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NETS AND SEPARATED S-POSETS

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Abstract. Nets, useful topological tools, used to generalize certain concepts that may only be general enough in the context of metric spaces. In this work we introduce this concept in an S-poset, a poset with an action of a posemigroup S on it which is a very useful structure in computer sciences and interesting for mathematicians, and give the concept of S-net. Using S-nets and its convergency we also give some characterizations of separated S-posets. Also, introducing the net-closure operators, we investigate the counterparts of topological separation axioms on S-posets and study their relation to separated S-posets.

1. Introduction

Nets, the useful topological tools, used to generalize certain concepts that may only be general enough in the context of metric spaces. Nets were first introduced by E. H. Moore and H. L. Smith in 1922, [9] to generalize the notion of a sequence in topological spaces. They considered arbitrary directed sets to define nets, rather than countable linearly ordered sets used to define sequences. This strong notion is helpful in topology, in particular to characterize Housdorff topological spaces.

General ordered algebraic structures play a role in a wide range of areas, including analysis, logic, theoretical computer science, and physics. One important of these

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structures is the category of S-posets, the representations of a posemigroup S by order-preserving maps of partially ordered sets, which is of interest to some mathematicians, see [2, 3, 4, 5].

A *separated S-poset* A is an S-poset in which any two distinct points a and b in A can be separated by at least one $s \in S$, by $sa \neq sb$. The class of separated S-posets, can be considered as a good counterpart of Housdorff topological spaces in the context of S-posets, since they imply the uniqueness of limits of S-nets.

In this paper we will have a closer look at the interesting class of separated S-posets, and give some characterizations of them by the help of S-nets. To do so, first we give some definitions and preliminaries and we then introduce the concept of S-nets in an S-poset where S is a posemigroup. In the pomonid case the results are in some sense trivial. Thus we will assume from now on that S is a posemigroup. Also we give a definition of the convergency of S-nets. Then we characterize the separated S-posets using the uniqueness of limits of convergent S-nets.

We also introduce a closure operator using convergent S-nets, and give another characterization of separated S-posets.

We then investigate the counterparts of topological separation axioms on S-posets and we see that many of them are surprisingly equivalent to each other and imply separateness of S-posets.

2. Preliminaries

In this section we briefly recall some preliminary notions which will be used in the sequel.

Definition 2.1. (i) For a semigroup S, a left S-*act* is a non-empty set A together with a mapping $S \times A \to A$ sending (s, a) to sa such that (1) $s(ta) = (st)a$, and (2) if S has an identity element 1, $1a = a$ for all $s, t \in S$ and $a \in A$.

(ii) Let A be a left S-act and $B \subseteq A$ be a non-empty subset of A. Then B is called a *subact* of A if $sb \in B$, for all $s \in S$ and $b \in B$.

(iii) For each two left S-acts A and B a mapping $f : A \to B$ is called an *act-morphism* (or briefly a *morphism*) if $f(sa) = sf(a)$, for all $s \in S$ and $a \in A$.

(iv) Given a left S-act A, an equivalence relation θ on A is called an S-*act congruence* (or briefly a *congruence* on A), if $a\theta a'$ implies $(sa)\theta(sa')$ for every $a, a' \in A$ and $s \in S$.

A recent and complete discussion of this area is contained in the monograph Monoids, Acts and Categories by M. Kilp, U. Knauer, and A.V. Mikhalev, see [7].

As the representation theory of a semigroup S by mappings of sets we study the category of S-acts. Now we consider the representations of a posemigroup S by order-preserving maps of partially ordered sets which give us S-posets. See the following definition.

Definition 2.2. (i) A partially ordered set (briefly, a poset) $(S, \leq)$ is said to be a *posemigroup* if it is a semigroup whose operation is order-preserving. That is for every $s, s_1, s_2 \in S$,

$$s_1 \leq s_2 \Rightarrow ss_1 \leq ss_2 \quad \text{and} \quad s_1 s \leq s_2 s.$$

(ii) Let $(S, \leq)$ be a posemigroup. Then a poset $(A, \leq)$ is called a (*left*) S*-poset* if A is a left S-act such that the action of S, $(s, a) \mapsto sa$, is monotone in both variables, that is:

$$a \leq b, s \leq t \Rightarrow sa \leq tb,$$

for all $a, b \in A$ and $s, t \in S$, see [5]. Since we consider left S-acts and left S-posets in this paper, the word ‘left’ in the following will be dropped.

(iii) A poset $(B, \leq_B)$ is said to be an S-subposet of an S-poset $(A, \leq_A)$ if B is a subact of A and $\leq_B = \leq_A \cap B^2$.

(iv) A morphism $f : A \to B$ from an S-poset A to an S-poset B is called an S*-poset morphism*, if it is order-preserving and S-act morphism. More explicitly, $f(sa) = sf(a)$ and $f(a) \leq f(a')$ in B, if $a \leq a'$ in A.

(v) For any binary relation θ on an S-poset A, one can define the relation $\leq_\theta$ on A as follows:

$$a \leq_\theta b \Leftrightarrow \exists a_1, \cdots, a_n, b_1, \cdots, b_n;\ a \leq a_1 \theta b_1 \leq \cdots \leq a_n \theta b_n \leq b.$$

Then, an S-act congruence θ on an S-poset A is an S*-poset congruence* if and only if $a\theta b$ whenever $a \leq_\theta b \leq_\theta a$.

The family of S-posets and the morphisms between them form a category which is denoted by $\mathsf{S\text{-}Pos}$.

Note 2.3. We should note that the category of S-posets is not obtained by taking acts in the category of posets. Because in this case we will have the family of unary operations

$\{\lambda_s : A \to A\}_{s\in S}$ in which each $\lambda_s : A \to A$ maps every $a \in A$ to sa, for every poset A, as the order-preserving maps with the property that $\lambda_s \circ \lambda_t = \lambda_{st}$, but there is no order-relation between λ_s's. That is, although by taking acts in the category of posets we will get the property that if $a \leq a' \in A$ then $sa \leq sa'$ for every $s \in S$ but not the property that if $s \leq t \in S$ then $sa \leq ta$, for every $a \in A$. But if one takes the action as $\lambda : S \times A \to A$ and defining the order on $S \times A$ component-wise, then for λ to be order preserving means:

$$[(s, a) \leq (t, b) \Leftrightarrow s \leq t \text{and } a \leq b] \Rightarrow [\lambda(s, a) \leq \lambda(t, b) \Leftrightarrow sa \leq tb]$$

3. Nets in the context of S-poset

In this section we are going to introduce the notion of S-nets in an S-poset, and study the concept of convergent S-nets

Definition 3.1. Let S be a posemigroup.

(i) A *monotone S-net* (or an *S-net*) in an S-poset A is a monotone S-act map α from the posemigroup S, as an S-poset, to the S-poset A. The posemigroup S is called the index set of the net. We sometimes write an S-net $\alpha : S \to A$ in the form $(a_s)_{s\in S}$, which expresses the fact that the element $s \in S$ is mapped to the element a_s in A by α. Hence, with this notation we have $a_s \leq a_t$ if $s \leq t$ and $a_{st} = sa_t$, for every $s, t \in S$.

(ii) Let S be a posemigroup and A be an S-poset. Then for every $a \in A$, consider $\rho_a : S \to A$ which maps every $s \in S$ to sa. Since $\rho_a : S \to A$ preserves the order and the S-action, $\rho_a : S \to A$ is an S-net and we call it the S-net *induced by a*.

(iii) We say that the S-net $(a_s)_{s\in S}$ *converges to* $a \in A$ whenever $sa = a_s$, for every $s \in S$. The element a is called the *limit* of $(a_s)_{s\in S}$ and let $\lim(a_s)_{s\in S}$ denote the set of all limits of the S-net $(a_s)_{s\in S}$.

Remark 3.2. (1) An S-net $(a_s)_{s\in S}$ does not necessarily have a unique limit. For example, if we take the semigroup S to be the set of natural number together with taking minimum as its operation, $(\mathbb{N}, min)$, and consider the chain of natural numbers $\mathbb{N}$ as an S-poset with the action $\mu : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ sending each (n, m) to $m - 1$ if $m \neq 1$ and $(n, 1)$ to 1, for every $n \in \mathbb{N}$. Here the constant $\mathbb{N}$-net $(a_n = 1)_{n\in\mathbb{N}}$ has two limits 1 and 2.

(2) Let A be an S-poset and $a \in A$. Then the S-net induced by a, as in Definition 3.1 (ii), converges to a. Also for every fixed (zero) element $a \in A$, the S-net induced by a is the constant S-net $(a_s = a)_{s\in S}$ converging to a.

(3) The convergency notion is compatible with the morphisms in the category **S-Pos**. In fact, if $f : A \to B$ is an S-poset morphism, then given any point a in A and any S-net in A converging to a, the composition of f with this S-net converges to $f(a)$.

4. Separated S-posets

In the present section we give a characterization of separated S-posets. We also show that how one can make an S-poset into a separated one. We then introduce the net-closure operator and give another characterization of separated S-posets.

In the following theorem we give a characterization of S-posets in which limits of S-nets are unique. But first recall that an S-poset A is called *separated* whenever for every $a, b \in A$ with $sa = sb$, for every $s \in S$, then $a = b$ [7].

Theorem 4.1. *Let S be a posemigroup and A be an S-poset. Then, the limit of an S-net in A is unique if and only if A is a separated S-poset.*

Proof. ($\Rightarrow$) Suppose that the limit of S-nets in A is unique and $sa = sb$ for a pair $a, b \in A$, for every $s \in S$. So the induced S-nets by a and by b, see Definition 3.1 (ii), coincide. Then the uniqueness of the limit implies that $a = b$.

($\Leftarrow$) Let a and b be two limits of an S-net $(a_s)_{s\in S}$ in a separated S-poset A. Then $sa = a_s = sb$, for every $s \in S$. Now since A is separated, $a = b$. □

Remark 4.2. Theorem 4.1 shows that if we are looking for a unique limit for an S-net in A, we should make A separated. The congruence defined by

$$a\theta b \Leftrightarrow sa = sb; \quad \forall s \in S$$

on an S-act A makes A/θ into a separated S-act. The following theorem shows how we can make an S-poset separated.

Theorem 4.3. *Let A be an S-poset. Then the above equivalence relation θ is a congruence on A and A/θ is a separated S-poset. Moreover, if $S = S^2$ then the set of limits of a given S-net $\alpha : S \to A$ in A is exactly the class $[a]_\theta$ in which $[a]_\theta$ is the unique limit of the S-net $\pi \circ \alpha : S \to A \to A/\theta$ in A/θ.*

Proof. First we note that θ is an S-poset congruence, see Definition 2.1 (v). Suppose that $a \leq_\theta b \leq_\theta a$, and therefore

$$a \leq a_1\theta b_1 \leq \cdots \leq a_n\theta b_n \leq b; \quad \text{for some} \quad a_1, \cdots, a_n, b_1, \cdots, b_n \quad \text{and,}$$

$$b \leq a'_1\theta b'_1 \leq \cdots \leq a'_n\theta b'_n \leq a; \quad \text{for some} \quad a'_1, \cdots, a'_n, b'_1, \cdots, b'_n.$$

So, for every $s \in S$ we have:

$$sa \leq sa_1 = sb_1 \leq \cdots \leq sa_n = sb_n \leq sb \quad \text{and,}$$

$$sb \leq sa'_1 = sb'_1 \leq \cdots \leq sa'_n = b'_n \leq sa.$$

That is $sa = sb$, for every $s \in S$, and hence $a\theta b$.

Now let $S = S^2$ and consider the S-nets $\alpha : S \to A$ in A and $\pi \circ \alpha : S \to A \to A/\theta$ in A/θ. Also let $[a]_\theta$ be the unique limit of $\pi \circ \alpha$. So $s[a]_\theta = [sa]_\theta = [a_s]_\theta$, for every $s \in S$. We show that each $b \in [a]_\theta$ is a limit of $(a_s)_{s\in S}$. Indeed, for every $s \in S$, since $s = s_1s_2$ we have $sb = s_1s_2b = s_1s_2a = s_1a_{s_2} = a_{s_1s_2} = a_s$. The second equation is true, since $[a]_\theta = [b]_\theta$ and the third is for $[s_2a]_\theta = [a_{s_2}]_\theta$. So $b \in \lim(a_s)_{s\in S}$. The converse is obviously true by the properties of congruences in S-posets. $\square$

Now we want to define the concept of the set of boundary points of a subset B of an S-post A as well as the boundary of a subset Y of a topological space X, and introduce the net-closure operator C_{net} on S-posets to construct a topology on an S-poset for investigating the separation axioms on that in the next section.

First recall the definition of the categorical *closure operator* from [10] on a category $\mathcal{A}$ which is given by a family $cl = (cl_A)_{A\in\mathcal{A}}$ of maps from the class of subojects of A to itself which satisfies the following conditions for all subobjects $X, Y \leq A$:

(i) $X \leq cl_A(X)$ (cl_A is extensive)

(ii) $X \leq Y \Rightarrow cl_A(X) \leq cl_A(Y)$ (cl_A is monoton)

(iii) $f(cl_A(X)) \leq cl_A(f(X))$ (cl_A is continuous)

in which f is a morphism from X to Y. Also the subobject X of A is called *cl-closed in* A whenever $cl_A(X) = X$.

Now we are ready to give the definition of net-closure as follows:

Definition 4.4. (1) The set of the *boundary points* of a subset X of an S-poset is defined to be the set

$$X' = \{a \in A \mid \text{there exists an } S\text{-net } \alpha : S \to X; a \in \lim(\alpha)\}.$$

(2) In the category $\mathsf{S\text{-}Pos}$ we define the family $C_{net} = (C_{netA})_{A\in \mathsf{S-Pos}}$ and call it a *net-closure* as follows: given S-poset A,

$$C_{netA} : S\text{-subposet}(A) \to S\text{-subposet}(A)$$

maps each S-subposet $X \leq A$ to $C_{netA}(X) = X \cup X'$.

Lemma 4.5. *The above defined C_{net} is a closure operator.*

Proof. First note that for every S-poset A and an S-subposet X of A, the set $X \cup X'$ is an S-subposet of A. Because, for every $a \in X'$ which is a limit of an S-net such as $\alpha : S \to X$ and every $t \in S$, ta is a limit of the net $\beta = \alpha \circ \lambda_t : S \to S \to X$ in which $\lambda_t : S \to S$ maps each $s \in S$ to st, since $\beta(s) = \alpha(st) = a_{st} = sa_t = sta$. That is $C_{netA}(X)$ is an S-subposet of A. Also the conditions (i) and (ii) for a closure operator are clearly satisfied. For the third condition, let $f : X \to Y$ be an S-poset morphism from an S-subposet X to an S-subposet Y and $a \in X'$. So there exists an S-net $\alpha : S \to X$ such that $a \in \lim(a_s)_{s\in S}$. Then $sf(a) = f(sa) = f(a_s) = \alpha \circ f(s)$. So $f(a)$ is a boundary point of $f(X)$. □

Remark 4.6. In [6, 8] where the authors take the semigroup S to be the chain of natural numbers $\mathbb{N}$ together with taking minimum as its operation, a closure operation C_s defined by $C_s(X) = \{a \in A \mid sa \in X, \text{ for all } s \in \mathbb{N}\}$, for every S-poset X of A. Thus every element a of $C_s(X)$ is a limit of the induced S-net by a in X, and every element a of $C_{net}(X)$ which is a limit of some S-net in X belongs to C_s. That is, in $\mathbb{N}$-posets, C_{net} is exactly C_s.

Given an S-poset A one can consider a *net-topology* with respect to the net-closed S-subposets. Also every directed complete poset, that is a poset containing the join of directed subsets, is equipped with a topology which is called the *Scott-topology* [1]. In a directed complete poset P, a subset A of P is *Scott-closed* if and only if it is a lower set and is closed under supremum of directed subsets. Now what is the relation between Scott-topology and the net-topology? See the following theorem.

Theorem 4.7. *Let S be a directed posemigroup and A be a directed S-poset. Then the Scott-topology is finer than the net-topology, that is, every Scott-closed S-subposet of A is a net-closed one, whenever $sa \leq b$, for every $s \in S$, if $a \leq b$.*

Proof. Let A be a directed S-poset, B be a Scott-closed S-subposet of A, $\alpha : S \to B$ be an S-net in B, and $a \in \lim(a_s)_{s\in S}$. Since $(a_s)_{s\in S}$ is a directed family and B is Scott-closed, the family $(a_s)_{s\in S}$ has a supremum in B, namely b. So $sa = a_s \leq b$ for every $s \in S$, and hence $a \in B$. □

The following theorem gives another characterization of separated S-posets.

Theorem 4.8. *An S-poset A is separated if and only if the diagonal $\Delta_A = \{(a,a) \mid a \in A\}$ is C_{net}-closed in $A \times A$.*

Proof. ($\Rightarrow$) Let A be separated, $\alpha : S \to \Delta_A$ be an S-net in Δ_A, and $(a,b) \in \lim(\alpha)$. Then $\alpha(s) = s(a,b) = (sa, sb)$, for each $s \in S$. Since α is an S-net in Δ_A, we have $(sa, sb) = \alpha(s) \in \Delta_A$, for each $s \in S$. Therefore $sa = sb$, for each $s \in S$ and hence $a = b$. That is $(a,b) \in \Delta_A$.

($\Leftarrow$) For the converse, let $sa = sb$, for every $s \in S$. Then consider the S-net induced by (a,b), that is $\rho_{(a,b)} : S \to \Delta_A$, which converges to (a,b). Now, since Δ_A is C_{net}-closed, $a = b$. □

As a corollary of the above theorem and Theorem 4.1, we have the following theorem:

Theorem 4.9. *For an S-poset A, the following are equivalent:*

(i) *Limits of nets in A are unique.*

(ii) *A is separated.*

(iii) *The diagonal Δ_A is C_{net}-closed S-subposet of $A \times A$.*

5. Separation axioms and Separated S-posets

In this section we want to study some of the counterparts of topological separation axioms for S-posets. But before that, we remark that since in this section we are working only with a fixed S-poset A, we denote C_{netA} simply by C_{net}.

Definition 5.1. An S-poset is said to be:

(1) T_0 if for every pair of distinct element $a, b \in A$ there exists a C_{net}-closed S-subposet F_a containing a but not b, or there exists a C_{net}-closed S-subposet F_b containing b but not a.

(2) T_1 if for every pair of distinct element $a, b \in A$ there exists a C_{net}-closed S-subposet F_a containing a but not b, and there exists a C_{net}-closed S-subposet F_b containing b but not a.

(3) T_2 if for every pair of distinct element $a, b \in A$ there exists a C_{net}-closed S-subposet F_a containing a but not b, and there exists a C_{net}-closed S-subposet F_b containing b but not a such that $F_a \cap F_b = \emptyset$.

(4) T_3 if for each $a \in A$, the S-subposet S^1a is the least C_{net}-closed S-subposet containing a, in which S^1a means the union of Sa and the singleton set $\{a\}$ which is an S-subposet.

(5) T_4 if for every S-subposet B and every $b \in B$ there exists a C_{net}-closed S-subposet F_b with $b \in F_b \subseteq B$.

Remark 5.2. One can see that:

$$T_4 \Leftrightarrow T_3 \quad \text{and} \quad T_2 \Rightarrow T_1 \Rightarrow T_0$$

The implication $T_4 \Rightarrow T_3$ is because of the fact that S^1a is the least S-subposet containing a, so, by T_4, it should be closed. The other implications are gotten directly by definitions.

For the converse of the other implications the following theorem is helpful.

Theorem 5.3. *For an S-poset A the following are equivalent:*

(1) *A is T_1.*

(2) *For distinct element a, b we have $a \notin C_{net}(Sb)$ and $b \notin C_{net}(Sa)$.*

(3) *For each element $a \in A$, $Sa = \{a\}$.*

(4) *The intersection of all the C_{net}-closed S-subposets containing an element $a \in A$ is $\{a\}$.*

Proof. (1) $\Rightarrow$ (2) We can consider the C_{net}-closed S-subposet F_a containing a but not b and the C_{net}-closed S-subposet F_b containing b but not a, since A is T_1. Then we have $Sa \subseteq S^1a \subseteq F_a$ and $b \notin F_a$. Hence $C_{net}(Sa) \subseteq F_a$ and $b \notin C_{net}(Sa)$. By the same way one can see that $a \notin C_{net}(Sb)$.

(2) $\Rightarrow$ (3) For each $a \in A$ and $s \in S$ if $sa \neq a$ then $sa \notin C_{net}(Sa)$ and this is a contradiction.

(3)$\Rightarrow$ (1) If each element of A is a fixed point then every singleton set $\{a\}$ is C_{net}-closed. Because, first of all $\{a\}$ is an S-subposet and the only net in $\{a\}$ is the constant

S-net $(a_s)_{s\in S} = (a)$ and the only limit of this S-net is a. So every distinct pair a, b in A can be separated by C_{net}-closed S-subposets.

(1) $\Rightarrow$ (4) Let the intersection of all the C_{net}-closed S-subposets containing an element $a \in A$ have another element such as $b \neq a$. Then, since A is T_1, there is a C_{net}-closed S-subposet F_a containing a but not b and this is a contradiction.

(4) $\Rightarrow$ (1) Since the intersection of C_{net}-closed S-subposets is a C_{net}-closed S-subposet, $\{a\}$ is C_{net}-closed and hence every two distinct points a, b of A can be separated by $\{a\}$ and $\{b\}$. $\square$

Corollary 5.4. (1) *Now the above theorem results that:*

$$T_4 \Leftrightarrow T_3 \Leftrightarrow T_2 \Leftrightarrow T_1 \Rightarrow T_0$$

For $T_3 \Leftrightarrow T_2$, we note that $T_3 \Leftrightarrow T_1 \Leftrightarrow T_2$.

(2) *The third statement of Theorem 5.3 ensures that every T_1 S-poset is separated.*

(3) *We should note that Axiom T_2 is not the topological Hausdorff property. Because the requirement for the topological Hausdorff axiom in terms of closed sets is that their union to be the whole space. Here using Theorem 4.9 we reserve the term of Hausdorff S-posets for the separated ones.*

Note 5.5. As it is mentioned in Corollary 5.4, every T_1 S-poset is T_0 but the converse is not necessarily true. For example, take S to be N and A to be the chain $2 = \{x, y\}$ in which $x \leq y$ and $sx = x$ for every $s \in S$ and $1y = x$ and $sy = y$ for each $1 \lneqq s$. Then A is T_0 but not T_1.

Theorem 5.6. *Let S be a commutative semigroup and A be a T_1 S-poset. Then the only S-nets in A are the constant S-nets. Moreover, every S-net has a unique limit.*

Proof. Let $\alpha : S \to A$ be a nonconstant S-net in A. Then for each pair $s, t \in S$, by the third statement in Theorem 5.3, we know that a_s and a_t are fixed. So we have $a_s = ta_s = a_{ts} = a_{st} = sa_t = a_t$. Hence α is a constant S-net $(a_s = a)_{s\in S}$, for some $a \in A$ and it converges to a. Further more, since every T_1 S-poset is separated, Corollary 5.4 (2), so the limit of S-nets are unique, Theorem 4.1. $\square$

Theorem 5.7. *Every T_0 S-poset is separated.*

Proof. To prove this, we use Theorem 4.1 and show that the limits of S-nets are unique. Suppose that $\alpha : S \to A$ is an S-net in the S-poset A which is T_0. Also suppose that

$\alpha : S \to A$ converges to a and b. If $a \neq b$ then there exists a C_{net}-closed F_a containing a but not b. Since a is a limit of α, so $a_s = sa \in F_a$, for every $s \in S$. That is $(a_s)_{s\in S}$ is an S-net in F_a. Now since F_a is C_{net}-closed and $b \in \lim(a_s)_{s\in S}$, we have $b \in F_a$ and this is a contradiction. Hence $a = b$. $\square$

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References

1. S. Abramsky, D. M. Gabbay and T. S. E. Maibaum, *Handbook of Logic in Computer Science. Vol.III*, Clarendon Press, Oxford, 1994.
2. S. Bulman-Fleming, Subpullback flat S-posets need not be subequlizer flat, *Semigroup Forum*, **78**(1) (2009), 27-33.
3. S. Bulman-Fleming, D. Gutermuth, A. Gilmour and M. Kilp, Flatness properties of S-posets, *Communications in Algebra*, **34** (2006), 1291-1317.
4. S. Bulman-Fleming and V. Laan, Lazards theorem for S-posets, *Math. Nachr*, **278**(15) (2005), 1743-1755.
5. S. Bulman-Fleming and M. Mahmoudi, The category of S-Posets, *Semigroup Froum*, **71** (2005), 443-461.
6. E. Giuli, on M-Separated projection space, *Appl. Categ. Structures*, **2** (1994), 91-99.
7. M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, New York, 2000.
8. M. Mahmoudi, Separation axioms on $\mathbb{N}^\infty$-systems, *Semigroup Froum* **70** (2005), 97-106.
9. E. H. Moore and H. L. Smith, A general theory of limits, *American Journal of Mathematics*, **44**(2) (1922), 102-121.
10. W. Tholen and D. Dikranjan, *Categorical Structure of Closure Operators*, Kluwer Academic Publishers University Press, 1995.

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