with one matrix argument ${}_rF_s^\beta$ is defined to be the real-analytic function on $\mathcal{C}_p^\beta$ given by the series

$${}_rF_s^\beta(a_1, \ldots, a_r; b_1, \ldots, b_s; \boldsymbol{X}) = \sum_{k=0}^{\infty}\sum_\kappa \frac{(a_1)_\kappa^\beta \ldots (a_r)_\kappa^\beta}{(b_1)_\kappa^\beta \ldots (b_s)_\kappa^\beta}\,\frac{C_\kappa^\beta(\boldsymbol{X})}{k!}.$$

For convergence properties see [9].
Then we have

$${}_0F_0^\beta(\boldsymbol{X}) = \sum_{k=0}^{\infty}\sum_\kappa \frac{C_\kappa^\beta(\boldsymbol{X})}{k!} = \text{etr}(\boldsymbol{X}).$$

For more details and results the interested reader is referred to [6], [3], and [9].

Analogous to [15], define the confluent hypergeometric function Ψ^β of $p \times p$ matrix $\boldsymbol{X} \in \mathcal{B}_p^\beta$ as

$$\begin{aligned}\Psi^\beta(a, c; \boldsymbol{X}) &= \frac{1}{\Gamma_p^\beta(a)} \int_{\boldsymbol{Y} \in \mathcal{B}_p^\beta} \operatorname{etr}(-\boldsymbol{X}\boldsymbol{Y}) \\ &\quad \times |\boldsymbol{Y}|^{a - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p + \boldsymbol{Y}|^{c - a - \frac{(p-1)\beta}{2} - 1} \mathrm{d}\boldsymbol{Y},\end{aligned}$$

where $\operatorname{Re}(a) > (p-1)\beta/2$.

2. Matrix Variate Distributions

In this study we consider some matrix variate distribution for our purpose. Consider the following definitions (see also [7] and [8]).

Definition 2.1. Let $\boldsymbol{X} \in \mathcal{L}_{p,n}^\beta$ be a random matrix, and $\boldsymbol{\Sigma} \in \mathcal{B}_p^\beta$ and $\boldsymbol{\Theta} \in \mathcal{B}_n^\beta$ be parameter matrices.

1. (**Matrix Variate Normal**) The random matrix $\boldsymbol{X}$ is said to have a matrix variate normal distribution denoted by $\boldsymbol{X} \sim N_{n\times p}^\beta(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Theta})$, with mean $\boldsymbol{\mu}$ and $\operatorname{Cov}(\operatorname{vec} \boldsymbol{X}^*) = \boldsymbol{\Theta} \otimes \boldsymbol{\Sigma}$, if its density function is given by

$$\frac{\beta^{\frac{\beta pn}{2}}}{(2\pi)^{\frac{\beta pn}{2}} |\boldsymbol{\Sigma}|^{\frac{\beta n}{2}} |\boldsymbol{\Theta}|^{\frac{\beta p}{2}}} \operatorname{etr}\left\{-\frac{\beta}{2} \boldsymbol{\Sigma}^{-1} (\boldsymbol{X} - \boldsymbol{\mu})^* \boldsymbol{\Theta}^{-1} (\boldsymbol{X} - \boldsymbol{\mu})\right\}.$$

2. (**Wishart**) Let $\boldsymbol{X} \sim N_{n\times p}^\beta(\boldsymbol{0}, \boldsymbol{\Sigma}, \boldsymbol{I}_n)$ and define $\boldsymbol{S} = \boldsymbol{X}^*\boldsymbol{X}$, then $\boldsymbol{S}$ is said to have a central Wishart distribution $\boldsymbol{S} \sim W_p^\beta(n, \boldsymbol{\Sigma})$ with n degrees of freedom and parameter matrix $\boldsymbol{\Sigma}$. Moreover, its density function is given by

$$\frac{\beta^{\frac{\beta pn}{2}}}{2^{\frac{\beta pn}{2}} \Gamma_p^\beta\left(\frac{\beta n}{2}\right) |\boldsymbol{\Sigma}|^{\frac{\beta n}{2}}} |\boldsymbol{S}|^{\frac{\beta(n-p+1)}{2} - 1} \operatorname{etr}\left\{-\frac{\beta}{2} \boldsymbol{\Sigma}^{-1} \boldsymbol{S}\right\},$$

where $n \geq p - 1$.

3. (**Matrix Variate T Type II**) The random matrix $\boldsymbol{X}$ is said to have a matrix variate T-distribution of type II denoted by $\boldsymbol{X} \sim T_{n,p}^\beta(\nu, \boldsymbol{I}_p)$, if its density function is given by

$$\frac{\Gamma_p^\beta\left(\frac{\beta(n+\nu)}{2}\right)}{\pi^{\frac{\beta pn}{2}} \Gamma_p^\beta\left(\frac{\beta \nu}{2}\right)} |\boldsymbol{I}_p + \boldsymbol{X}^*\boldsymbol{X}|^{-\frac{\beta(n+\nu)}{2}}, \quad \nu > p.$$

This distribution is also termed as matrix variate Pearson Type VII distribution.

4. (**Gegenbauer Type II**) The random matrix $\boldsymbol{X}$ is said to have a Gegenbauer Type II distribution $\boldsymbol{X} \sim G^{\beta}_{n,p}(\nu, \boldsymbol{I}_p)$, if its density function is given by

$$\frac{\Gamma_p^{\beta}\left(\frac{\beta(n+\nu)}{2}\right)}{\pi^{\frac{\beta pn}{2}}\Gamma_p^{\beta}\left(\frac{\beta\nu}{2}\right)}\,|\boldsymbol{I}_p - \boldsymbol{X}^*\boldsymbol{X}|^{\frac{\beta(\nu-p+1)}{2}-1}, \quad \nu > p-1.$$

   This distribution is known in statistical bibliography as the matrix variate inverted T or matrix variate Pearson Type II distribution.

5. (**T-Laguerre Type II Ensemble**) The random matrix $\boldsymbol{X}$ is said to have a T-Laguerre Type II ensemble distribution $\boldsymbol{X} \sim TL^{\beta}_{n,p}(\nu)$, if its density function is given by

$$\frac{1}{B_p^{\beta}\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}\,|\boldsymbol{X}^*\boldsymbol{X}|^{\frac{\beta(n-p+1)}{2}-1}|\boldsymbol{I}_p + \boldsymbol{X}^*\boldsymbol{X}|^{-\frac{\beta(n+\nu)}{2}},$$

   where $\nu \geq p-1, \quad n \geq p-1$.

   This distribution is also known as the Studentized Wishart distribution

6. (**Gegenbauer-Laguerre Type II Ensemble**) The random matrix $\boldsymbol{X}$ is said to have a Gegenbauer-Laguerre type II ensemble distribution $\boldsymbol{X} \sim GL^{\beta}_{n,p}(\nu)$, if its density function is given by

$$\frac{1}{B_p^{\beta}\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}\,|\boldsymbol{X}^*\boldsymbol{X}|^{\frac{\beta(n-p+1)}{2}-1}|\boldsymbol{I}_p - \boldsymbol{X}^*\boldsymbol{X}|^{\frac{\beta(\nu-p+1)}{2}-1},$$

   where $\nu \geq p-1, \quad n \geq p-1$.

The following result directly obtains from Definition 2.1 and Proposition 4 of [6].

Theorem 2.2. (1) *Let* $\boldsymbol{X} \sim T^{\beta}_{n,p}(\nu, \boldsymbol{I}_p)$. *Define* $\boldsymbol{S}_1 = \boldsymbol{X}^*\boldsymbol{X} \in \mathcal{B}^{\beta}_p$. *Then* $\boldsymbol{S}_1$ *has the following density*

$$\frac{1}{B_p^{\beta}\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}\,|\boldsymbol{S}_1|^{\frac{\beta(n-p+1)}{2}-1}|\boldsymbol{I}_p + \boldsymbol{S}_1|^{-\frac{\beta(n+\nu)}{2}},$$

where $n \geq p$.

(2) *Let* $\boldsymbol{Y} \sim G^{\beta}_{n,p}(\nu, \boldsymbol{I}_p)$. *Define* $\boldsymbol{S}_2 = \boldsymbol{Y}^*\boldsymbol{Y} \in \mathcal{B}^{\beta}_p$. *Then* $\boldsymbol{S}_2$ *has the following density*

$$\frac{1}{B_p^{\beta}\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}\,|\boldsymbol{S}_2|^{\frac{\beta(n-p+1)}{2}-1}|\boldsymbol{I}_p - \boldsymbol{S}_2|^{\frac{\beta(\nu-p+1)}{2}-1},$$

where $n \geq p$.

The following definitions are analogous to the results of [16] and [17] respectively for real normed division algebras.

Definition 2.3. (**Kummer-Beta Type I**) The random matrix $\boldsymbol{X}$ is said to have a Kummer-beta type I distribution $\boldsymbol{X} \sim KB1_p^\beta(\alpha_1, \alpha_2, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} \in \mathcal{B}_p^\beta$ if its density function is given by

$$K_1(\alpha_1, \alpha_2, \beta, \boldsymbol{\Sigma}) \operatorname{etr}(-\boldsymbol{\Sigma}\boldsymbol{X}) |\boldsymbol{X}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p - \boldsymbol{X}|^{\alpha_2 - \frac{(p-1)\beta}{2} - 1},$$

where $\boldsymbol{0} < \boldsymbol{X} < \boldsymbol{I}_p$, $\alpha_1 \geq (p-1)\beta/2$ and $\alpha_2 \geq (p-1)\beta/2$. The normalizing constant is given by

$$\begin{aligned} &\{K_1(\alpha_1, \alpha_2, \beta, \boldsymbol{\Sigma})\}^{-1} \\ = \int_{\boldsymbol{0}<\boldsymbol{Y}<\boldsymbol{I}_p} \operatorname{etr}(-\boldsymbol{\Sigma}\boldsymbol{Y}) |\boldsymbol{Y}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} &|\boldsymbol{I}_p - \boldsymbol{Y}|^{\alpha_2 - \frac{(p-1)\beta}{2} - 1} \mathrm{d}\boldsymbol{Y} \\ &= B_p^\beta(\alpha_1, \alpha_2)\ {}_1F_1^\beta(\alpha_1, \alpha_1 + \alpha_2; -\boldsymbol{\Sigma}). \end{aligned}$$

Definition 2.4. (**Kummer-Beta Type II**) The random matrix $\boldsymbol{X}$ is said to have a Kummer-beta type II distribution $\boldsymbol{X} \sim KB2_p^\beta(\alpha_1, \alpha_2, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} \in \mathcal{B}_p^\beta$ if its density function is given by

$$K_2(\alpha_1, \alpha_2, \beta, \boldsymbol{\Sigma}) \operatorname{etr}(-\boldsymbol{\Sigma}\boldsymbol{X}) |\boldsymbol{X}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p + \boldsymbol{X}|^{-\alpha_2}, \quad \boldsymbol{X} > \boldsymbol{0},$$

where using the confluent hypergeometric function, the normalizing constant is given by

$$\begin{aligned} \{K_2(\alpha_1, \alpha_2, \beta, \boldsymbol{\Sigma})\}^{-1} &= \int_{\boldsymbol{Y} \in \mathcal{B}_p^\beta} \operatorname{etr}(-\boldsymbol{\Sigma}\boldsymbol{Y}) \\ &\quad \times |\boldsymbol{Y}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p + \boldsymbol{Y}|^{-\alpha_2} \mathrm{d}\boldsymbol{Y} \\ &= \Gamma_p^\beta(\alpha_1) \Psi^\beta \left(\alpha_1, \alpha_1 - \alpha_2 \frac{(p-1)\beta}{2} + 1; \boldsymbol{\Sigma} \right). \end{aligned}$$

The following result is an extension to Theorem 3.1 of [16] for real normed division algebras.

Lemma 2.5. *Let* $\boldsymbol{U} \sim KB1_p^\beta(\alpha_1, \alpha_2, \boldsymbol{\Sigma})$. *Then for the given matrices* $\boldsymbol{\Psi} \in \mathcal{C}_p^\beta$, $\boldsymbol{\Omega} \in \mathcal{B}_p^\beta$ *and* $\boldsymbol{\Omega} - \boldsymbol{\Psi} \in \mathcal{B}_p^\beta$ $(\boldsymbol{\Omega} > \boldsymbol{\Psi})$, *the random matrix* $\boldsymbol{X}$ *defined by*

$$\boldsymbol{X} = (\boldsymbol{\Omega} - \boldsymbol{\Psi})^{\frac{1}{2}} \boldsymbol{U} (\boldsymbol{\Omega} - \boldsymbol{\Psi})^{\frac{1}{2}} + \boldsymbol{\Psi}$$

has generalized matrix variate Kummer-beta type I distribution with the following density function

$$C_1(\alpha_1,\alpha_2,\beta,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi})\,\mathrm{etr}(-\boldsymbol{\Theta}\boldsymbol{X})|\boldsymbol{X}-\boldsymbol{\Psi}|^{\alpha_1-\frac{(p-1)\beta}{2}-1}|\boldsymbol{\Omega}-\boldsymbol{X}|^{\alpha_2-\frac{(p-1)\beta}{2}-1},$$

where $\boldsymbol{\Psi}<\boldsymbol{X}<\boldsymbol{\Omega}$, $\boldsymbol{\Theta}=(\boldsymbol{\Omega}-\boldsymbol{\Psi})^{-\frac{1}{2}}\boldsymbol{\Sigma}(\boldsymbol{\Omega}-\boldsymbol{\Psi})^{-\frac{1}{2}}$ *and*

$$\begin{aligned}C_1(\alpha_1,\alpha_2,\beta,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi}) &= K_1(\alpha_1,\alpha_2,\beta,(\boldsymbol{\Omega}-\boldsymbol{\Psi})^{\frac{1}{2}}\boldsymbol{\Theta}(\boldsymbol{\Omega}-\boldsymbol{\Psi})^{\frac{1}{2}})\\ &\quad\times\mathrm{etr}(\boldsymbol{\Theta}\boldsymbol{\Psi})|\boldsymbol{\Omega}-\boldsymbol{\Psi}|^{-(\alpha_1+\alpha_2)+\frac{(p-1)\beta}{2}+1}.\end{aligned}$$

In this case we use the notation $\boldsymbol{X}\sim GKB1_p^{\beta}(\alpha_1,\alpha_2,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi})$.

Proof. The proof directly follows by Definition 2.3 and the fact that the Jacobian of transformation is $J(\boldsymbol{U}\to\boldsymbol{X})=|\boldsymbol{\Omega}-\boldsymbol{\Psi}|^{-\frac{\beta(p-1)}{2}-1}$. □

The following result is an extension to Theorem 2.2 of [17] for real normed division algebras.

Lemma 2.6. *Let* $\boldsymbol{U}\sim KB2_p^{\beta}(\alpha_1,\alpha_2,\boldsymbol{\Sigma})$. *Then for the given matrices* $\boldsymbol{\Psi}\in\mathcal{C}_p^{\beta}$, $\boldsymbol{\Omega}\in\mathcal{B}_p^{\beta}$ *and* $\boldsymbol{\Omega}+\boldsymbol{\Psi}\in\mathcal{B}_p^{\beta}$, *the random matrix* $\boldsymbol{X}$ *defined by*

$$\boldsymbol{X}=(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}}\boldsymbol{U}(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}}+\boldsymbol{\Psi}$$

has generalized matrix variate Kummer-beta type II with the following density function

$$C_2(\alpha_1,\alpha_2,\beta,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi})\,\mathrm{etr}(-\boldsymbol{\Theta}\boldsymbol{X})|\boldsymbol{X}-\boldsymbol{\Psi}|^{\alpha_1-\frac{(p-1)\beta}{2}-1}|\boldsymbol{\Omega}+\boldsymbol{X}|^{-\alpha_2},$$

where $\boldsymbol{\Psi}<\boldsymbol{X}$, $\boldsymbol{\Theta}=(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{-\frac{1}{2}}\boldsymbol{\Sigma}(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{-\frac{1}{2}}$ *and*

$$\begin{aligned}C_2(\alpha_1,\alpha_2,\beta,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi}) &= K_2(\alpha_1,\alpha_2,\beta,(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}}\boldsymbol{\Theta}(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}})\\ &\quad\times\mathrm{etr}(\boldsymbol{\Theta}\boldsymbol{\Psi})|\boldsymbol{\Omega}+\boldsymbol{\Psi}|^{-\alpha_1+\alpha_2}.\end{aligned}$$

In this case we use the notation $\boldsymbol{X}\sim GKB2_p^{\beta}(\alpha_1,\alpha_2,\boldsymbol{\Theta},\boldsymbol{\Omega},\boldsymbol{\Psi})$.

Proof. The proof directly follows by Definition 2.4 and the fact that the Jacobian of transformation is $J(\boldsymbol{U}\to\boldsymbol{X})=|\boldsymbol{\Omega}+\boldsymbol{\Psi}|^{-\frac{\beta(p-1)}{2}-1}$. □

3. Selberg Type Square Matrices Integrals

In this section we are interested in evaluating the integrals of the form

$$\int h(\mathbf{\Lambda}) \prod_{i<j}^{p} (\lambda_i - \lambda_j)^{\beta} \mathrm{d}\mathbf{\Lambda}, \tag{3.1}$$

where $\mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \ldots, \lambda_p)$, $\lambda_1 > \ldots > \lambda_p > 0$, for some function h. In this regard, we need the following essential result due to [7]. Note that similar results can also be found in [8] for general form of the distributions defined in Definition 2.1. However the purpose of this study is to evaluate integrals of the form (3.1).

Theorem 3.1. *Let $\boldsymbol{X} \in \mathcal{B}_p^{\beta}$ be a random matrix with density function $f(\boldsymbol{X})$. Then the joint density function of the eigenvalues $\lambda_1, \ldots, \lambda_p$ of $\boldsymbol{X}$ is*

$$g(\lambda_1, \ldots, \lambda_p) \quad = \quad \frac{\pi^{\frac{1}{2}p^2\beta+\varrho}}{\Gamma_p^{\beta}\left(\frac{p\beta}{2}\right)} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^{\beta} \int_{\boldsymbol{H} \in \mathcal{U}^{\beta}(p)} f(\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*)(\mathrm{d}\boldsymbol{H}),$$

where $(\mathrm{d}\boldsymbol{H})$ is the normalized Haar measure and

$$\varrho = \begin{cases} 0, & \beta = 1; \\ -p, & \beta = 2; \\ -2p, & \beta = 4; \\ -4p, & \beta = 8. \end{cases}$$

Lemma 3.2. *Let $\boldsymbol{X} \in \mathcal{B}_p^{\beta}$ be a random matrix with density function $f(\boldsymbol{X})$. Then we have*

$$\int_{\boldsymbol{H} \in \mathcal{U}^{\beta}(p)} \left(\int_{\mathbf{\Lambda}} f(\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*) \prod_{i<j}^{p} (\lambda_i - \lambda_j)^{\beta} \mathrm{d}\mathbf{\Lambda} \right) (\mathrm{d}\boldsymbol{H}) = \frac{\Gamma_p^{\beta}\left(\frac{p\beta}{2}\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}}.$$

Proof. Since $\int_{\mathbf{\Lambda}} g(\lambda_1, \ldots, \lambda_p) \mathrm{d}\mathbf{\Lambda} = 1$, by making use of Theorem 3.1 and changing the order of integration, the result follows. □

In sequel we proceed by giving some examples of gamma and beta integrals for real normed division algebras.

3.1. **Examples.** In this part, we give some examples of Selberg-type integrals.

Example 3.3. Based on Definition 2.1, and the fact that $\boldsymbol{H}^*\boldsymbol{H} = \boldsymbol{I}_p$, we have the following results using Lemma 3.1

(1:1) Suppose that $\boldsymbol{X} \sim W_p^\beta(n, \boldsymbol{I}_p)$, then using the density of $\boldsymbol{S}$ given in Definition 2.1(2), we have (extension to [14])

$$\int_{\boldsymbol{\Lambda}} |\boldsymbol{\Lambda}|^{\frac{\beta(n-p+1)}{2}-1} \operatorname{etr}\left\{-\frac{\beta}{2}\boldsymbol{\Lambda}\right\} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\boldsymbol{\Lambda} = \frac{2^{\frac{\beta p n}{2}} \left[\Gamma_p^\beta\left(\frac{p\beta}{2}\right)\right]^2}{\beta^{\frac{\beta p n}{2}} \pi^{\frac{1}{2}p^2\beta+\varrho}}$$

(1:2) Suppose that $\boldsymbol{X} \sim T_{n,p}^\beta(\nu, \boldsymbol{I}_p)$, then from the density of $\boldsymbol{S}_1 = \boldsymbol{X}^*\boldsymbol{X}$ given in Definition 2.1(3), we have (extension to [11])

$$\int_{\boldsymbol{\Lambda}} |\boldsymbol{\Lambda}|^{\frac{\beta(n-p+1)}{2}-1} |\boldsymbol{I}_p + \boldsymbol{\Lambda}|^{-\frac{\beta(n+\nu)}{2}} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\boldsymbol{\Lambda} = \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}}.$$

In the same fashion

(1:3) Suppose that $\boldsymbol{X} \sim G_{n,p}^\beta(\nu, \boldsymbol{I}_p)$, then from the density of $\boldsymbol{S}_2 = \boldsymbol{X}^*\boldsymbol{X}$ given in Definition 2.1(4), we have

$$\int_{\boldsymbol{\Lambda}} |\boldsymbol{\Lambda}|^{\frac{\beta(n-p+1)}{2}-1} |\boldsymbol{I}_p - \boldsymbol{\Lambda}|^{\frac{\beta(\nu-p+1)}{2}-1} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\boldsymbol{\Lambda} = \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta\left(\frac{\beta\nu}{2}, \frac{\beta n}{2}\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}}.$$

Example 3.4. (2:1) Suppose that $\boldsymbol{X} \sim KB1_p^\beta(\alpha_1, \alpha_2, \boldsymbol{\Sigma})$, then by Definition 2.3 we get

$$\int_{\boldsymbol{H}} \int_{\boldsymbol{\Lambda}} \operatorname{etr}\left(\boldsymbol{\Sigma}\boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}^*\right) |\boldsymbol{I}_p - \boldsymbol{\Lambda}|^{\alpha_1 - \frac{(p-1)\beta}{2}-1} |\boldsymbol{\Lambda}|^{\alpha_2 - \frac{(p-1)\beta}{2}-1} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\boldsymbol{\Lambda}(\mathrm{d}\boldsymbol{H})$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta\left(\alpha_1, \alpha_2\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}} \, {}_1F_1^\beta(\alpha_1, \alpha_1 + \alpha_2; -\boldsymbol{\Sigma}). \tag{3.2}$$

It is easily seen that taking $\boldsymbol{\Sigma} = \boldsymbol{0}$, $\alpha_1 = \frac{\nu\beta}{2}$ and $\alpha_2 = \frac{n\beta}{2}$ in (3.2) gives item (1:3) in the above. Further, for the case $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, we have that

$$\int_{\boldsymbol{\Lambda}} \operatorname{etr}\left(\boldsymbol{\Lambda}\right) |\boldsymbol{I}_p - \boldsymbol{\Lambda}|^{\alpha_1 - \frac{(p-1)\beta}{2}-1} |\boldsymbol{\Lambda}|^{\alpha_2 - \frac{(p-1)\beta}{2}-1} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\boldsymbol{\Lambda}$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta\left(\alpha_1, \alpha_2\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}} \, {}_1F_1^\beta(\alpha_1, \alpha_1 + \alpha_2; -\boldsymbol{I}_p).$$

Note that the integral in (3.2) is not of required form, thus making use of (see [7])

$$\int_{\boldsymbol{H}} \operatorname{etr}\left(\boldsymbol{\Sigma}\boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}^*\right)(\mathrm{d}\boldsymbol{H}) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\boldsymbol{H}} C_\kappa(\boldsymbol{\Sigma}\boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}^*)(\mathrm{d}\boldsymbol{H})$$

$$= \sum_{k=0}^{\infty} \frac{C_\kappa(\mathbf{\Sigma})C_\kappa(\mathbf{\Lambda})}{k!C_\kappa(\boldsymbol{I}_p)} \tag{3.3}$$

we obtain

$$\sum_{k=0}^{\infty} \frac{C_\kappa(\mathbf{\Sigma})}{k!C_\kappa(\boldsymbol{I}_p)} \int_{\mathbf{\Lambda}} |\boldsymbol{I}_p - \mathbf{\Lambda}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\mathbf{\Lambda}|^{\alpha_2 - \frac{(p-1)\beta}{2} - 1}$$

$$\times C_\kappa(\mathbf{\Lambda}) \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\mathbf{\Lambda}$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta(\alpha_1, \alpha_2)}{\pi^{\frac{1}{2}p^2\beta + \varrho}} {}_1F_1^\beta(\alpha_1, \alpha_1 + \alpha_2; -\mathbf{\Sigma}). \tag{3.4}$$

(2:2) Suppose that $\boldsymbol{X} \sim KB2_p^\beta(\alpha_1, \alpha_2, \mathbf{\Sigma})$, then by Definition 2.4 we get

$$\int_{\boldsymbol{H}} \int_{\mathbf{\Lambda}} \mathrm{etr}\left(\mathbf{\Sigma}\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*\right) |\mathbf{\Lambda}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p + \mathbf{\Lambda}|^{-\alpha_2} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\mathbf{\Lambda}(\mathrm{d}\boldsymbol{H})$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right)\Gamma_p^\beta(\alpha_1)}{\pi^{\frac{1}{2}p^2\beta + \varrho}} \Psi^\beta\left(\alpha_1, \alpha_1 - \alpha_2 \frac{(p-1)\beta}{2} + 1; \mathbf{\Sigma}\right).$$

It is also interesting to see that for the case $\mathbf{\Sigma} = \boldsymbol{I}_p$ we get

$$\int_{\mathbf{\Lambda}} \mathrm{etr}\left(\mathbf{\Lambda}\right) |\mathbf{\Lambda}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1} |\boldsymbol{I}_p + \mathbf{\Lambda}|^{-\alpha_2} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\mathbf{\Lambda}$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right)\Gamma_p^\beta(\alpha_1)}{\pi^{\frac{1}{2}p^2\beta + \varrho}} \Psi^\beta\left(\alpha_1, \alpha_1 - \alpha_2 \frac{(p-1)\beta}{2} + 1; \boldsymbol{I}_p\right).$$

(2:3) Suppose that $\boldsymbol{X} \sim GKB1_p^\beta(\alpha_1, \alpha_2, \mathbf{\Theta}, \mathbf{\Omega}, \mathbf{\Psi})$, then by Lemma 2.5 we have

$$\int_{\boldsymbol{H}} \int_{\mathbf{\Lambda}} \mathrm{etr}\left(-\mathbf{\Theta}\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*\right) |\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^* - \mathbf{\Psi}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1}$$

$$|\mathbf{\Omega} - \boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*|^{\alpha_2 - \frac{(p-1)\beta}{2} - 1} \prod_{i<j}^{p} (\lambda_i - \lambda_j)^\beta \mathrm{d}\mathbf{\Lambda}(\mathrm{d}\boldsymbol{H})$$

$$= \frac{\Gamma_p^\beta\left(\frac{p\beta}{2}\right) B_p^\beta(\alpha_1, \alpha_2)}{\pi^{\frac{1}{2}p^2\beta + \varrho}} {}_1F_1^\beta(\alpha_1, \alpha_1 + \alpha_2; -\mathbf{\Sigma})$$

$$\times \mathrm{etr}(-\mathbf{\Theta}\mathbf{\Psi}) |\mathbf{\Omega} - \mathbf{\Psi}|^{\alpha_1 + \alpha_2 - \frac{(p+1)\beta}{2} - 1},$$

where $\mathbf{\Sigma} = (\mathbf{\Omega} - \mathbf{\Psi})^{\frac{1}{2}} \mathbf{\Theta} (\mathbf{\Omega} - \mathbf{\Psi})^{\frac{1}{2}}$.

(2:4) Suppose that $\boldsymbol{X} \sim GKB2_p^\beta(\alpha_1, \alpha_2, \mathbf{\Theta}, \mathbf{\Omega}, \mathbf{\Psi})$, then by Lemma 2.6 we have

$$\int_{\boldsymbol{H}} \int_{\mathbf{\Lambda}} \mathrm{etr}\left(-\mathbf{\Theta}\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^*\right) |\boldsymbol{H}\mathbf{\Lambda}\boldsymbol{H}^* - \mathbf{\Psi}|^{\alpha_1 - \frac{(p-1)\beta}{2} - 1}$$

$$|\boldsymbol{\Omega}+\boldsymbol{H\Lambda H}^*|^{-\alpha_2}\prod_{i<j}^{p}(\lambda_i-\lambda_j)^{\beta}\mathrm{d}\boldsymbol{\Lambda}(\mathrm{d}\boldsymbol{H})$$

$$=\frac{\Gamma_p^{\beta}\left(\frac{p\beta}{2}\right)\Gamma_p^{\beta}\left(\alpha_1\right)}{\pi^{\frac{1}{2}p^2\beta+\varrho}}\operatorname{etr}(-\boldsymbol{\Theta\Psi})|\boldsymbol{\Omega}+\boldsymbol{\Psi}|^{\alpha_1-\alpha_2}$$

$$\times\Psi^{\beta}\left(\alpha_1,\alpha_1-\alpha_2\frac{(p-1)\beta}{2}+1;(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}}\boldsymbol{\Theta}(\boldsymbol{\Omega}+\boldsymbol{\Psi})^{\frac{1}{2}}\right).$$

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