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f-DERIVATIONS AND (f,g)-DERIVATIONS OF MV-ALGEBRAS

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Abstract. In this paper, we extend the notion of derivation of MV-algebras and give some illustrative examples. Moreover, as a generalization of derivation of MV-algebras we introduce the notion of f-derivations and (f,g)-derivations of MV-algebras. Also, we investigate some properties of them.

1. Introduction

In [7], Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Recently, the algebraic theory of MV-algebras is intensively studied, for example see [17, 18, 19]. The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic systems. Several authors [3, 9, 16] studied derivations in rings and near-rings. Jun and Xin [11] applied the notion of derivation to BCI-algebras. In [20], Szász introduced the concept of derivation for lattices and investigated some of its properties. Also, in [21], Xin et al. improved derivation for a lattice and discussed some related properties. They gave some equivalent conditions under which a derivation is isotone for lattices with a greatest modular lattices and distributive lattices, also see [15]. After these studies the f-derivation and symmetric bi derivation of lattices were defined and studied in [5, 6]. Ozbal and Firat in [14] introduced the notion of symmetric f-bi-derivation of a lattice. They characterized

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the distributive lattice by symmetric f-bi-derivation. In [11], Jun and Xin introduced the notion of derivation in BCI-algebras, which is defined in a way similar to the notion in ring theory, and investigated some properties related to this concept. In [22], Zhan and Liu introduced the notion of f-derivation in BCI-algebras. In [4], Ceran and Aşci defined the symmetric bi-(σ, τ) derivations on prime and semiprime Gamma rings. In [2], Alshehri applied the notion of derivation to MV-algebras and investigated some of its properties.

Now, in this paper, we extend the notion of derivation of MV-algebras. Moreover, as a generalization of derivation of MV-algebras we introduce the notion of $f-$derivations and (f, g)-derivations of MV-algebras.

2. Preliminaries

In this section, we recall the notion of an MV-algebra and then we review some definitions and properties which we will need in the next section.

Definition 2.1. An *MV-algebra* is a structure $(M, \oplus, *, 0)$ where M is a non-empty set, $\oplus$ is a binary operation, $*$ is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in M$

$(MV1)$ $(M, \oplus, 0)$ is a commutative monoid;
$(MV2)$ $(a^*)^* = a$;
$(MV3)$ $0^* \oplus a = 0^*$;
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

We define the constant $1 = 0^*$ and the auxiliary operations $\odot, \ominus, \vee$ and $\wedge$ by

$$a \odot b = (a^* \oplus b^*)^*, \ a \ominus b = a \odot b^*, \ a \vee b = a \oplus (b \odot a^*), \ a \wedge b = a \odot (b \oplus a^*).$$

Example 2.2. Any boolean algebra is an MV-algebra.

Example 2.3. The real unit interval $[0, 1]$ with operations $\oplus$ and $*$ defined by

$$x \oplus y = \min\{1, x + y\} \text{ and } x^* = 1 - x$$

is an MV-algebra

Theorem 2.4. [7] *Let $(M, \oplus, *, 0)$ be an MV-algebra. The following properties hold for all $x \in M$*

(1) $x \oplus 1 = 1$;

(2) $x \oplus x^* = 1$;

(3) $x \odot 0 = 0$ *and* $x \odot x^* = 0$.

Moreover, $(M, \odot, 1)$ *is a commutative monoid* [7].

Theorem 2.5. [7] *Let* $(M, \oplus, *, 0)$ *be an* MV*-algebra. The following properties hold for all* $x \in M$

(1) *If* $x \oplus y = 0$, *then* $x = y = 0$;

(2) *If* $x \odot y = 1$, *then* $x = y = 1$;

(3) $x \oplus y = y$ *if and only if* $x \odot y = x$;

(4) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$.

Let $(M, \oplus, *, 0)$ be an MV-algebra. The partial ordering $\leq$ on M is defined by

$$x \leq y \Longleftrightarrow x \wedge y = x, \text{for all } x, y \in M.$$

$x \wedge y = x$ is equivalent to $x \vee y = y$. The structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. If the order relation $\leq$, defined over M, is total, then we say that M is *linearly ordered*.

Theorem 2.6. [7] *Let* $(M, \oplus, *, 0)$ *be an* MV*-algebra. The following properties hold for all* $x \in M$

(1) *If* $x \leq y$, *then* $x \vee z \leq y \vee z$ *and* $x \wedge z \leq y \wedge z$;

(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$;

(3) $x \leq y$ *if and only if* $y^* \leq x^*$.

Theorem 2.7. [7] *For all* $x, y \in M$, *the following conditions are equivalent:*

(1) $x \leq y$;

(2) $y \oplus x^* = 1$;

(3) $x \odot y^* = 0$.

Theorem 2.8. [7] *Let* M *be a linearly ordered* MV*-algebra. Then,* $x \oplus y = x \oplus z$ *and* $x \oplus z \neq 1$ *implies that* $y = z$.

Let M and N be two MV-algebras. The function $f : M \longrightarrow N$ is called a *homomorphism* if it satisfies the following conditions:

(1) $f(0_M) = 0_N$;

(2) $f(x \oplus_M y) = f(x) \oplus_N f(y)$;
(3) $f(x^*) = f(x)^*$;

for all $x, y \in M$. If f is a homomorphism, then $f(1_M) = 1_N$ and $f(x \odot_M y) = f(x) \odot_N f(y)$. A homomorphism f is called an *isomorphism* if it is one to one and onto.

Let M be an MV-algebra and I be a non-empty subset of M. Then, we say that I is an *ideal* if the following conditions are satisfied:

(1) $0 \in I$;
(2) $x, y \in I$ imply $x \oplus y \in I$;
(3) $x \in I$ and $y \leq x$ imply $y \in I$.

Lemma 2.9. *Let I be an ideal of MV-algebra M and $f : M \longrightarrow M$ be an isomorphism. Then, $f(I)$ is an ideal, too.*

Proof. It is obvious. □

Let $B(M) = \{x \in M \mid x \oplus x = x\} = \{x \in M \mid x \odot x = x\}$. Then, $(B(M), \oplus, *, 0)$ is both a largest subalgebra of M and a Boolean algebra.

Definition 2.10. A *BCI-algebra* X is an abstract algebra $(X, *, 0)$ of type $(2, 0)$, satisfying the following conditions, for all $x, y, z \in X$,

$(BCI1)$ $((x * y) * (x * z)) * (z * y) = 0$;
$(BCI2)$ $(x * (x * y)) * y = 0$;
$(BCI3)$ $x * x = 0$;
$(BCI4)$ $x * y = 0$ and $y * x = 0$ imply that $x = y$.

A non-empty subset S of a BCI-algebra X is called a *subalgebra* of X if $x * y \in S$, for all $x, y \in S$. In any BCI-algebra X, one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$. A BCI-algebra X satisfying $0 \leq x$, for all $x \in X$, is called a *BCK-algebra*.

Let $(M, \oplus, *, 0)$ be an MV-algebra. Then, the structure $(M, \ominus, 0)$ is a bounded $BCI \setminus BCK$-algebra.

3. Derivations of MV-Algebras

Definition 3.1. Let $(M, \oplus, *, 0)$ be an MV-algebra. Then, the map $D : M \longrightarrow M$ is called

(1) a *derivation of type* 1, if $D(x \odot y) = (D(x) \odot y) \oplus (x \odot D(y))$, for all $x, y \in M$ [2];
(2) a *derivation of type* 2, if $D(x \wedge y) = (D(x) \wedge y) \vee (x \wedge D(y))$, for all $x, y \in M$;
(3) a *derivation of type* 3, if $D(x \ominus y) = (D(x) \ominus y) \wedge (x \ominus D(y))$, for all $x, y \in M$.

If MV-algebra M is a Boolean algebra, then for all $x, y \in M$, $x \oplus y = x \vee y$ and $x \odot y = x \wedge y$. So, in this case, every derivation of type 1 on M is coincide with derivation of type 2 on M.

Let $(M, \oplus, *, 0)$ be an MV-algebra. Then, the definition of derivation of type 2 on $(M, \oplus, *, 0)$ is coincide with the definition of derivation on lattice $(M, \wedge, \vee, 0, 1)$. Also, the definition of derivation of type 3 on $(M, \oplus, *, 0)$ is coincide with the definition of derivation on bounded $BCI \setminus BCK$-algebra $(M, \ominus, 0)$.

Let M be an MV-algebra and $D : M \longrightarrow M$ be a derivation of type 1 (2 and 3, respectively). Then, for convenience, we denote D by D^1 (D^2 and D^3, respectively).

Theorem 3.2. *Let $(M, \oplus, *, 0)$ be an MV-algebra and D^i be a derivation of type i on M, $1 \leq i \leq 3$. Then, for all $1 \leq i \leq 3$, we have*

(1) $D^i(0) = 0$;
(2) $D^i(x) \leq x$, *for all* $x \in M$.

Proof. (1) It is proved in [2] that $D^1(0) = 0$. We have $D^2(0) = D^2(0 \wedge 0) = (D^2(0) \wedge 0) \vee (0 \wedge D^2(0)) = 0$ and $D^3(0) = D^3(x \ominus 1) = (D^3(x) \ominus 1) \wedge (x \ominus D^3(1)) = 0$, for all $x \in M$.

(2) It is proved in [2] that $D^1(x) \leq x$. We have $D^2(x) = D^2(x \wedge x) = (D^2(x) \wedge x) \vee (x \wedge D^2(x)) = D^2(x) \wedge x$. So, $D^2(x) \leq x$.

Also, we have $D^3(x) = D^3(x \ominus 0) = (D^3(x) \ominus 0) \wedge (x \ominus D^3(0)) = D^3(x) \wedge x$. So, $D^3(x) \leq x$. □

Let M be an MV-algebra. The function $D : M \longrightarrow M$, defined by $D(x) = 0$, for all $x \in M$, is a derivation of type 1, 2 and 3 on M. We denote it by $D = 0$.

Also, the function $D : M \longrightarrow M$, defined by $D(x) = x$, for all $x \in M$, is a derivation of type 2 and 3 on M. We denote it by $D = I$.

Example 3.3. Let $M = \{0, 1\}$. Consider the following tables:

$\oplus$	0	1
0	0	1
1	1	1

$*$	0	1
	1	0

Then, $(M, \oplus, *, 0)$ is an MV-algebra. It is only MV-algebra of order 2. The functions $D_1 = 0$ and $D_2 = I$ are only derivations of type 1. Also, they are only derivations of type 2 and 3.

Example 3.4. Let $M = \{0, x_1, 1\}$. Consider the following tables:

$\oplus$	0	x_1	1
0	0	x_1	1
x_1	x_1	1	1
1	1	1	1

$*$	0	x_1	1
	1	x_1	0

Then, $(M, \oplus, *, 0)$ is an MV-algebra. It is only MV-algebra of order 3. By calculation, we obtain Figure 1.

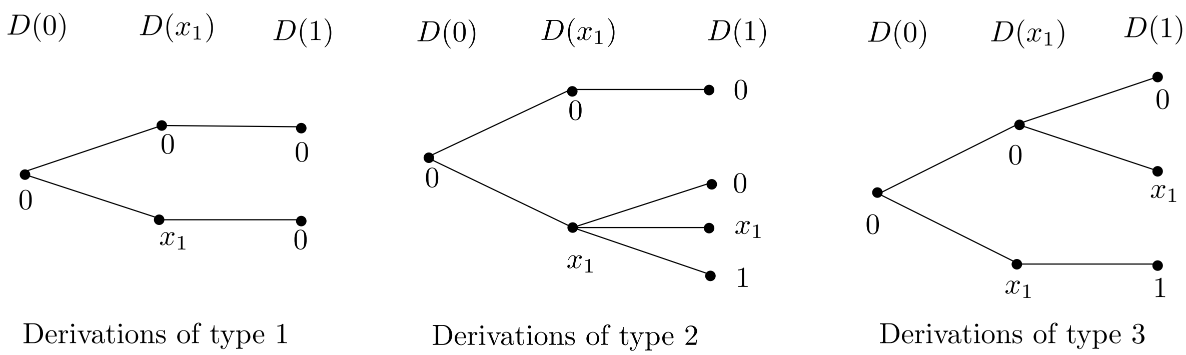

FIGURE 1. Derivations of type 1, 2 and 3 for Example 3.4.

Thus, we have only two derivations of type 1 on M. They are as follows:

$$D_1^1 = 0 \quad \text{and} \quad D_2^1(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1. \end{cases}$$

We have only four derivations of type 2 on M. They are as follows:

$$D_1^2 = 0,\ D_2^2 = I,\ D_3^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \end{cases} \quad \text{and}$$

$$D_4^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, 1. \end{cases}$$

We have only three derivations of type 3 on M. They are as

$$D_1^3 = 0, \quad D_2^3 = I \quad \text{and} \quad D_3^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_1 & \text{if } x = 1. \end{cases}$$

It is clear that D_2^1 is not a derivation of type 3, because $x_1 = D_2^1(x_1) = D_2^1(1 \ominus x_1) \neq (D_2^1(1) \ominus x_1) \wedge (1 \ominus D_2^1(x_1)) = 0$. Also, D_3^3 is not a derivation of type 1, because $x_1 = D_3^3(1 \odot 1) \neq (D_3^3(1) \odot 1) \oplus (1 \odot D_3^3(1)) = 1$. So, derivation of type 1 and 3 are independent.

It is clear that D_3^2 is not a derivation of type 3, because $x_1 = D_3^2(1 \ominus x_1) \neq (D_3^2(1) \ominus x_1) \wedge (1 \ominus D_3^2(x_1)) = 0$. Also, D_3^3 is not a derivation of type 2, because $0 = D_3^3(x_1 \wedge 1) \neq (D_3^3(x_1) \wedge 1) \vee (x_1 \wedge D_3^3(1)) = x_1$. So, derivation of type 2 and 3 are independent.

We have only two MV-algebras of order 4. They are considered in the next two examples.

Example 3.5. Let $M = \{0, x_1, x_2, 1\}$. Consider the following tables:

$\oplus$	0	x_1	x_2	1
0	0	x_1	x_2	1
x_1	x_1	x_2	1	1
x_2	x_2	1	1	1
1	1	1	1	1

$*$	0	x_1	x_2	1
	1	x_2	x_1	0

Then, $(M, \oplus, *, 0)$ is an MV-algebra. By calculation, we get Figure 2.

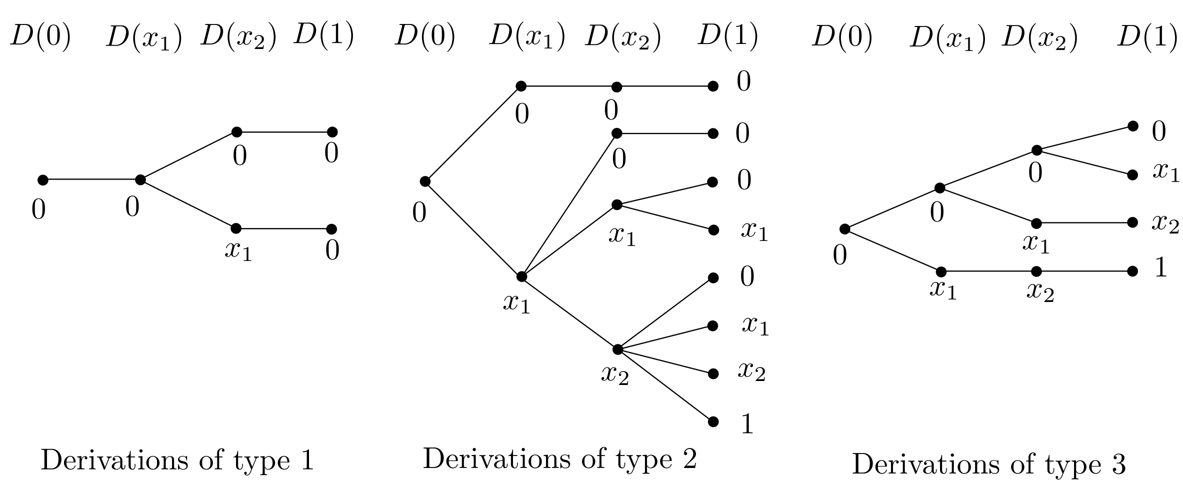

FIGURE 2. Derivations of type 1, 2 and 3 for Example 3.5.

Thus, we have only two derivations of type 1 on M. They are as follows:

$$D_1^1 = 0 \text{ and } D_2^1(x) = \begin{cases} 0 & \text{if } x = 0, x_1, 1 \\ x_1 & \text{if } x = x_2. \end{cases}$$

We have only eight derivations of type 2 on M. They are as follows:

$$D_1^2 = 0, \qquad D_2^2 = I,$$

$$D_3^2(x) = \begin{cases} 0 & \text{if } x = 0, x_2, 1 \\ x_1 & \text{if } x = x_1, \end{cases} \qquad D_4^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1, x_2, \end{cases}$$

$$D_5^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, x_2, 1, \end{cases} \qquad D_6^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, \end{cases}$$

$$D_7^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, 1 \\ x_2 & \text{if } x = x_2, \end{cases} \qquad D_8^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, 1. \end{cases}$$

We have only four derivations of type 3 on M. They are as follows:

$$D_1^3 = 0, \quad D_2^3 = I, \quad D_3^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_2 \\ x_1 & \text{if } x = 1 \end{cases} \quad \text{and}$$

$$D_4^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_1 & \text{if } x = x_2 \\ x_2 & \text{if } x = 1. \end{cases}$$

It is clear that D_2^1 is not a derivation of type 2, because $0 = D_2^1(x_1) = D_2^1(x_1 \wedge x_2) \neq (D_2^1(x_1) \wedge x_2) \vee (x_1 \wedge D_2^1(x_2)) = x_1$. Also, D_5^2 is not a derivation of type 1, because $x_1 = D_5^2(1) = D_5^2(1 \odot 1) \neq (D_5^2(1) \odot 1) \oplus (1 \odot D_5^2(1)) = x_1 \oplus x_1 = x_2$. So, derivation of type 1 and 2 are independent.

Example 3.6. Let $M = \{0, x_1, x_2, 1\}$. Consider the following tables:

$\oplus$	0	x_1	x_2	1
0	0	x_1	x_2	1
x_1	x_1	x_1	1	1
x_2	x_2	1	x_2	1
1	1	1	1	1

$*$	0	x_1	x_2	1
	1	x_2	x_1	0

Then, $(M, \oplus, *, 0)$ is a Boolean MV-algebra. So, derivation of type 1 is coincide with derivation of type 2. By calculation, we get Figure 3.

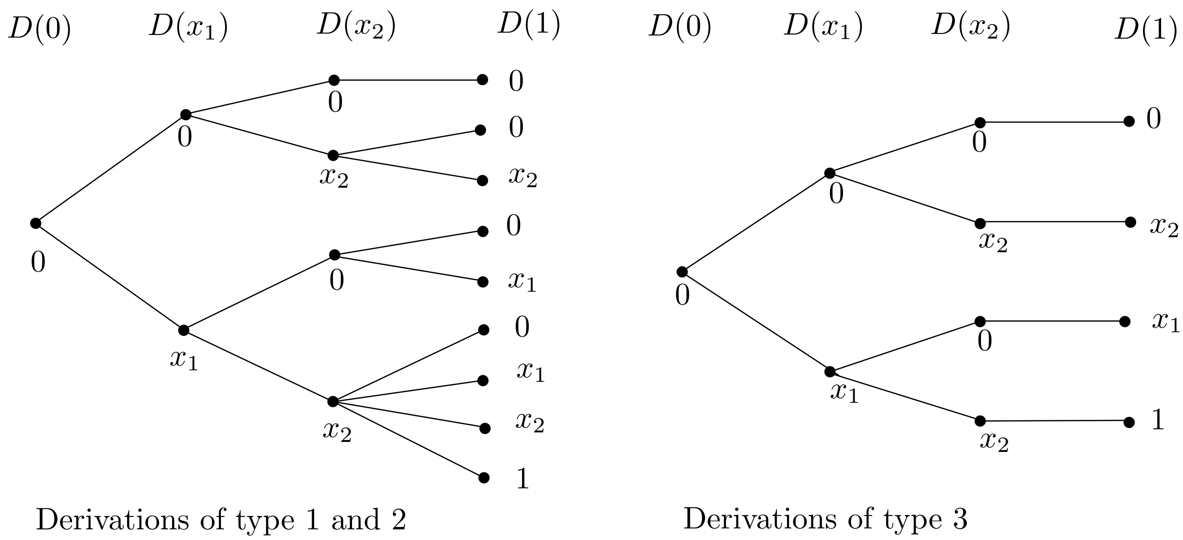

FIGURE 3. Derivations of type 1, 2 and 3 for Example 3.6.

Hence, we have only nine derivations of type 1 on M. They are as follows:

$$D_1^1 = 0,\ D_2^1 = I, \qquad D_3^1(x) = \begin{cases} 0 & \text{if } x = 0, x_1, 1 \\ x_2 & \text{if } x = x_2, \end{cases}$$

$$D_4^1(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_2 & \text{if } x = x_2, 1, \end{cases} \quad D_5^1(x) = \begin{cases} 0 & \text{if } x = 0, x_2, 1 \\ x_1 & \text{if } x = x_1, \end{cases}$$

$$D_6^1(x) = \begin{cases} 0 & \text{if } x = 0, x_2 \\ x_1 & \text{if } x = x_1, 1, \end{cases} \quad D_7^1(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2 \end{cases}$$

$$D_8^1(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, 1 \\ x_2 & \text{if } x = x_2, \end{cases} \quad D_9^1(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, 1. \end{cases}$$

We have only four derivations of type 3 on M. They are as $D_1^3 = 0$, $D_2^3 = I$, $D_3^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_2 & \text{if } x = x_2, 1 \end{cases}$ and $D_4^3(x) = \begin{cases} 0 & \text{if } x = 0, x_2 \\ x_1 & \text{if } x = x_1, 1 \end{cases}$.

We have one MV-algebra of order 5. It is considered in the next example.

Example 3.7. Let $M = \{0, x_1, x_2, x_3, 1\}$. Consider the following tables:

$\oplus$	0	x_1	x_2	x_3	1
0	0	x_1	x_2	x_3	1
x_1	x_1	x_2	x_3	1	1
x_2	x_2	x_3	1	1	1
x_3	x_3	1	1	1	1
1	1	1	1	1	1

$*$	0	x_1	x_2	x_3	1
	1	x_3	x_2	x_1	0

Then, $(M, \oplus, *, 0)$ is an MV-algebra. By calculation, we get Figure 4.

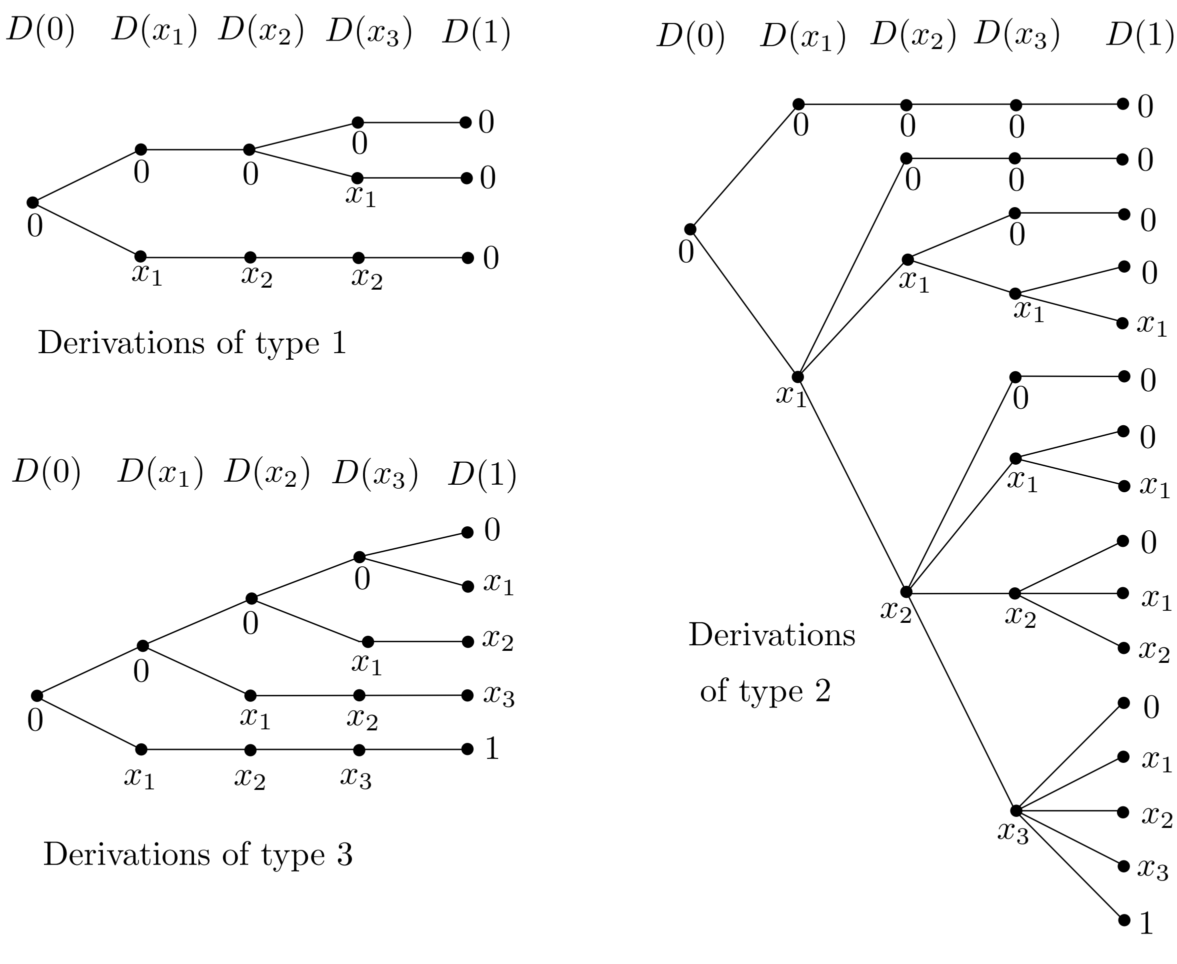

FIGURE 4. Derivations of type 1, 2 and 3 for Example 3.7.

Thus, we have only three derivations of type 1 on M. They are as follows:

$$D_1^1 = 0, \quad D_2^1(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_2, 1 \\ x_1 & \text{if } x = x_3 \end{cases} \quad \text{and}$$

$$D_3^1(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, x_3. \end{cases}$$

We have only sixteen derivations of type 2 on M. They are as follows:

$$D_1^2 = 0, \qquad D_2^2 = I,$$

$$D_3^2(x) = \begin{cases} 0 & \text{if } x = 0, x_2, x_3, 1 \\ x_1 & \text{if } x = x_1, \end{cases} \qquad D_4^2(x) = \begin{cases} 0 & \text{if } x = 0, x_3, 1 \\ x_1 & \text{if } x = x_1, x_2, \end{cases}$$

$$D_5^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1, x_2, x_3, \end{cases} \qquad D_6^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, x_2, x_3, 1, \end{cases}$$

$$D_7^2(x) = \begin{cases} 0 & \text{if } x = 0, x_3, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, \end{cases} \qquad D_8^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1, x_3 \\ x_2 & \text{if } x = x_2, \end{cases}$$

$$D_9^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, x_3, 1 \\ x_2 & \text{if } x = x_2, \end{cases} \qquad D_{10}^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, x_3, \end{cases}$$

$$D_{11}^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, 1 \\ x_2 & \text{if } x = x_2, x_3, \end{cases} \qquad D_{12}^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, x_3, 1, \end{cases}$$

$$D_{13}^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2 \\ x_3 & \text{if } x = x_3, \end{cases} \qquad D_{14}^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1, 1 \\ x_2 & \text{if } x = x_2 \\ x_3 & \text{if } x = x_3, \end{cases}$$

$$D_{15}^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2, 1 \\ x_3 & \text{if } x = x_3, \end{cases} \qquad D_{16}^2(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_1 & \text{if } x = x_1 \\ x_2 & \text{if } x = x_2 \\ x_3 & \text{if } x = x_3, 1. \end{cases}$$

We have only five derivations of type 3 on M. They are as follows:

$$D_1^3 = 0, \quad D_2^3 = I, \quad D_3^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_2, x_3 \\ x_1 & \text{if } x = 1, \end{cases}$$

$$D_4^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_2 \\ x_1 & \text{if } x = x_3 \\ x_2 & \text{if } x = 1 \end{cases} \quad \text{and} \quad D_5^3(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_1 & \text{if } x = x_2 \\ x_2 & \text{if } x = x_3 \\ x_3 & \text{if } x = 1 \end{cases}$$

Let M_1 and M_2 be two MV-algebras. Then $M_1 \times M_2$ is an MV-algebra. Also, let D_1 and D_2 be derivations of type 1 (2 and 3, respectively) on M_1 and M_2, respectively. Then, $D = D_1 \times D_2 : M_1 \times M_2 \longrightarrow M_1 \times M_2$ defined by $D((x, y)) = (D_1(x), D_2(y))$, for all $x \in M_1$, $y \in M_2$, is a derivation of type 1 (2 and 3, respectively). But, all of derivations of type 1 (2 and 3, respectively) on $M_1 \times M_2$ are not as form $D_1 \times D_2$, where D_1 and D_2 are derivations of type 1 (2 and 3, respectively) on M_1 and M_2, respectively. The following example shows this matter.

Example 3.8. Consider the MV-algebra M, defined in Example 3.6. Then $M \cong S_1 \times S_1$, where S_1 is the MV-algebra defined in Example 3.3. By Example 3.6, $M \cong S_1 \times S_1$ has nine derivations of type 1. But, only four derivations of them are as form $D_1 \times D_2$, where D_1 and D_2 are derivations of type 1 on S_1, since S_1 has two derivations of type 1. They are D_1^1, D_2^1, D_4^1 and D_6^1.

Definition 3.9. Let M be an MV-algebra. Then, a function $f : M \longrightarrow M$ is called *additive*, if $f(x \oplus y) = f(x) \oplus f(y)$, for all $x, y \in M$.

Example 3.10. The functions $D = 0$ and $D = I$ are always additive. In Examples 3.3, 3.4, 3.5 and 3.7, among derivations of type i, $1 \leq i \leq 3$, only $D = 0$ and $D = I$ are additive. In Example 3.6, among derivations of type i, $1 \leq i \leq 3$, only $D_1^1 = D_1^3 = 0$, $D_2^1 = D_2^3 = I$, $D_4^1 = D_3^3$ and $D_6^1 = D_4^3$ are additive.

Definition 3.11. Let M be an MV-algebra. Then, a function $f : M \longrightarrow M$ is called *isoton*, if $x \leq y$ implies that $f(x) \leq f(y)$, for all $x, y \in M$.

Example 3.12. The functions $D = 0$ and $D = I$ are always isoton. In Example 3.4, among derivations of type i, $1 \leq i \leq 3$, only $D_1^1 = D_1^2 = D_1^3 = 0$, $D_2^2 = D_2^3 = I$, D_4^2 and D_3^3 are isoton. In Example 3.5, among derivations of type i, $1 \leq i \leq 3$, only

$D_1^1 = D_1^2 = D_1^3 = 0$, $D_2^2 = D_2^3 = I$, D_5^2, D_8^2, D_3^3 and D_4^3 are isoton. In Example 3.6, among derivations of type i, $1 \le i \le 3$, only $D_1^1 = D_1^3 = 0$, $D_2^1 = D_2^3 = I$, $D_4^1 = D_3^3$ and $D_6^1 = D_4^3$ are isoton. In Example 3.7, among derivations of type i, $1 \le i \le 3$, only $D_1^1 = D_1^2 = D_1^3 = 0$, $D_2^2 = D_2^3 = I$, D_6^2, D_{12}^2, D_{16}^2, D_3^3, D_4^3 and D_5^3 are isoton.

Example 3.13. Let $S_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\}$, $n \in N$. Then, $(S_n, \oplus, *, 0, 1)$ is an MV-algebra with $n+1$ elements, where operations $\oplus$ and $*$ are defined as Example 2.3. Note that auxiliary operations $\odot$, $\ominus$, $\vee$ and $\wedge$ are as follows:

$$\begin{aligned} a \odot b &= \max\{0, a+b-1\}, \\ a \ominus b &= \max\{0, a-b\}, \\ a \vee b &= \max\{a, b\}, \\ a \wedge b &= \min\{a, b\} \end{aligned}$$

and the relation $\le$ is simply the natural ordering of real numbers. The MV-algebras defined in Examples 3.3, 3.4, 3.5 and 3.7 are S_1, S_2, S_3 and S_4, respectively. Let $n > 1$ be a fix positive integer. Define $D^1 : S_n \longrightarrow S_n$ by

$$D^1(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{n-1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

It is easily to check that D^1 is a derivation of type 1. D^1 is not additive because $D^1(1 \oplus \frac{n-1}{n}) = D^1(1) = 0$ but $D^1(1) \oplus D^1(\frac{n-1}{n}) = 0 \oplus \frac{1}{n} = \frac{1}{n}$. Also, D^1 is not isoton, because $\frac{n-1}{n} \leqslant 1$ but $\frac{1}{n} = D^1(\frac{n-1}{n}) \nleq D^1(1) = 0$.

Define $D^2 : S_n \longrightarrow S_n$ by

$$D^2(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

It is easily to check that D^2 is a derivation of type 2. D^2 is not additive because $D^2(1 \oplus \frac{1}{n}) = D^2(1) = 0$ but $D^2(1) \oplus D^2(\frac{1}{n}) = 0 \oplus \frac{1}{n} = \frac{1}{n}$. Also, D^2 is not isoton, because $\frac{1}{n} \leqslant 1$ but $\frac{1}{n} = D^2(\frac{1}{n}) \nleq D^2(1) = 0$.

Define $D^3 : S_n \longrightarrow S_n$ by

$$D^3(x) = \begin{cases} \frac{1}{n} & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easily to check that D^3 is a derivation of type 3. D^3 is not additive because $D^3(1 \oplus 1) = D^3(1) = \frac{1}{n}$ but $D^3(1) \oplus D^3(1) = \frac{2}{n}$. Note that D^3 is isoton.

4. f-Derivations and (f, g)-Derivations of MV-Algebras

In this section, we introduce the notion of f-derivations and (f, g)-derivations of type i, $1 \leq i \leq 3$, of MV-algebras.

Definition 4.1. Let M be an MV-algebra and $f, g : M \longrightarrow M$ be homomorphisms. A function $D : M \longrightarrow M$ is called

(1) an (f, g)-*derivation of type* 1, if $D(x \odot y) = (D(x) \odot f(y)) \oplus (g(x) \odot D(y))$, for all $x, y \in M$;
(2) an (f, g)-*derivation of type* 2, if $D(x \wedge y) = (D(x) \wedge f(y)) \vee (g(x) \wedge D(y))$, for all $x, y \in M$;
(3) an (f, g)-*derivation of type* 3, if $D(x \ominus y) = (D(x) \ominus f(y)) \wedge (g(x) \ominus D(y))$, for all $x, y \in M$.

In the above definition, if the function g is equal to the function f, then an (f, g)-derivation of type 1 (2 and 3, respectively) is called an f-derivation of type 1 (2 and 3, respectively). It is obvious that if we choose the functions f and g as the identity functions, then the (f, g)-derivation of type 1 (2 and 3, respectively) is ordinary derivation of type 1 (2 and 3, respectively).

Theorem 4.2. *Let M be an MV-algebra and f, g be homomorphisms on M. Also, let D be an (f, g)-derivation of type 1 and 3 on M. Then, for all $x, y \in M$*

$$((D(x) \ominus f(y)) \wedge (g(x) \ominus D(y))) \leq ((D(x) \odot f(y^*)) \oplus (g(x) \odot D(y^*))) .$$

Proof. We have

$$\begin{aligned}
&((D(x) \ominus f(y)) \wedge (g(x) \ominus D(y)))^* \oplus ((D(x) \odot f(y)^*) \oplus (g(x) \odot D(y^*))) \\
&= ((D(x) \ominus f(y))^* \vee (g(x) \ominus D(y))^*) \\
&\quad \oplus ((D(x) \ominus f(y)) \oplus (g(x) \ominus D(y^*)^*)) \\
&= ((D(x) \ominus f(y))^* \oplus (D(x) \ominus f(y)) \oplus (g(x) \ominus D(y^*)^*)) \\
&\quad \vee ((g(x) \ominus D(y))^* \oplus (D(x) \ominus f(y)) \oplus (g(x) \ominus D(y^*)^*)) = 1.
\end{aligned}$$

So, the statement is valid. □

Let M be an MV-algebra and $f, g : M \longrightarrow M$ be homomorphisms. A function $D : M \longrightarrow M$ is an (f, g)-derivation of type 1 (2, respectively) if and only if it is an (g, f)-derivation of type 1 (2, respectively).

Example 4.3. Let M be an MV-algebra and $f, g : M \longrightarrow M$ be homomorphisms on M. The function $D : M \longrightarrow M$ defined by $D = 0$ is an (f,g)-derivation of type 1, 2 and 3.

Example 4.4. For every MV-algebra, if we set $D = f = I$ and $g = 0$, then f, g are homomorphisms and D is (f,g)-derivation of type 1 and 2.

Example 4.5. Let M be as in Example 3.6. Then, every (f,g)-derivation (f-derivation, respectively) of type 1 on M is coincide with (f,g)-derivation (f-derivation, respectively) of type 2 on M. Define maps $f, g : M \longrightarrow M$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x_2 & \text{if } x = x_1 \\ x_1 & \text{if } x = x_2 \\ 1 & \text{if } x = 1 \end{cases} \quad \text{and } g(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ 1 & \text{if } x = x_2, 1. \end{cases}$$

Then, f and g are homomorphisms. Now, we define $D_1, D_2 : M \longrightarrow M$ by

$$D_1(x) = \begin{cases} 0 & \text{if } x = 0, x_1, 1 \\ x_1 & \text{if } x = x_2 \end{cases} \quad \text{and } D_2(x) = \begin{cases} 0 & \text{if } x = 0, x_1 \\ x_1 & \text{if } x = x_2, 1. \end{cases}$$

It is easily to check that D_1 is an f-derivation and an (f,g)-derivation of type 1 of M. But, it is not an f-derivation of type 3, because $x_1 = D_1(1 \ominus x_1) \neq (D_1(1) \ominus f(x_1)) \wedge (g(1) \ominus D_1(x_1)) = 0$. Similarly, one can show that D_1 is not an (f,g)-derivation of type 3. Note that D_1 is not additive. Also, it is not isotone. D_2 is additive, isotone and an f-derivation of type 1 and 3. Also, it is an (f,g)-derivation of type 1 and 3.

Example 4.6. Let $M = \{0, x_1, x_2, x_3, x_4, 1\}$. Consider the following tables:

$\oplus$	0	x_1	x_2	x_3	x_4	1
0	0	x_1	x_2	x_3	x_4	1
x_1	x_1	x_3	x_4	x_3	1	1
x_2	x_2	x_4	x_2	1	x_4	1
x_3	x_3	x_3	1	x_3	1	1
x_4	x_4	1	x_4	1	1	1
1	1	1	1	1	1	1

$*$	0	x_1	x_2	x_3	x_4	1
	1	x_4	x_3	x_2	x_1	0

Then, $(M, \oplus, *, 0)$ is an MV-algebra. Define maps $f, g : M \longrightarrow M$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_3 \\ 1 & \text{if } x = x_2, x_4, 1 \end{cases} \quad \text{and } g = I.$$

Then, f, g are homomorphisms on M. Now, we define

$$D_1(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_3, x_4, 1 \\ x_2 & \text{if } x = x_2, \end{cases}$$

$$D_2(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_3, 1 \\ x_2 & \text{if } x = x_2, x_4 \end{cases}$$

$$D_3(x) = \begin{cases} 0 & \text{if } x = 0, x_1, x_3 \\ x_2 & \text{if } x = x_2, x_4, 1. \end{cases}$$

D_1 is an f-derivation and an (f,g)-derivation of type 2. But, it is not an f-derivation of type 1, since $x_2 = D_1(x_2) = D_1(x_4 \odot x_4) \neq (D_1(x_4) \odot f(x_4)) \oplus (f(x_4) \odot D_1(x_4)) = 0$. Also, it is not an (f,g)-derivation of type 1, since $x_2 = D_1(x_2) = D_1(x_4 \odot x_4) \neq (D_1(x_4) \odot f(x_4)) \oplus (g(x_4) \odot D_1(x_4)) = 0$. D_1 is not an f-derivation of type 3, since $x_2 = D_1(1 \ominus x_3) \neq (D_1(1) \ominus f(x_3)) \wedge (f(1) \ominus D_1(x_3)) = 0$. Also, D_1 is not an (f,g)-derivation of type 3, since $x_2 = D_1(x_2) = D_1(x_4 \ominus x_1) \neq (D_1(x_4) \ominus f(x_1)) \wedge (g(x_4) \ominus D_1(x_1)) = 0$.

D_2 is an f-derivation of type 1 and 2. Also, it is an (f,g)-derivation of type 1 and 2. But, it is not f-derivation of type 3, since $x_2 = D_2(1 \ominus x_1) \neq (D_2(1) \ominus f(x_1)) \wedge (f(1) \ominus D_2(x_1)) = 0$. Also, D_2 is not an (f,g)-derivation of type 3, since $x_2 = D_2(1 \ominus x_1) \neq (D_2(1) \ominus f(x_1)) \wedge (g(1) \ominus D_2(x_1)) = 0$.

D_3 is an f-derivation and an (f,g)-derivation of type 1, 2 and 3.

The properties of f-derivation and (f,g)-derivation of type 2 (3, respectively) on MV-algebras is similar to the properties of f-derivation and (f,g)-derivation on lattices ($BCI \setminus BCK$-algebras, respectively). For more details, we refer reader to [1, 5] ([10, 12, 13, 22], respectively). So, we study the properties of f-derivation and (f,g)-derivation of type 1 on MV-algebras. We prove next theorems only for (f,g)-derivations of type 1. Putting the function g equal to the function f, then the results are satisfied for f-derivations of type 1.

In sequence, by an (f,g)-derivation we mean an (f,g)-derivation of type 1.

Theorem 4.7. *Let M be an MV-algebra and D be an (f,g)-derivation on M. Then, the following conditions hold:*

(1) $D(0) = 0$;

(2) $D(x) \odot f(x^*) = f(x) \odot D(x^*) = D(x) \odot g(x^*) = g(x) \odot D(x^*) = 0$;

(3) $D(x) \leq f(x), g(x)$;

(4) $D(x) = D(x) \oplus (g(x) \odot D(1))$;

for all $x, y \in M$.

Proof. (1) If $x \in M$, then

$$D(0) = D(x \odot 0) = (D(x) \odot f(0)) \oplus (g(x) \odot D(0)) = g(x) \odot D(0).$$

Putting $x = 0$, we obtain $D(0) = g(0) \odot D(0) = 0 \odot D(0) = 0$.

(2) If $x \in M$, then by Theorem 4.6 (3), we obtain $0 = D(0) = D(x \odot x^*) = (D(x) \odot f(x^*)) \oplus (g(x) \odot D(x^*))$. By Theorem 2.5 (1), we obtain $D(x) \odot f(x^*) = 0$ and $g(x) \odot D(x^*) = 0$. Similarly, we can prove $f(x) \odot D(x^*) = 0$ and $D(x) \odot g(x^*) = 0$.

(3) Since f and g are homomorphisms, by using (2), we have $D(x) \odot f(x)^* = D(x) \odot g(x)^* = 0$. Now, Theorem 2.6 implies that $D(x) \leq f(x), g(x)$.

(4) $D(x) = D(x \odot 1) = (D(x) \odot f(1)) \oplus (f(x) \odot D(1)) = D(x) \oplus (f(x) \odot D(1))$. □

Lemma 4.8. *Let M be an MV-algebra, D be an (f,g)-derivation on M such that f, g be isomorphisms and I be an ideal of M. Then, $D(I) \subseteq f(I) \cap g(I)$.*

Proof. If $y \in D(I)$, then there is $x \in I$ such that $y = D(x)$. Now, by Theorem 4.7 (3), we obtain $y = D(x) \leq f(x) \in f(I)$ and $y = D(x) \leq g(x) \in g(I)$. Since I is an ideal, by Lemma 2.9, $f(I)$ and $g(I)$ are ideals, too. Thus, $y \in f(I) \cap g(I)$. Therefore, $D(I) \subseteq f(I) \cap g(I)$. □

Theorem 4.9. *Let D be an (f,g)-derivation of an MV-algebra M and $x, y \in M$. If $x \leq y$, then the following hold:*

(1) $D(x \odot y^*) = 0$;

(2) $D(x) \leq f(y), g(y)$ *and* $D(y^*) \leq f(x)^*, g(x)^*$;

(3) $D(x) \odot D(y^*) = 0$.

Proof. (1) Suppose that $x \leq y$. Then, by Theorem 2.7, we have $x \odot y^* = 0$. Now, by Theorem 4.7 (1), we obtain $D(x \odot y^*) = D(0) = 0$.

(2) According to (1), we have $0 = D(x \odot y^*) = (D(x) \odot f(y^*)) \oplus (g(x) \odot D(y^*))$. Now, by Theorem 2.5, we have $D(x) \odot f(y^*) = 0$ and $g(x) \odot D(y^*) = 0$. Then, by Theorem 2.7, $D(x) \leq f(y), D(y^*) \leq g(x)^*$. Moreover, $0 = D(y^* \odot x) = (D(y^*) \odot f(x)) \oplus (g(y^*) \odot D(x))$. Hence, $D(y^*) \odot f(x) = 0$ and $g(y^*) \odot D(x) = 0$. Therefore, by using Theorem 2.5, we

get $D(y^*) \leq f(x)^*$ and $D(x) \leq g(y)$.

(3) Since f is a homomorphism, $x \leq y$ implies that $f(x) \leq f(y)$. By Theorem 4.7 (3), we have $D(x) \leq f(x) \leq f(y)$. Then, $D(x) \odot D(y^*) \leq f(y) \odot D(y^*) \leq f(y) \odot f(y^*) = f(y) \odot f(y)^* = 0$. Therefore, $D(x) \odot D(y^*) = 0$. □

Theorem 4.10. *Let M be an MV-algebra and D be an (f, g)-derivation on M. Then, the following hold:*

(1) $D(x) \odot D(x^*) = 0$;

(2) $D(x^*) = D(x)^*$ *if and only if* $D(x) = f(x)$ *or* $D(x) = g(x)$.

Proof. (1) Since $x \leq x$, by putting $y = x$ in Theorem 4.9, we get (1).

(2) Let $D = f$. We have $f(x^*) = f(x)^*$, for all $x \in M$, since f is a homomorphism. Hence, $D(x^*) = D(x)^*$.

Conversely, let $D(x^*) = D(x)^*$. By Theorem 4.7 (2), $D(x) \odot D(x^*) = 0$ which implies that $f(x) \odot D(x)^* = 0$. Hence, $f(x) \leq D(x)$. On the other hand, by Theorem 4.7 (3), we have $D(x) \leq f(x)$. Therefore, $D(x) = f(x)$. Similarly, we can prove that if $D(x^*) = D(x)^*$, then $D(x) = g(x)$. □

Proposition 4.11. *Let M be an MV-algebra and D be an (f, g)-derivation of M. If $D(x^*) = D(x)$, for all $x \in M$, then the following conditions hold:*

(1) $D(1) = 0$;

(2) $D(x) \odot D(x) = 0$;

(3) *If D is isotone, then $D = 0$.*

Proof. (1) By Theorem 4.7 (1), we have $D(1) = D(0^*) = D(0) = 0$.

(2) It follows from Theorem 4.10 (1).

(3) Since $x \leq 1$, for all $x \in M$, and D is isotone, we have $D(x) \leq D(1) = 0$, for all $x \in M$. Therefore, $D = 0$. □

Proposition 4.12. *Let M be an MV-algebra and D be a non-zero additive (f, g)-derivation of M. Then, $D(B(M)) \subseteq B(M)$.*

Proof. Suppose that $y \in D(B(M))$. Then, there exists $x \in B(M)$ such that $y = D(x)$. So, $y \oplus y = D(x) \oplus D(x) = D(x \oplus x) = D(x) = y$. Therefore, $y \in B(M)$. □

Theorem 4.13. *Let D be an additive (f, g)-derivation of a linearly ordered MV-algebra M. Then, either $D = 0$ or $D(1) = 1$.*

Proof. Suppose that D is an additive (f,g)-derivation of a linearly ordered MV-algebra M and $D(1) \neq 1$. Then, for all $x \in M$, we have $D(1) = D(x \oplus x^*) = D(x) \oplus D(x^*)$. On the other hand $D(1) = D(x \oplus 1) = D(x) \oplus D(1)$. Therefore, $D(1) = D(x) \oplus D(x^*) = D(x) \oplus D(1)$. Hence, by the additive cancellative law of MV-algebras, $D(x^*) = D(1)$, since $D(1) \neq 1$. By putting $x = 1$, we get $0 = D(0) = D(1)$. So, for all $x \in M$, $0 = D(1) = D(x \oplus 1) = D(x) \oplus D(1) = D(x)$. Therefore, $D = 0$. □

Theorem 4.14. *Let M be a linearly ordered MV-algebra and g be an isomorphism. Also, let D_1, D_2 be additive (f,g)-derivations of M. We define $D_1D_2(x) = D_1(D_2(x))$, for all $x \in M$. If $D_1D_2 = 0$, then $D_1 = 0$ or $D_2 = 0$.*

Proof. Suppose that $D_1D_2 = 0$ and $D_2 \neq 0$. Then, by Theorems 4.7 (4) and 4.13, for all $x \in M$, we obtain

$$\begin{aligned} 0 &= D_1D_2(x) = D_1(D_2(x)) = D_1(D_2(x) \oplus (g(x) \odot D_2(1))) \\ &= D_1D_2(x) \oplus D_1(g(x) \odot D_2(1)) = D_1D_2(x) \oplus D_1(g(x)) = D_1(g(x)). \end{aligned}$$

Thus, $D_1(g(x)) = 0$, for all $x \in M$. Hence, $D_1(x) = 0$, for all $x \in M$, since g is an isomorphism. Therefore, $D_1 = 0$. □

Theorem 4.15. *Let M be a linearly ordered MV-algebra and D be a non-zero additive (f,g)-derivation of M. Then, $D(x \odot x) = (D(x) \odot f(x)) \oplus g(x)$.*

Proof. By Theorem 4.7 (4), we have $D(x) = D(x) \oplus (g(x) \odot D(1))$, for all $x \in M$. By Theorem 4.13, $D(1) = 1$, since $D \neq 0$. Therefore $D(x) = D(x) \oplus g(x)$. Thus, by Theorem 2.5 (3), we have $D(x) \odot g(x) = g(x)$. Then,

$$D(x \odot x) = (D(x) \odot f(x)) \oplus (g(x) \odot D(x)) = (D(x) \odot f(x)) \oplus g(x),$$

and the proof completes. □

Theorem 4.16. *Every non-zero additive (f,g)-derivation of a linearly ordered MV-algebra M is isotone.*

Proof. Let D be a non-zero additive (f,g)-derivation of a linearly ordered MV-algebra M and $x, y \in M$ be arbitrary. If $x \leq y$, then $x^* \oplus y = 1$. Now, by Theorem 4.13, $D(1) = 1$, since $D \neq 0$. Therefore, $1 = D(1) = D(x^* \oplus y) = D(x^*) \oplus D(y)$ which implies that $(D(x^*))^* \leq D(y)$. On the other hand, by Theorem 4.7 (3), $D(x^*) \leq (f(x))^*$ implies that $f(x) \leq (D(x^*))^*$. So, $f(x) \leq (D(x^*))^* \leq D(y)$. Also, we have $D(x) \leq f(x)$, by Theorem 4.7 (3). Therefore, $D(x) \leq f(x) \leq D(y)$ which implies that $D(x) \leq D(y)$. □

Theorem 4.17. *Let M be a linearly ordered MV-algebra and D be a non-zero additive (f,g)-derivation. Then,*

$$D^{-1}(0) = \{x \in M : D(x) = 0\}$$

is an ideal of M.

Proof. By Theorem 4.7 (1), we have $D(0) = 0$. Then, $0 \in D^{-1}(0)$. Now, suppose that $x, y \in D^{-1}(0)$. Then, $D(x \oplus y) = D(x) \oplus D(y) = 0 \oplus 0 = 0$ which implies that $x \oplus y \in D^{-1}(0)$. Now, suppose that $x \in D^{-1}(0)$ and $y \leq x$. Then, $D(x) = 0$. Hence, by Theorem 4.16, we have $D(y) \leq D(x) = 0$ which implies that $D(y) = 0$. Therefore, $y \in D^{-1}(0)$. □

References

1. M. Aşci, O. Kecilioglu and B. Davvaz, On symetric f bi-derivations of lattices, *J. Combin. Math. Combin. Comput.* **83** (2012), 243–253.
2. N. O. Alshehri, Derivations of MV-algebras, *Int. J. Math. Math. Sci.* (2010), Art. ID 312027, 7 pp.
3. H. E. Bell and L.-C. Kappe, Rings in which derivations satisfy certain algebraic conditions, *Acta Math. Hungar.* **53** (1989), 339–346.
4. S. Ceran and M. Aşci, Symmetric bi-(σ, τ) derivations of prime and semi prime gamma rings, *Bull. Korean Math. Soc.* **43** (2006), 9–16.
5. Y. Çeven and M.A. Öztürk, On f-derivations of lattices, *Bull. Korean Math. Soc.* **45** (2008), 701–707.
6. Y. Çeven, Symmetric bi-derivations of lattices, *Quaest. Math.* **32** (2009), 241–245.
7. C. C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* **88** (1958), 467–490.
8. R. Cignoli, I. Do˒Ottaviano and D. Mundici, Algebraic foundations of many-valued reasoning, Kluwer Academic, Dodrecht, The Netherlands, 2000.
9. B. Davvaz, L. Zareyan and V. Leoreanu-Fotea, $(3,3)$-ary differential rings, *Mediterr. J. Math.* **9** (2012), 357–378.
10. Y. B. Javed, M. Aslam, A note on f-derivations of BCI-algebras, *Commun. Korean Math. Soc.* **24** (2009), 321–331.
11. Y. B. Jun and X. L. Xin, On derivations of BCI-algebras, *Inform. Sci.* **159** (2004), 167–176.
12. F. Nisar, Characterization of f-derivations of BCI-algebras, *EAMJ* **25** (2009), 69–87.
13. F. Nisar, On f-derivations of BCI-algebras, *J. Prime Res. Math.* **5** (2009), 176–191.

14. S. A. Ozbal and A. Firat, Symmetric f bi derivations of lattices, *ARS Combinatoria* **97** (2010), 471–477.

15. M. A. Ozturk, H. Yazarh and K.H. Kim, Permuting tri-derivations in lattices, *Quaest. Math.* **32** (2009), 415-425.

16. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093–1100.

17. S. Rasouli, D. Heidari and B. Davvaz, η-relations and transitivity conditions of η on hyper MV-algebra, *J. Mult.-Valued Logic Soft Comput.* **15** (2009), 517–524.

18. S. Rasouli and B. Davvaz, Roughness in MV-algebras, *Inform. Sci.* **180** (2010), 737–747.

19. S. Rasouli and B. Davvaz, Homomorphisms, ideals and binary relations on hyper-MV algebras, *J. Mult.-Valued Logic Soft Comput.* **17** (2011), 47–68.

20. G. Szász, Derivations of lattices, *Acta Sci. Math.* **37** (1975), 149–154.

21. X. L. Xin, T. Y. Li and J. H. Lu, On derivations of lattices, *Inform. Sci.* **178** (2008), 307–316.

22. J. Zhan, Y. L. Liu, On f-derivations of BCI-algebras, *Int. J. Math. Math. Sci.* **11** (2005), 1675–1684.

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