MULTIPLICATION MODULES THAT ARE FINITELY GENERATED

Y. TOLOOEI

ABSTRACT. Let $R$ be a commutative ring with identity and $M$ be a unitary $R$-module. An $R$-module $M$ is called a multiplication module if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is shown that over a Noetherian domain $R$ with $\dim(R) \leq 1$, multiplication modules are cyclic or isomorphic to an invertible ideal of $R$. Moreover, we give a characterization of finitely generated multiplication modules.

1. Introduction

All rings in this article are commutative with identity and modules are unitary. For a ring $R$, we denote by $\dim(R)$ the classical Krull dimension of $R$ and for a module $M$, we denote by $\text{Ann}(M)$ the annihilator of $M$. A ring $R$ is called semilocal whenever $R/J(R)$ is a semisimple ring. A module $M$ is called multiplication module whenever for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. Examples of multiplication modules are every ring, every cyclic module, every ideal in a Dedekind domain \cite[page 38]{7} and every ideal in a regular ring. In \cite{4}, equivalent conditions for multiplication modules to be finitely generated is given, and here we give more equivalent conditions for finitely generated multiplication modules. Multiplication modules are generalized to the non-commutative ring by Tuganbaev \cite{9}. Recently in \cite{1}, rings in which every ideal is multiplication as a multiplication ring, is studied. Also, Perez et al. \cite{6},

---

Keywords: Multiplication module, Noetherian ring, faithful module.
Received: 14 July 2019, Accepted: 19 October 2019.
*Corresponding author.
generalized the multiplication modules, Azizi and Jayaram work on
the multiplication module which they called principal multiplication
modules [2] and Smith [8] work on the fully invariant multiplication
modules. We bring here some results from [3] and [4] which will be
useful throughout the paper.

**Theorem 1.1.** [3, Theorem 4.] Let $R$ be a semi-local ring. Then an
$R$-module $M$ is a multiplication module if and only if it is cyclic.

**Lemma 1.2.** [4, Lemma 3.6] Let $R$ be a domain and $M$ a faithful
multiplication $R$-module. Then there exists an invertible ideal $I$ of $R$
such that $M \cong I$.

**Proposition 1.3.** [4, Proposition 3.4.] Let $M$ be a faithful multiplica-
tion $R$-module. Then $M$ is finitely generated $R$-module if and only if
$M \neq PM$ for all minimal prime ideals $P$ of $R$.

## 2. Multiplication modules

Let $M$ be an $R$-module. Torsion subset of $M$, denoted by $T(M)$,
is defined as $T(M) = \{x \in M | \text{Ann}(x) \neq 0 \}$. Note that $T(M)$ is not
necessary a submodule of $M$, unless $R$ is a domain. Recall that a ring
$R$ is a Dedekind ring if and only if every ideal of $R$ is invertible. Also
the Krull dimension of Dedekind domain is less than 1. In the follow-
ning theorem, we characterize multiplication modules over a Noetherian
domain $R$ with the Krull dimension less than or equal to 1.

**Theorem 2.1.** Let $R$ be a Noetherian domain with $\dim(R) \leq 1$. Then
$M$ is a multiplication $R$-module if and only if either $M \cong I$ for some
invertible ideal $I$ of $R$ or $M$ is a cyclic $R$-module.

**Proof.** ($\Rightarrow$). Let $M$ be a multiplication $R$-module. We consider the
following two cases:

Case 1: If $T(M) = 0$, then by Lemma 1.2, $M \cong I$ for some invertible
ideal $I$ of $R$.

Case 2: Let $T(M) \neq 0$. We claim that $\text{Ann}(M) \neq 0$. Let $0 \neq m \in T(M)$, then $K = \text{Ann}(m) \neq 0$ and also $Rm = LM$ for some non-
zero ideal $L$ of $R$. Thus $KLM = (0)$ and since $R$ is a domain, $KL \neq (0)$
and so $\text{Ann}(M) \neq 0$ (i.e., $M = T(M)$). Let $I = \text{Ann}(M)$. Since $R$ is
Noetherian, $R/I$ is also a Noetherian ring with $\dim(R/I) = 0$, i.e $R/I$
is an Artinian ring. On the other hand, $M$ is an $R/I$-multiplication
module. Thus by Theorem 1.1, $M$ is a cyclic $R/I$-module. Hence it is
a cyclic $R$-module, as desired.

($\Leftarrow$). It is evident. \qed
Theorem 2.2. Let $M$ be a multiplication $R$-module. For every maximal ideal $P$ of $R$, $PM \neq M$ if and only if $M_P \neq 0$. Furthermore, for every prime ideal $P$ of $R$, $M_P \neq 0$ implies $PM \neq M$.

Proof. ($\Rightarrow$). Let $PM \neq M$ and $x \in M \setminus PM$. If $\text{Ann}(x) \nsubseteq P$ then $Rx = (\text{Ann}(x) + P)x = Px \subseteq PM$, which is a contradiction. Hence $\text{Ann}(x) \subseteq P$ and it conclude that $0 \neq R_p x \subseteq M_P$. ($\Leftarrow$). Let $M_P \neq 0$ and $0 \neq x \in M_P$, hence $\text{Ann}(x) \subseteq P$. Suppose that $Rx = IM$ for some ideal $I$ of $R$. Now, if $PM = M$ then 

$$Rx = IM = I(PI) = P(IM) = P(Rx) = Px.$$ 

So $(1 - p)x = 0$ for some $p \in P$ and hence $(1 - p) \in \text{Ann}(x) \subseteq P$, but this is a contradiction. Therefore $PM \neq M$. \hfill \Box

Theorem 2.3. Let $M$ be a multiplication $R$-module. If $M_P \neq 0$ for every maximal (minimal) ideal $P$ of $R$, then $M$ is finitely generated.

Proof. If $M_P \neq 0$ for every maximal (minimal) ideal $P$ of $R$, by Theorem 2.2, $PM \neq M$ and by [4, Theorem 3.1] (Proposition 1.3), $M$ is a finitely generated $R/\text{Ann}(M)$-module, so $M$ is a finitely generated $R$-module. \hfill \Box

Recall that an $R$-module $M$ is called locally free (locally cyclic) if $M_P$ is a free (cyclic) $R_P$-module for any maximal (prime) ideal $P$ of $R$ (See [7, Exercise 2.21]). Note that for every $R$-module $M$ if $I = \text{Ann}(M)$, then $M$ is a faithful $R/I$-module, hence we can assume that every multiplication module is a faithful module. Barnard [3] proved that if $M$ is finitely generated $R$-module, then $M$ is a multiplication module if and only if $M$ is a locally cyclic $R$-module [3, Proposition 5]. The following lemma shows that a faithful multiplication module is locally free. We note that the following lemma exists in the literature, however we give here its proof for the sake of completeness and convenience of the reader.

Lemma 2.4. If $M$ is a faithful multiplication $R$-module, then $M$ is a locally free $R$-module.

Proof. Let $P$ be a maximal ideal of $R$. If $M_P = 0$ then it is free with empty basis. So, let $M_P \neq 0$. Thus by Theorem 2.2, $PM \neq M$. 


Let $x \in M \setminus PM$. Therefore there exists an ideal $I$ of $R$ such that $IM = Rx$. It is easy to see that $I \not\subseteq P$ thus $I_P = R_P$ and we have,

$$R_Px = R_P(IM)_P = R_PI_PM_P = R_PM_P = M_P.$$  

We shall show that $\text{Ann}_{R_P}(x) = 0$. Suppose that $a/b \in \text{Ann}_{R_P}(x)$. So

$$ax/b = 0 \implies \exists u \in R \setminus P \text{ such that } uax = 0.$$  

Hence $0 = Ruax = Rua(IM) = (Iua)M$. Since $M$ is faithful, $Iua = 0$ and so $0 = I_Pua = R_Pua = R_pa$. Hence $a/b = 0$, as wanted. 

\[ \square \]

**Theorem 2.5.** Let $M$ be a faithful multiplication $R$-module. Then the following statements are equivalent:

(a) $M$ is finitely generated.
(b) $M_P \neq 0$ for every prime ideal $P$ of $R$.
(c) $M_P \neq 0$ for all maximal ideal $P$ of $R$.
(d) $M_p \cong R_p$ for every $P \in \text{Spec}(R)$.
(e) $\text{Hom}(M, R/P) \neq 0$ for every maximal ideal $P$ of $R$.

**Proof.**

(a) $\Rightarrow$ (b) $\Rightarrow$ (d) Let $P$ be a prime ideal of $R$ and $M = Rx_1 + \cdots + Rx_n$. If $M_P = 0$, then there exists $s_1, \ldots, s_n \in R \setminus P$ such that $s_1x_1 = s_2x_2 = \cdots = s_nx_n = 0$. Thus $0 \neq s_1s_2\cdots s_n \in \text{Ann}(M)$, that is a contradiction. Therefore $M_P \neq 0$ for every prime ideal $P$ of $R$.

Hence, by Theorem 1.1 $M_P$ is cyclic $R_P$-module and by Lemma 2.4, $M_P$ is a free $R_P$-module and thus $M_P \cong R_P$.

(d) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (a) It follows by Theorem 2.3.

(c) $\Rightarrow$ (e) By Theorem 2.2, $PM \neq M$ for every maximal ideal $P$ of $R$. Therefore $\text{Hom}(M, R/P) \neq 0$.

(e) $\Rightarrow$ (c) Let $\text{Hom}(M, R/P) \neq 0$ for a maximal ideal $P$ of $R$. Then it is clear that $PM \neq M$ and thus by Theorem 2.2, we get $M_P \neq 0$. \[ \square \]

Note that there exists an example of a multiplication module that is not cyclic and satisfy in Theorem 2.5. Let $R$ be a domain and $Q$ be the fraction field of $R$. An ideal $I$ of $R$ is called *invertible ideal* whenever $IJ = R$ for some subset $J$ of $Q$. Let $I := \langle 3, 2 + \sqrt{-5} \rangle \subset \mathbb{Z}[\sqrt{-5}]$. It is easy to check that $3^{-1}(2 - \sqrt{-5}) \in I^{-1}$ and $II^{-1} = R$. Thus $I$ is an invertible ideal and so it is multiplication module which is not cyclic.

The following corollary is proved by Nauom [5] and it is extended to duo rings by Tuganbaev [10].

**Corollary 2.6.** Every faithful multiplication $R$-module $M$ is a flat $R$-module.
Proof. Let $M$ be a faithful multiplication $R$-module. By Lemma 2.4, $M$ is locally free and by [7, Exercise 4.14], $M$ is flat. □

**Corollary 2.7.** Every faithful multiplication module over a Noetherian ring is projective.

Proof. Note that by [4, Corollary 3.3], a faithful multiplication module over a Noetherian ring is a finitely generated module. Now, the result follows from Lemma 2.4 and [7, Exercise 4.15]. □

**Acknowledgments**

The author would like to thank the referee for his/her careful reading of the paper and valuable suggestions to improve the presentation of this paper.

**References**


**Yaser Tolooei**

Department of Mathematics, Faculty of Science, Razi University, Kermanshah, 67149-67346, Iran.
y.toloei@razi.ac.ir
MULTIPLICATION MODULES THAT ARE FINITELY GENERATED

Y. TOLOOEI

مدولهای ضریبی متناهی مولد

گروه ریاضی، دانشکده علوم، دانشگاه رازی، کرمانشاه، ایران

فرض کنید $R$ یک حلقه جایگاهی یک دار و $M$ یک $R$-مدول یکاتی باشد. $R$-مدول $M$ را ضریبی میدهیم که $I$ یک مانند هرگاه باشد $R$ یک ایدهآل از $N$ از $M$ داشته باشد به گونه‌ای که $\dim(R) \leq 1$. نتیجه می‌گیریم که برای یک ایده آل نوتری $R$ دارد، $IM = N$.

همین‌طور که در مقاله انتخاب شده، ما یک مشخصه‌سازی برای مدولهای ضریبی متناهی مولد ارائه خواهیم کرد.

کلمات کلیدی: مدول ضریبی، حلقه نوتری، مدول وفادار.