A NEW CHARACTERIZATION OF ABSOLUTELY PO-PURE AND ABSOLUTELY PURE S-POSETS

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Abstract. In this paper, we investigate po-purity using finitely presented $S$-posets, and give some equivalent conditions under which an $S$-poset is absolutely po-pure. We also introduce strongly finitely presented $S$-posets to characterize absolutely pure $S$-posets. Similar to the acts, every finitely presented cyclic $S$-posets is isomorphic to a factor $S$-poset of a pomonoid $S$ by a finitely generated right congruence on $S$. Finally, the relationships between regular injectivity and absolute po-purity are considered.

1. Introduction

A pomonoid $S$ is a monoid which it is also a poset whose partial order $\leq$ is compatible with the binary operation on $S$. A right $S$-poset $A_S$ is a right $S$-act $A_S$ equipped with a partial order $\leq$ and, in addition, for all $s, t \in S$ and $a, b \in A_S$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. A sub $S$-poset $B_S$ of a right $S$-poset $A_S$ is a subposet of $A_S$ that is closed under the $S$-action. In this case, $A_S$ is said to be an extension of $B_S$. Moreover, $S$-morphisms are the functions that preserve both the action and the order. The class of right $S$-posets and $S$-morphisms form a category, denoted by $\text{POS}-S$, which comprises the main background of this work. For an account on this category and categorical notions used in this paper, the reader is referred to [3]. An $S$-morphism $\iota : A_S \longrightarrow B_S$ is a regular monomorphism if and only if it is an order-embedding, i.e., $a \leq a' \iff \iota(a) \leq \iota(a')$, for all $a, a' \in A_S$.

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Let \( S \) be a pomonoid and \( I \) be a nonempty subset of \( S \). Then \( I \) is said to be a \textit{right ideal} of \( S \), if \( IS \subseteq S \) (not necessarily ordered right ideal). A \textit{right poideal} of a pomonoid \( S \) is a nonempty subset \( I \) of \( S \) which is both a right ideal \((IS \subseteq I)\) and a poset ideal (that is, \( a \leq b \) and \( b \in I \) imply \( a \in I \)).

Let \( A_S \) be a right \( S \)-poset. An \( S \)-poset \textit{congruence} \( \theta \) on \( A \) is a right \( S \)-act congruence with the property that the \( S \)-act \( A/\theta \) can be made into an \( S \)-poset in such a way that the natural map \( A_S \rightarrow A/\theta \) is an \( S \)-poset map. For an \( S \)-act congruence \( \theta \) on \( A \) we write \( a \leq_\theta a' \) if the so-called \( \theta \)-chain

\[
a \leq a_1 \theta b_1 \leq a_2 \theta b_2 \leq \ldots \leq a_n \theta b_n \leq a',
\]

for \( a \) to \( a' \) exists in \( A_S \), where \( a_i, b_i \in A \), \( 1 \leq i \leq n \). It can be shown that an \( S \)-act congruence \( \theta \) on a right \( S \)-poset \( A_S \) is an \( S \)-poset congruence if and only if \( a \theta a' \) whenever \( a \leq_\theta a' \leq_\theta a \). Let \( H \subseteq A \times A \). Then \( a \leq a_0(H) \ b \) if and only if \( a \leq b \) or there exist \( n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n \) such that

\[
a \leq c_1 s_1 \ d_1 s_1 \leq c_2 s_2 \ldots \ d_n s_n \leq b.
\]

The relation \( \nu(H) \) given by \( a \nu(H) b \) if and only if \( a \leq a_0(H) \ b \leq a_0(H) \ a \) is the \( S \)-poset congruence \textit{induced} by \( H \). Moreover, \([a]_{\nu(H)} = [b]_{\nu(H)} \) if and only if \( a \leq a_0(H) \ b \). The relation \( \theta(H) = \nu(H \cup H^{-1}) \) is the \( S \)-poset congruence \textit{generated} by \( H \). A congruence \( \rho \) on an \( S \)-poset \( A_S \) is called \textit{finitely induced} (finitely generated) if \( \rho = \nu(H) (\rho = \theta(H)) \) for some finite subset \( H \) of \( A \times A \).

Recall that an \( S \)-poset \( A_S \) is \textit{regular injective} if for each regular monomorphism \( g : B_S \rightarrow C_S \) and \( S \)-morphism \( f : B_S \rightarrow A_S \), there exists an \( S \)-morphism \( \overline{f} : C_S \rightarrow A_S \) such that \( \overline{f} g = f \). An \( S \)-poset \( A_S \) is \textit{weakly regular injective} (\( f g \)-weakly regular injective, \textit{principally weakly regular injective}) if every \( S \)-morphism \( f : I_S \rightarrow A_S \) from a (finitely generated, principal) right ideal \( I \) of \( S \) can be extended to an \( S \)-morphism \( \overline{f} : S_S \rightarrow A_S \). By a \textit{retract} of \( A_S \), we mean a sub \( S \)-poset \( B_S \) of \( A_S \) together with an \( S \)-morphism from \( A_S \) to \( B_S \) which maps \( B_S \) identically. Clearly, a retract of a regular injective \( S \)-poset is also regular injective. Moreover, \( A_S \) is called an \textit{absolute retract} if \( A_S \) is a retract of each of its extensions. In [8], it is shown that all regular injective \( S \)-posets are absolute retract. An \( S \)-poset \( E(A_S) \) is called a \textit{regular injective envelope} of an \( S \)-poset \( A_S \) if \( E(A_S) \) is regular injective and does not contain a proper sub \( S \)-poset \( B_S \) which is a regular injective extension of \( A_S \). In [8], it is proved that for each \( S \)-poset there exists a regular injective envelope. In light of [8, Proposition 2.2], the following corollary is clear which will be needed in the sequel.
Corollary 1.1. If $\rho$ is a congruence relation on $E(A_S)$ with $\rho \neq \Delta_{E(A_S)}$, then $\leq_{\rho} |A| \neq |A|$.

In the category of $S$-acts, absolutely pure acts were first considered by Normak [7] and then studied by Gould in [4]. Moreover, Gould introduced absolutely 1-pure acts under the name of almost pure acts in [5]. For $S$-posets, recently in [11], the authors generalized purity on $S$-acts into the theory of $S$-posets and introduced the properties of (1-)pure and absolutely (1-)pure $S$-posets regardless of their order. Then in [9], they introduced po-purity of $S$-posets and characterized absolutely 1-po-pure $S$-posets. In the following, we study strongly finitely presented cyclic $S$-posets. In Section 2, some general properties of po-purity and absolute-po-purity for $S$-posets are studied. Then, we investigate absolutely po-pure $S$-posets using finitely presented $S$-posets. Finally, the relationships between regular injectivity and absolute po-purity are discussed.

An $S$-poset $A_S$ is free on a set $X$ if and only if $A_S \cong \bigcup_{x \in X} xS$ where for all $x, y \in X$ and $s, t \in S$, $xs \leq yt$ if and only if $x = y$ and $s \leq t$. The concept of finitely presented $S$-poset was introduced in [2] which we recall it. It was mentioned by the notion of semi-finitely presented in [9]. An $S$-poset $A_S$ is said to be finitely presented if it is isomorphic to a quotient $S$-poset of a finitely generated free $S$-poset by a finitely induced $S$-poset congruence. In the category of $S$-acts, finitely presented $S$-acts was introduced as a factor $S$-act of finitely generated free $S$-acts by a finitely generated right congruence. Now, we define it in the category of $S$-posets as follows.

Definition 1.2. An $S$-poset $A_S$ is said to be strongly finitely presented if it is isomorphic to a quotient $S$-poset of a finitely generated free $S$-poset by a finitely induced $S$-poset congruence.

Proposition 1.3. Let $A_S$ be a cyclic $S$-poset. Then $A_S$ is strongly finitely presented if and only if it is isomorphic to a factor $S$-poset of $S_S$ by a finitely generated right congruence on $S$.

Proof. Necessity. Let $F_S$ be a free $S$-poset generated by $\{x_1, \ldots, x_n\}$ and let $\rho$ be a congruence on $F_S$ generated by

$$H = \{(x_{m_1}s_1, x_{n_1}t_1), \ldots, (x_{m_k}s_k, x_{n_k}t_k)\},$$

so that $F_S/\rho$ is cyclic. Assume that $F_S/\rho = [x_1 u]_{\rho} S$ for some $u \in S$. 

Let \( [x_i]_\rho = [x_1 u]_\rho \cdot z_i, \) \( z_i \in S, 1 \leq i \leq n. \) Set
\[
p_i = \begin{cases} 
  s_i & m_i = 1 \\
  u z_m, s_i & m_i \neq 1
\end{cases}
\]
and \( q_i = \begin{cases} 
  t_i & n_i = 1 \\
  u z_n, t_i & n_i \neq 1
\end{cases} \)
for every \( 1 \leq i \leq k. \) Consider the right congruence
\[
\sigma = \theta((p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k))
on S. \) We shall prove that \( F_S / \rho \cong S / \sigma, \) dividing the proof into three parts:

(a) First, we show that \( x_1 p_i \rho x_1 q_i \) for every \( 1 \leq i \leq k. \) If \( m_i = 1, \) clearly \( x_m s_i = x_1 p_i, \) otherwise using the equalities \( [x_m]_\rho = [x_1 u]_\rho \cdot z_m, \) we get that \( [x_m, s_i]_\rho = [x_m]_\rho \cdot s_i = [x_1 u z_m, s_i] = [x_1]_\rho \cdot p_i. \) This means that \( x_m s_i \rho x_1 p_i. \) Analogously one can prove that \( x_n t_i \rho x_1 q_i. \)

(b) Second, we show that if \( x_1 s \leq \rho x_1 t \) for some elements \( s, t \in S, \) then \( s \leq x \leq t. \) From \( x_1 s \leq \rho x_1 t \) it follows that either \( x_1 s \leq x_1 t \) and therefore \( s \leq t \) or there exist \( m \geq 1, c_i, d_i \in F_S, w_i \in S, 1 \leq i \leq m \) such that \( (c_i, d_i) \in H \cup H^{-1} \) and
\[
x_1 s \leq c_1 w_1 d_1 w_1 \leq c_2 w_2 \ldots d_m w_m \leq x_1 t.
\]
From the inequality \( x_1 s \leq c_1 w_1 \) we obtain that \( c_1 \in x_1 S. \) Then
\[
(c_1, d_1) = (x_m s_j, x_n t_j) \text{ or } (x_n t_j, x_m s_j).
\]
In the first case, \( m_j = 1 \) and \( s \leq s_j w_1. \) The second case implies \( n_j = 1, \) and so \( s \leq q_j w_1. \)

If \( d_1 = x_n t_j, \) from the inequality \( d_1 w_1 \leq c_2 w_2 \) we get that \( c_2 \in x_n j, \) and if \( d_1 = x_m s_j, \) then \( c_2 \in x_m j. \) Now we have again two cases, \( (c_2, d_2) = (x_m j, s_j, x_n, t_j) \text{ or } (x_n, t_j, x_m, s_j) \) for some \( 1 \leq j' \leq k. \) Four cases may occur:

(i) If \( d_1 = x_n t_j \) and \( c_2 = x_m j, s_j, \) then \( m_j = n_j. \) Then we have
\[
t_j w_1 \leq s_j w_2. \] Multiplying the last inequality from the left by \( u z_m, \) we get the inequality \( q_j w_1 \leq p_j w_2. \) So \( s \leq p_j w_1 q_j w_1 \leq p_j w_2. \)

(ii) If \( d_1 = x_m j, s_j \) and \( c_2 = x_m j, s_j, \) then \( m_j = m_j. \) Then we obtain
\[
s_j w_1 \leq s_j w_2, \text{ and so } w_1 \leq w_2. \text{ Thus } s \leq q_j w_1 p_j w_1 \leq p_j w_2.
\]

(iii) If \( d_1 = x_n t_j \) and \( c_2 = x_n t_j, \) then \( n_j = n_j. \) Hence \( t_j w_1 \leq t_j w_2, \) and so \( n_j = n_j. \) We get \( t_j w_1 \leq t_j w_2, \text{ and so } w_1 \leq w_2. \) Consequently, \( s \leq p_j w_1 q_j w_1 \leq q_j w_2. \)

(iv) If \( d_1 = x_m j, s_j \) and \( c_2 = x_m j, t_j, \) then \( n_j = m_j. \) So \( s_j w_1 \leq t_j w_2. \) Multiplying the last inequality from the left by \( u z_m, \) we get the inequality \( p_j w_1 \leq q_j w_2. \) Thus \( s \leq q_j w_1 p_j w_1 \leq q_j w_2. \)

Continuing in this process we reach to the sequence of inequalities
\[
s \leq c_1 w_1 d_1 w_1 \leq c_2 w_2 \ldots d_m w_m \leq t,
\]
where for every \( 1 \leq i \leq m, \) \( (c_i, d_i) = (p_j, q_j) \text{ or } (q_j, p_j) \) for some \( 1 \leq j \leq k \) which means that \( s \leq t. \)
(c) Finally, we will prove that $S_S/\sigma \cong F_S/\rho$. Since $[x_1]_\rho = [x_1u]_\rho z_1$ using part (b) we have $[1]_\rho = [u]_\rho z_1$ which means that $S_S/\sigma = [u]_\sigma S$. Define a mapping $f : S_S/\sigma \longrightarrow F_S/\rho$ by $f([u]_\sigma s) = [x_1u]_\rho s$ for every $s \in S$. Suppose $[u]_\sigma s \leq [u]_\sigma t$ for $s, t \in S$, i.e. $us \leq ut$. Then either $us \leq ut$ and therefore $(x_1us) \leq \rho (x_1ut)$ or $us \leq c_1 w_1 d_1 w_1 \leq c_2 w_2 \ldots d_m w_m \leq ut,$ where for every $1 \leq i \leq m$, $(c_i, d_i) = (p_j, q_j)$ or $(q_j, p_j)$ for some $1 \leq j \leq k$. Consider elements $(c_i, d_i) = (p_j, q_j)$ or $(q_j, p_j)$, it follows from part (a) that $c_1 w_1 \leq \rho d_1 w_1$. We get $x_1 us \leq x_1 c_1 w_1 \leq \rho x_1 d_1 w_1 \leq x_1 c_2 w_2 \leq \rho \ldots \leq \rho x_1 d_m w_m \leq x_1 ut$. This means that $f$ is well-defined. Clearly, $f$ is a surjective $S$-morphism.

Suppose $f([u]_\sigma s) \leq f([u]_\sigma t)$, $s, t \in S$, i.e. $[x_1u]_\rho s \leq [x_1u]_\rho t$ or $x_1 us \leq \rho x_1 ut$. By part (b), $[u]_\sigma s \leq [u]_\sigma t$. Hence $f$ is order-embedding and therefore an isomorphism.

Sufficiency is obvious. \qed

2. Absolutely pure and (1-)po-pure

In this section, we investigate (po-)pure properties. First we give some general properties of $S$-posets satisfying such properties. Then, we use finitely presented $S$-posets to give a necessary and sufficient condition for a right $S$-poset to be absolutely pure or absolutely po-pure. We say that two elements $x, y$ of an $S$-poset $A_S$ are comparable if $x \leq y$ or $y \leq x$ and denote this relation by $x \parallel y$. Let us recall from [9] and [11] the notions related to (1-) po-purity and purity.

**Definition 2.1.** Let $A_S$ be an $S$-poset.

(i) Consider the system $\Sigma$ consisting of inequations of the following four forms

$$xs \leq xt, \; xs \leq yt, \; xs \leq a, \; a \leq xs,$$

where $s, t \in S$ and $a \in A_S$ and $x, y \in X$, where $X$ is a set. We call $x, y$ variables, $s, t$ coefficients, $a$ a constant and $\Sigma$ a system of inequations with constants from $A_S$. We briefly use $xs \parallel a$ for two last inequations. Systems of inequations will be written as

$$\Sigma = \{xs_i \parallel a_i | \; s_i \in S, \; a_i \in A, \; 1 \leq i \leq n\}.$$ 

If we can map the variables of $\Sigma$ onto a subset of an $S$-poset $B_S$ such that the inequations turn into inequalities in $B_S$ then such subset of $B_S$ is called a solution of the system $\Sigma$ in $B_S$. In this case, $\Sigma$ is called solvable in $B_S$.

(ii) If $\Sigma$ has a solution in an $S$-poset $B_S$ containing $A_S$ then $\Sigma$ is called a consistent system of inequations.
(iii) A sub \( S \)-poset \( A_S \) of an \( S \)-poset \( B_S \) is called \textit{po-pure} in \( B_S \) if every finite system of inequations with constants from \( A_S \) which has a solution in \( B_S \) has a solution in \( A_S \). An \( S \)-poset \( A_S \) is called \textit{absolutely po-pure} if every finite consistent system of inequations with constants from \( A_S \) has a solution in \( A_S \).

(iv) A sub \( S \)-poset \( A_S \) of an \( S \)-poset \( B_S \) is called \textit{1-po-pure} in \( B_S \) if every finite system of inequations in one variable with constants from \( A_S \) which has a solution in \( B_S \) has a solution in \( A_S \). An \( S \)-poset \( A_S \) is called \textit{absolutely 1-po-pure} if every finite consistent system of inequations in one variable with constants from \( A_S \) has a solution in \( A_S \).

Replacing the term inequations by equations in the foregoing definition the concept of pure, absolutely pure and absolutely 1-pure can be defined, as \[11\], Definitions 6,7,8]. In our opinion the term extension po-pure would be more appropriate in the ordered case, and we first study some properties of po-purity.

By \[9\], Proposition 2.1], we deduce the following corollary.

**Corollary 2.2.** If an \( S \)-poset \( A_S \) is po-pure (1-po-pure) in its regular injective envelope \( E(A_S) \), then \( A_S \) is absolutely po-pure (1-po-pure).

By \[11\], Proposition 16], we get the following result is.

**Lemma 2.3.** If an \( S \)-poset \( A_S \) is absolutely 1-po-pure, then for any \( s_1, \ldots, s_n \in S \) there exists \( a \in A_S \) such that \( a = as_1 = \cdots = as_n \).

**Definition 2.4.** We say that a pomonoid \( S \) has \textit{local left zeros} if for any \( s_1, \ldots, s_n \in S \) there exists \( s \in S \) such that \( s = ss_1 = \cdots = ss_n \).

The following lemma is a direct consequence of Lemma 2.3.

**Lemma 2.5.** If \( S_S \) is absolutely 1-po-pure then \( S \) has local left zeros.

**Lemma 2.6.** The following hold for a pomonoid \( S \).

(i) \( \Theta \) is absolutely (1-) po-pure.

(ii) A retract of an absolutely (1-) po-pure \( S \)-poset is absolutely (1-) po-pure.

**Proof.** (i) is obvious. (ii). Let \( B_S \) be a retract of \( A_S \) by an \( S \)-morphism \( g : A_S \rightarrow B_S \) and \( A_S \) is absolutely po-pure. Clearly \( E(B_S) \) is a sub \( S \)-poset of \( E(A_S) \). Suppose that \( \Sigma \) is a finite system of inequations with constants from \( B_S \) which has a solution in \( E(B_S) \). So \( \Sigma \) has a solution in \( E(A_S) \). Since \( A_S \) is absolutely po-pure, \( \Sigma \) has a solution in \( A_S \). If \( \{a_1, \ldots, a_n\} \) is a solution of \( \Sigma \) in \( A_S \), then \( \{g(a_1), \ldots, g(a_n)\} \) is a solution of \( \Sigma \) in \( B_S \). Therefore, \( B_S \) is absolutely po-pure. □
Now, we consider the relationship between po-purity and tensor products.

**Proposition 2.7.** [9, Proposition 2.20] If $A_S$ is a po-pure sub $S$-poset of an $S$-poset $B_S$, then the mapping $A_S \otimes_S C \rightarrow B_S \otimes_S C$ is a regular monomorphism for every left $S$-poset $sC$.

Using the previous proposition we get the following corollary.

**Corollary 2.8.** If all right $S$-posets are absolutely po-pure, then all left $S$-posets are po-flat.

To give an equivalent condition for absolutely po-purity, we need the conditions under which an $S$-poset is po-pure in its extensions.

**Proposition 2.9.** An $S$-poset $A_S$ is a po-pure sub $S$-poset of $B_S$ if and only if for every finitely presented $S$-poset $C_S$, every $S$-morphism $\varphi : C_S \rightarrow B_S$ and every finite subset $\{c_1, \ldots, c_n| \varphi(c_i) \parallel a_i \in A\}$ of $C_S$ there exists an $S$-morphism $\psi : C_S \rightarrow A_S$ such that $\psi(c_i) \parallel a_i$ for $i = 1, \ldots, n$.

**Proof.** Necessity. Suppose that $A_S$ is a po-pure sub $S$-poset of $B_S$. Let $C_S$ be finitely presented and $\varphi : C_S \rightarrow B_S$ be such that $c_1, \ldots, c_n \in C$ and $\varphi(c_i) \parallel a_i \in A_S$. Without loss of generality assume that $C_S = F/\rho$ where $F$ is a free $S$-poset generated by $\{f_1, \ldots, f_m\}$ and

$$\rho = \nu(\{(f_{k_1}s_1, f_{l_1}t_1), \ldots, (f_{k_r}s_r, f_{l_r}t_r)\}).$$

Let $c_i = [f_{q_i}]p_i$ for $1 \leq i \leq n$, and $\varphi(c_i) \parallel a_i \in A$. If $\varphi([f_j]) = b_j$ for $j = 1, \ldots, m$, then $b_{k_j}s_j = \varphi([f_{k_j}])s_j \leq \varphi([f_{q_i}])t_j = b_{q_i}t_j$ and $a_i \parallel \varphi(c_i) = \varphi([f_{q_i}])p_i = b_{q_i}p_i$. Hence there exist $a'_j \in A_S$, $1 \leq j \leq m$, such that $a'_{k_j}s_j \leq a'_{q_i}t_j$ for $1 \leq j \leq r$ and $a_i \parallel a'_i p_i$ for $1 \leq i \leq n$. Now define a mapping $\psi : C_S \rightarrow A_S$ by $\psi([f_i]) = a'_i$. It is easily checked that $\psi$ is an $S$-morphism such that $\psi(c_i) = \psi([f_{q_i}])p_i = a'_i p_i \parallel a_i$ for $i = 1, \ldots, n$.

Sufficiency. Suppose that $\Sigma = \{x_{k_j}s_j \leq x_{l_j}t_j, a_i \parallel x_{q_i}p_i\} \leq n$, $1 \leq j \leq r\}$ is a system of inequalities which has a solution $\{b_1, \ldots, b_m\}$. Let $F_S$ be a free $S$-posets generated by $\{f_1, \ldots, f_m\}$, and

$$\rho = \nu(\{(f_{k_1}s_1, f_{l_1}t_1), \ldots, (f_{k_r}s_r, f_{l_r}t_r)\}).$$

So $C = F/\rho$ is finitely presented. Define $\varphi : C_S \rightarrow B_S$ by $\varphi([f_{j_i}]) = b_{j_i}$. It is clear that $\varphi$ is an $S$-morphism and $\varphi(c_i) \parallel a_i \in A_S$ where $c_i = [f_{q_i}]p_i$ for $1 \leq i \leq n$. By assumption there exists an $S$-morphism $\psi : C_S \rightarrow A_S$ such that $\psi(c_i) \parallel a_i$ for $i = 1, \ldots, n$. Therefore, $\{\psi([f_{i_1}]), \ldots, \psi([f_{i_m}])\}$ is a solution of $\Sigma$ in $A_S$, as desired.

Replacing $\nu(H)$ and $\parallel$ by $\theta(H)$ and $\vdash$, respectively, in the proof of the previous proposition, one can prove the following proposition.
**Proposition 2.10.** An S-poset $A_S$ is a pure sub S-poset of $B_S$ if and only if for every $C_S = F_S/\rho$ where $F_S$ is a finitely generated free S-poset and $\rho$ is a finitely generated congruence on $F_S$, for every $S$-morphism $\varphi : C_S \to B_S$ and for every finite subset $\{c_1, \ldots, c_n\} \varphi(c_i) = a_i \in A_S$ of $C_S$ there exists an $S$-morphism $\psi : C_S \to A_S$ such that $\psi(c_i) = \varphi(c_i)$ for $i = 1, \ldots, n$.

The following two theorems give some equivalent conditions for absolute purity and absolute po-purity

**Theorem 2.11.** The following statements are equivalent for any S-poset $A_S$:

(i) $A_S$ is absolutely pure;

(ii) for every strongly finitely presented S-poset $M_S = F_S/\rho$, every finitely generated S-poset $N_S$, every regular monomorphism $\iota : N_S \to M_S$, and every $S$-morphism $f : N_S \to A_S$ there exists an $S$-morphism $g : M_S \to A_S$ such that $g \iota = f$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose that $M_S$, $N_S$, $\iota : N_S \to M_S$, and $f : N_S \to A_S$ are as stated in the assumption of part (ii). Consider $A_S$ as a sub S-poset of $E(A_S)$, we have $f : N_S \to E(A_S)$. Regular injectivity of $E(A_S)$ implies the existence of $h : M_S \to E(A_S)$ such that $h \iota = f$. Assume that $N_S$ is generated by $\{b_1, \ldots, b_n\}$. So $h(b_i) \in A_S$ for each $1 \leq i \leq n$. Now, applying Proposition 2.10, we get $g : M_S \to A_S$ such that $g(b_i) = h(b_i)$ for each $1 \leq i \leq n$. Hence $g \iota = f$ and we have done.

(i) $\Rightarrow$ (ii). It suffices to show that $A_S$ is pure in $E(A_S)$. Using Proposition 2.10, suppose that $C_S = F_S/\rho$ where $F_S$ is a finitely generated free S-poset and $\rho$ is a finitely generated congruence on $F_S$, $\varphi : C_S \to E(A_S)$ is an $S$-morphism and $\{c_1, \ldots, c_n\} \varphi(c_i) \in A_S \subseteq C_S$. Let $N_S$ be generated by $\{c_1, \ldots, c_n\}$. Then $f = \varphi|_N : N_S \to A_S$ and by assumption there exists an $S$-morphism $g : C_S \to A_S$ such that $g \iota = f$. Thus $g(c_i) = f(c_i) = \varphi(c_i)$ for $i = 1, \ldots, n$, and the result follows. □

**Theorem 2.12.** The following statements are equivalent for any S-poset $A_S$:

(i) $A_S$ is absolutely po-pure;

(ii) for every finitely presented S-poset $M_S$, every finitely generated sub S-poset $N_S \subseteq M_S$ and every $S$-morphism $f : N_S \to E(A_S)$ such that $\text{Im}(f) \subseteq \{c | c \nmid a \in A\}$ there exists an $S$-morphism $g : M_S \to A_S$ such that for each $b \in N$ we have $g(b) \nmid a \nmid f(b)$ for some $a \in A_S$.

**Proof.** (i) $\Rightarrow$ (ii). Let $M_S$ be a finitely presented S-poset, $N_S$ be its finitely generated sub S-poset and $f : N_S \to E(A_S)$ an $S$-morphism
such that $\text{Im}(f) \subseteq \{c \mid c \parallel a \in A\}$. Regular injectivity of $E(AS)$ implies the existence of $h : M_S \to E(AS)$ such that $h|_M = f$. Let $L = \{b_1, \ldots, b_n\}$ be a finite set of generating elements of $N_S$. Now $h(b_i) \parallel a_i \in AS$ and Proposition 2.9 implies the existence of an $S$-morphism $g : M_S \to AS$ with $g(b_i) \parallel a_i$ for any $1 \leq i \leq n$. So for each $b_is \in N$ we have $g(b_is) \parallel a_is \parallel f(b_is)$.

(ii) $\Rightarrow$ (i). By assumption and using Proposition 2.9, $AS$ is po-pure in $E(AS)$, and so $AS$ is absolutely po-pure. □

We conclude this section by considering the relationship between regular injectivity and absolute po-purity. In [9], the authors gave another characterization of regular injective S-poses.

**Proposition 2.13.** [9, Theorem 2.5] An S-poset is regular injective if and only if any consistent system of inequations with constants from $AS$ has a solution in $AS$.

In view of the previous proposition we deduce that every regular injective S-poset is absolutely po-pure. Recall from [10] that a pomonoid $S$ is called right (po-)Noetherian if it satisfies the ascending chain condition on right (po)ideals. Equivalently, all right (po)ideals of $S$ are finitely generated.

In [9] it is shown that if every absolutely po-pure S-poset is weakly regular injective, then the pomonoid $S$ is right po-Noetherian.

**Proposition 2.14.** Every absolutely 1-po-pure S-poset over a right po-Noetherian pomonoid is regular injective.

**Proof.** Let $S$ be a po-Noetherian pomonoid and $AS$ be absolutely po-pure. To reach the contrary, suppose that $b \in E(AS) \setminus AS$. Let $I = \{s \in S \mid (\exists a \in A)(bs \leq a)\}$. If $I = \emptyset$, then $\leq_{\rho_B}\Big|_A = \leq\Big|_A$ where $B = [bs]$, is the convex ideal generated by $b$, and $\rho_B$ is a Rees congruence on $B$, which is contradiction to Corollary 1.1. Now, suppose that $I \neq \emptyset$. Clearly, $I$ is a poideal of $S$. Since $S$ is po-Noetherian, we may assume that $I$ is generated by the set $\{s_1, \ldots, s_n\}$. Now, consider the finite system $\Sigma = \{xs_i \leq bs_i \mid 1 \leq i \leq n\}$ of inequations which has a solution $b$ in $E(AS)$. So the system $\Sigma$ has a solution $a \in A$. Take $\sigma = \nu(a,b)$. Let $a_1, a_2 \in A$ such that $a_1 \leq_a a_2$. Then $a \leq at_1 bt_1 \leq at_2 bt_2 \leq at_3 \ldots bt_m \leq a_2$, where $t_i \in S$ for $1 \leq i \leq m$. It is obvious that $t_i \in I$ which implies that $at_i \leq bt_i$, and so $a_1 \leq a_2$. Thus $\leq_a\Big|_A = \leq\Big|_A$, which is again a contradiction by Corollary 1.1. Therefore, $AS = E(AS)$ is regular injective. □
In [9, Corollary 2.5], it is shown that absolute 1-purity implies fg-
weakly regular injectivity.

The following examples illustrate that weak regular injectivity does
not imply absolute 1-po-purity and also absolute po-purity does not
imply weak regular injectivity.

**Example 2.15.** Weak regular injectivity does not imply absolute 1-
po-purity. Similar to [6, Example 3.6.17], let \( S = T^1 \), where \( T = \{x, y\} \)
is the two-element right zero semigroup with trivial order, then \( S \) is
weakly regular injective. But since \( S \) does not have any local left zeros,
\( S \) cannot be absolutely 1-po-pure.

**Example 2.16.** Absolute po-purity does not imply weak regular
injectivity. Indeed, let \( S = (N, \min) \cup \varepsilon \), where \( \varepsilon \) denotes the externally
adjointed identity with the order \( 1 < 2 < 3 < \cdots < \varepsilon \). Then \( K_S = S \setminus \{\varepsilon\} \) is a right ideal of \( S \) which is absolutely po-pure, but \( K \) is not
weakly regular injective.

The following relations exist between absolute purity properties and
regular injectivity of \( S \)-posets.

\[
\text{regular injective } \Rightarrow \text{ abs. po } - \text{ pure } \Rightarrow \text{ abs. } 1 - \text{ po } - \text{ pure } \\
\downarrow \quad \quad \downarrow \\
\text{abs. pure } \Rightarrow \text{ abs. } 1 - \text{ pure } \\
\downarrow \\
\text{fg } - \text{ w. regular injective}
\]

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A NEW CHARACTERIZATION OF ABSOLUTELY PO-PURE AND ABSOLUTELY PURE S-POSETS

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