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A NEW CHARACTERIZATION OF ABSOLUTELY PO-PURE AND ABSOLUTELY PURE S-POSETS

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ABSTRACT. In this paper, we investigate po-purity using finitely presented S-posets, and give some equivalent conditions under which an S-poset is absolutely po-pure. We also introduce strongly finitely presented S-posets to characterize absolutely pure S-posets. Similar to the acts, every finitely presented cyclic S-posets is isomorphic to a factor S-poset of a pomonoid S by a finitely generated right congruence on S. Finally, the relationships between regular injectivity and absolute po-purity are considered.

1. Introduction

A *pomonoid* S is a monoid which it is also a poset whose partial order $\leq$ is compatible with the binary operation on S. A *right S-poset* A_S is a right S-act A_S equipped with a partial order $\leq$ and, in addition, for all $s, t \in S$ and $a, b \in A_S$, if $s \leq t$ then $as \leq at$, and if $a \leq b$ then $as \leq bs$. A *sub S-poset* B_S of a right S-poset A_S is a subposet of A_S that is closed under the S-action. In this case, A_S is said to be an *extension* of B_S. Moreover, *S-morphisms* are the functions that preserve both the action and the order. The class of right S-posets and S-morphisms form a category, denoted by **POS**-S, which comprises the main background of this work. For an account on this category and categorical notions used in this paper, the reader is referred to [3]. An S-morphism $\iota : A_S \longrightarrow B_S$ is a *regular monomorphism* if and only if it is an *order-embedding*, i.e., $a \leq a' \Leftrightarrow \iota(a) \leq \iota(a')$, for all $a, a' \in A_S$.

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Let S be a pomonoid and I be a nonempty subset of S. Then I is said top be a *right ideal* of S, if $IS \subseteq S$ (not necessarily ordered right ideal). A *right poideal* of a pomonoid S is a nonempty subset I of S which is both a right ideal ($IS \subseteq I$) and a poset ideal (that is, $a \leq b$ and $b \in I$ imply $a \in I$).

Let A_S be a right S-poset. An *S-poset congruence* θ on A is a right S-act congruence with the property that the S-act A/θ can be made into an S-poset in such a way that the natural map $A_S \longrightarrow A/\theta$ is an S-poset map. For an S-act congruence θ on A_S we write $a \leq_\theta a'$ if the so-called θ-chain

$$a \leq a_1 \theta b_1 \leq a_2 \theta b_2 \leq \ \ldots \ \leq a_n \theta b_n \leq a',$$

from a to a' exists in A_S, where $a_i, b_i \in A,\ 1 \leq i \leq n$. It can be shown that an S-act congruence θ on a right S-poset A_S is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. Let $H \subseteq A \times A$. Then $a \leq_{\alpha(H)} b$ if and only if $a \leq b$ or there exist $n \geq 1, (c_i, d_i) \in H, s_i \in S, 1 \leq i \leq n$ such that

$$a \leq c_1 s_1 \ d_1 s_1 \leq c_2 s_2 \ \ldots \ d_n s_n \leq b.$$

The relation $\nu(H)$ given by $a \ \nu(H) \ b$ if and only if $a \leq_{\alpha(H)} b \leq_{\alpha(H)} a$ is the S-poset congruence *induced* by H. Moreover, $[a]_{\nu(H)} \leq [b]_{\nu(H)}$ if and only if $a \leq_{\alpha(H)} b$. The relation $\theta(H) = \nu(H \cup H^{-1})$ is the S-poset congruence *generated* by H. A congruence ρ on an S-poset A_S is called *finitely induced (finitely generated)* if $\rho = \nu(H)$ ($\rho = \theta(H)$) for some finite subset H of $A \times A$.

Recall that an S-poset A_S is *regular injective* if for each regular monomorphism $g : B_S \longrightarrow C_S$ and S-morphism $f : B_S \longrightarrow A_S$, there exists an S-morphism $\overline{f} : C_S \longrightarrow A_S$ such that $\overline{f}g = f$. An S-poset A_S is *weakly regular injective (fg-weakly regular injective, principally weakly regular injective)* if every S-morphism $f : I_S \longrightarrow A_S$ from a (finitely generated, principal) right ideal I of S can be extended to an S-morphism $\overline{f} : S_S \longrightarrow A_S$. By a *retract* of A_S, we mean a sub S-poset B_S of A_S together with an S-morphism from A_S to B_S which maps B_S identically. Clearly, a retract of a regular injective S-poset is also regular injective. Moreover, A_S is called an *absolute retract* if A_S is a retract of each of its extensions. In [8], it is shown that all regular injective S-posets are absolute retract. An S-poset $E(A_S)$ is called a *regular injective envelope* of an S-poset A_S if $E(A_S)$ is regular injective and does not contain a proper sub S-poset B_S which is a regular injective extension of A_S. In [8], it is proved that for each S-poset there exists a regular injective envelope. In light of [8, Proposition 2.2], the following corollary is clear which will be needed in the sequel.

Corollary 1.1. *If ρ is a congruence relation on $E(A_S)$ with $\rho \neq \Delta_{E(A_S)}$, then $\leq_\rho |_A \neq \leq |_A$.*

In the category of S-acts, absolutely pure acts were first considered by Normak [7] and then studied by Gould in [4]. Moreover, Gould introduced absolutely 1-pure acts under the name of almost pure acts in [5]. For S-posets, recently in [11], the authors generalized purity on S-acts into the theory of S-posets and introduced the properties of (1-)pure and absolutely (1-)pure S-posets regardless of their order. Then in [9], they introduced po-purity of S-posets and characterized absolutely 1-po-pure S-posets. In the following, we study strongly finitely presented cyclic S-posets. In Section 2, some general properties of po-purity and absolute-po-purity for S-posets are studied. Then, we investigate absolutely po-pure S-posets using finitely presented S-posets. Finally, the relationships between regular injectivity and absolute po-purity are discussed.

An S-poset A_S is *free* on a set X if and only if $A_S \cong \bigcup_{x\in X} xS$ where for all $x, y \in X$ and $s, t \in S$, $xs \leq yt$ if and only if $x = y$ and $s \leq t$. The concept of finitely presented S-poset was introduced in [2] which we recall it. It was mentioned by the notion of semi-finitely presented in [9]. An S-poset A_S is said to be *finitely presented* if it is isomorphic to a quotient S-poset of a finitely generated free S-poset by a finitely induced S-poset congruence. In the category of S-acts, finitely presented S-acts was introduced as a factor S-act of finitely generated free S-acts by a finitely generated right congruence. Now, we define it in the category of S-posets as follows.

Definition 1.2. An S-poset A_S is said to be *strongly finitely presented* if it is isomorphic to F/ρ, where F_S is a finitely generated free S-poset and $\rho = \theta(H)$ for some finite subset $H \subseteq F \times F$, i.e. ρ is a finitely generated congruence on F_S.

In the category of S-acts, every finitely presented cyclic S-act is isomorphic to a factor S-act of S by a finitely generated right congruence on S. The following result shows that it is also valid for S-posets, which is needed to characterize absolutely 1-po-pure S-posets.

Proposition 1.3. *Let A_S be a cyclic S-poset. Then A_S is strongly finitely presented if and only if it is isomorphic to a factor S-poset of S_S by a finitely generated right congruence on S.*

Proof. Necessity. Let F_S be a free S-poset generated by $\{x_1, \ldots, x_n\}$ and let ρ be a congruence on F_S generated by

$$H = \{(x_{m_1}s_1, x_{n_1}t_1), \ldots, (x_{m_k}s_k, x_{n_k}t_k)\},$$

so that F_S/ρ is cyclic. Assume that $F_S/\rho = [x_1u]_\rho S$ for some $u \in S$.

Let $[x_i]_\rho = [x_1u]_\rho z_i$, $z_i \in S$, $1 \leq i \leq n$. Set

$$p_i = \begin{cases} s_i & m_i = 1 \\ uz_{m_i}s_i & m_i \neq 1 \end{cases} \text{ and } q_i = \begin{cases} t_i & n_i = 1 \\ uz_{n_i}t_i & n_i \neq 1 \end{cases}$$

for every $1 \leq i \leq k$. Consider the right congruence

$$\sigma = \theta(\{(p_1, q_1), (p_2, q_2), \ldots, (p_k, q_k)\})$$

on S. We shall prove that $F_S/\rho \cong S/\sigma$, dividing the proof into three parts:

(a) First, we show that $x_1p_i\rho x_1q_i$ for every $1 \leq i \leq k$. If $m_i = 1$, clearly $x_{m_i}s_i = x_1p_i$, otherwise using the equalities $[x_{m_i}]_\rho = [x_1u]_\rho z_{m_i}$, we get that $[x_{m_i}s_i]_\rho = [x_{m_i}]_\rho s_i = [x_1uz_{m_i}]_\rho s_i = [x_1]_\rho uz_{m_i}s_i = [x_1]_\rho p_i$. This means that $x_{m_i}s_i\rho x_1p_i$. Analogously one can prove that $x_{n_i}t_i\rho x_1q_i$. Since $x_{m_i}s_i\rho x_{n_i}t_i$ we have $x_1p_i\rho x_1q_i$.

(b) Second, we show that if $x_1s \leq_\rho x_1t$ for some elements $s, t \in S$, then $s \leq_\sigma t$. From $x_1s \leq_\rho x_1t$ it follows that either $x_1s \leq x_1t$ and therefore $s \leq t$ or there exist $m \geq 1, c_i, d_i \in F_S, w_i \in S, 1 \leq i \leq m$ such that $(c_i, d_i) \in H \cup H^{-1}$ and

$$x_1s \leq c_1w_1\ d_1w_1 \leq c_2w_2\ ...\ d_mw_m \leq x_1t.$$

From the inequality $x_1s \leq c_1w_1$ we obtain that $c_1 \in x_1S$. Then $(c_1, d_1) = (x_{m_j}s_j, x_{n_j}t_j)$ or $(x_{n_j}t_j, x_{m_j}s_j)$. In the first case, $m_j = 1$ and so $s \leq s_jw_1$. The second case implies $n_j = 1$, and so $s \leq q_jw_1$. If $d_1 = x_{n_j}t_j$, from the inequality $d_1w_1 \leq c_2w_2$ we get that $c_2 \in x_{n_j}S$, and if $d_1 = x_{m_j}s_j$, then $c_2 \in x_{m_j}S$. Now we have again two cases, $(c_2, d_2) = (x_{m_{j'}}s_{j'}, x_{n_{j'}}t_{j'})$ or $(x_{n_{j'}}t_{j'}, x_{m_{j'}}s_{j'})$ for some $1 \leq j' \leq k$. Four cases may occur:

(i) If $d_1 = x_{n_j}t_j$ and $c_2 = x_{m_{j'}}s_{j'}$, then $m_{j'} = n_j$. Then we have $t_jw_1 \leq s_{j'}w_2$. Multiplying the last inequality from the left by uz_{n_j} we get the inequality $q_jw_1 \leq p_{j'}w_2$. So $s \leq p_jw_1\ q_jw_1 \leq p_{j'}w_2$.

(ii) If $d_1 = x_{m_j}s_j$ and $c_2 = x_{m_{j'}}s_{j'}$, then $m_{j'} = m_j$. Then we obtain $s_jw_1 \leq s_jw_2$, and so $w_1 \leq w_2$. Thus $s \leq q_jw_1\ p_jw_1 \leq p_{j'}w_2$.

(iii) If $d_1 = x_{n_j}t_j$ and $c_2 = x_{n_{j'}}t_{j'}$, then $n_{j'} = n_j$. Hence $t_jw_1 \leq t_{j'}w_2$, and so $n_{j'} = n_j$. We get $t_jw_1 \leq t_jw_2$, and so $w_1 \leq w_2$. Consequently, $s \leq p_jw_1\ q_jw_1 \leq q_{j'}w_2$.

(iv) If $d_1 = x_{m_j}s_j$ and $c_2 = x_{n_{j'}}t_{j'}$, then $n_{j'} = m_j$. So $s_jw_1 \leq t_{j'}w_2$. Multiplying the last inequality from the left by uz_{m_j} we get the inequality $p_jw_1 \leq q_{j'}w_2$. Thus $s \leq q_jw_1\ p_jw_1 \leq q_{j'}w_2$.

Continuing in this process we reach to the sequence of inequalities

$$s \leq c'_1w_1\ d'_1w_1 \leq c'_2w_2\ ...\ d'_mw_m \leq t,$$

where for every $1 \leq i \leq m$, $(c'_i, d'_i) = (p_j, q_j)$ or (q_j, p_j) for some $1 \leq j \leq k$ which means that $s \leq_\sigma t$.

(c) Finally, we will prove that $S_S/\sigma \cong F_S/\rho$. Since $[x_1]_\rho = [x_1u]_\rho z_1$ using part (b) we have $[1]_\sigma = [u]_\sigma z_1$ which means that $S_S/\sigma = [u]_\sigma S$. Define a mapping $f : S_S/\sigma \longrightarrow F_S/\rho$ by $f([u]_\sigma s) = [x_1u]_\rho s$ for every $s \in S$. Suppose $[u]_\sigma s \leq [u]_\sigma t$ for $s, t \in S$, i.e. $us \leq_\sigma ut$. Then either $us \leq ut$ and therefore $(x_1us) \leq_\rho (x_1ut)$ or

$$us \leq c_1w_1\ d_1w_1 \leq c_2w_2\ \ldots\ d_mw_m \leq ut,$$

where for every $1 \leq i \leq m$, $(c_i, d_i) = (p_j, q_j)$ or (q_j, p_j) for some $1 \leq j \leq k$. Consider elements $(c_i, d_i) = (p_j, q_j)$ or (q_j, p_j), it follows from part (a) that $c_1w_1 \leq_\rho d_1w_1$. We get

$$x_1us \leq x_1c_1w_1 \leq_\rho x_1d_1w_1 \leq x_1c_2w_2 \leq_\rho \cdots \leq_\rho x_1d_mw_m \leq x_1ut.$$

This means that f is well-defined. Clearly, f is a surjective S-morphism. Suppose $f([u]_\sigma s) \leq f([u]_\sigma t)$, $s, t \in S$, i.e. $[x_1u]_\rho s \leq [x_1u]_\rho t$ or $x_1us \leq_\rho x_1ut$. By part (b), $[u]_\sigma s \leq [u]_\sigma t$. Hence f is order-embedding and therefore an isomorphism.

Sufficiency is obvious. □

2. Absolutely pure and (1-)po-pure

In this section, we investigate (po-)pure properties. First we give some general properties of S-posets satisfying such properties. Then, we use finitely presented S-posets to give a necessary and sufficient condition for a right S-poset to be absolutely pure or absolutely po-pure. We say that two elements x, y of an S-poset A_S are comparable if $x \leq y$ or $y \leq x$ and denote this relation by $x \nparallel y$. Let us recall from [9] and [11] the notions related to (1-) po-purity and purity.

Definition 2.1. Let A_S be an S-poset.

(i) Consider the system Σ consisting of inequations of the following four forms

$$xs \leq xt,\ xs \leq yt,\ xs \leq a,\ a \leq xs,$$

where $s, t \in S$ and $a \in A_S$ and $x, y \in X$, where X is a set. We call x, y variables, s, t coefficients, a a constant and Σ a system of inequations with constants from A_S. We briefly use $xs \nparallel a$ for two last inequations. Systems of inequations will be written as

$$\Sigma = \{xs_i \nparallel a_i |\ s_i \in S,\ a_i \in A,\ 1 \leq i \leq n\}.$$

If we can map the variables of Σ onto a subset of an S-poset B_S such that the inequations turn into inequalities in B_S then such subset of B_S is called a *solution* of the system Σ in B_S. In this case, Σ is called solvable in B_S.

(ii) If Σ has a solution in an S-poset B_S containing A_S then Σ is called a *consistent* system of inequations.

(iii) A sub S-poset A_S of an S-poset B_S is called *po-pure* in B_S if every finite system of inequations with constants from A_S which has a solution in B_S has a solution in A_S. An S-poset A_S is called *absolutely po-pure* if every finite consistent system of inequations with constants from A_S has a solution in A_S.

(iv) A sub S-poset A_S of an S-poset B_S is called *1-po-pure* in B_S if every finite system of inequations in one variable with constants from A_S which has a solution in B_S has a solution in A_S. An S-poset A_S is called *absolutely 1-po-pure* if every finite consistent system of inequations in one variable with constants from A_S has a solution in A_S.

Replacing the term inequations by equations in the foregoing definition the concept of pure, absolutely pure and absolutely 1-pure can be defined, as [11, Definitions 6,7,8]. In our opinion the term extension po-pure would be more appropriate in the ordered case, and we first study some properties of po-purity.

By [9, Proposition 2.1], we deduce the following corollary.

Corollary 2.2. *If an S-poset A_S is po-pure (1-po-pure) in its regular injective envelope $E(A_S)$, then A_S is absolutely po-pure (1-po-pure).*

By [11, Proposition 16], we get the following result is.

Lemma 2.3. *If an S-poset A_S is absolutely 1-po-pure, then for any $s_1, \ldots, s_n \in S$ there exists $a \in A_S$ such that $a = as_1 = \cdots = as_n$.*

Definition 2.4. We say that a pomonoid S has *local left zeros* if for any $s_1, \ldots, s_n \in S$ there exists $s \in S$ such that $s = ss_1 = \cdots = ss_n$.

The following lemma is a direct consequence of Lemma 2.3.

Lemma 2.5. *If S_S is absolutely 1 -po-pure then S has local left zeros.*

Lemma 2.6. *The following hold for a pomonoid S.*

(i) *Θ is absolutely (1-) po-pure.*

(ii) *A retract of an absolutely (1-) po-pure S-poset is absolutely (1-) po-pure.*

Proof. (i) is obvious. (ii). Let B_S be a retract of A_S by an S-morphism $g : A_S \longrightarrow B_S$ and A_S is absolutely po-pure. Clearly $E(B_S)$ is a sub S-poset of $E(A_S)$. Suppose that Σ is a finite system of inequations with constants from B_S which has a solution in $E(B_S)$. So Σ has a solution in $E(A_S)$. Since A_S is absolutely po-pure, Σ has a solution in A_S. If $\{a_1, ...a_n\}$ is a solution of Σ in A_S, then $\{g(a_1), ..., g(a_n)\}$ is a solution of Σ in B_S. Therefore, B_S is absolutely po-pure.

□

Now, we consider the relationship between po-purity and tensor products.

Proposition 2.7. [9, Proposition 2.20] *If A_S is a po-pure sub S-poset of an S-poset B_S, then the mapping $A_S \otimes_S C \longrightarrow B_S \otimes_S C$ is a regular monomorphism for every left S-poset ${}_SC$.*

Using the previous proposition we get the following corollary.

Corollary 2.8. *If all right S-posets are absolutely po-pure, then all left S-posets are po-flat.*

To give an equivalent condition for absolutely po-purity, we need the conditions over which an S-poset is po-pure in its extensions.

Proposition 2.9. *An S-poset A_S is a po-pure sub S-poset of B_S if and only if for every finitely presented S-poset C_S, every S-morphism $\varphi : C_S \longrightarrow B_S$ and every finite subset $\{c_1, \ldots, c_n |\ \varphi(c_i) \nparallel a_i \in A\}$ of C_S there exists an S-morphism $\psi : C_S \longrightarrow A_S$ such that $\psi(c_i) \nparallel a_i$ for $i = 1, \ldots, n$.*

Proof. Necessity. Suppose that A_S is a po-pure sub S-poset of B_S. Let C_S be finitely presented and $\varphi : C_S \longrightarrow B_S$ be such that $c_1, \ldots, c_n \in C$ and $\varphi(c_i) \nparallel a_i \in A_S$. Without loss of generality assume that $C_S = F/\rho$ where F is a free S-poset generated by $\{f_1, \ldots, f_m\}$ and

$$\rho = \nu(\{(f_{k_1}s_1, f_{l_1}t_1), \ldots, (f_{k_r}s_r, f_{l_r}t_r)\}).$$

Let $c_i = [f_{q_i}]p_i$ for $1 \leq i \leq n$, and $\varphi(c_i) \nparallel a_i \in A$. If $\varphi([f_j]) = b_j$ for $j = 1, \ldots, m$, then $b_{k_j}s_j = \varphi([f_{k_j}])s_j \leq \varphi([f_{l_j}])t_j = b_{l_j}t_j$ and $a_i \nparallel \varphi(c_i) = \varphi([f_{q_i}]p_i) = b_{q_i}p_i$. Hence there exist $a'_j \in A_S$, $1 \leq j \leq m$, such that $a'_{k_j}s_j \leq a'_{l_j}t_j$ for $1 \leq j \leq r$ and $a_i \nparallel a'_{q_i}p_i$ for $1 \leq i \leq n$. Now define a mapping $\psi : C_S \longrightarrow A_S$ by $\psi([f_is]) = a'_is$. It is easily checked that ψ is an S-morphism such that $\psi(c_i) = \psi([f_{q_i}]p_i) = a'_{q_i}p_i \nparallel a_i$ for $i = 1, ..., n$.

Sufficiency. Suppose that $\Sigma = \{x_{k_j}s_j \leq x_{l_j}t_j, a_i \nparallel x_{q_i}p_i |\ 1 \leq i \leq n,\ 1 \leq j \leq r\}$ is a system of inequations which has a solution $\{b_1, \ldots, b_m\}$. Let F_S be a free S-posets generated by $\{f_1, \ldots, f_m\}$, and

$$\rho = \nu(\{(f_{k_1}s_1, f_{l_1}t_1), \ldots, (f_{k_r}s_r, f_{l_r}t_r\}).$$

So $C = F/\rho$ is finitely presented. Define $\varphi : C_S \longrightarrow B_S$ by $\varphi([f_js]) = b_js$. It is clear that φ is an S-morphism and $\varphi(c_i) \nparallel a_i \in A_S$ where $c_i = [f_{q_i}]p_i$ for $1 \leq i \leq n$. By assumption there exists an S-morphism $\psi : C_S \longrightarrow A_S$ such that $\psi(c_i) \nparallel a_i$ for $i = 1, \ldots, n$. Therefore, $\{\psi([f_1]), \ldots, \psi([f_m])\}$ is a solution of Σ in A_S, as desired. □

Replacing $\nu(H)$ and $\nparallel$ by $\theta(H)$ and $=$, respectively, in the proof of the previous proposition, one can prove the following proposition.

Proposition 2.10. *An S-poset A_S is a pure sub S-poset of B_S if and only if for every $C_S = F_S/\rho$ where F_S is a finitely generated free S-poset and ρ is a finitely generated congruence on F_S, for every S-morphism $\varphi : C_S \longrightarrow B_S$ and for every finite subset $\{c_1, \ldots, c_n | \ \varphi(c_i) = a_i \in A_S\}$ of C_S there exists an S-morphism $\psi : C_S \longrightarrow A_S$ such that $\psi(c_i) = \varphi(c_i)$ for $i = 1, \ldots, n$.*

The following two theorems give some equivalent conditions for absolute purity and absolute po-purity

Theorem 2.11. *The following statements are equivalent for any S-poset A_S:*

(i) *A_S is absolutely pure;*
(ii) *for every strongly finitely presented S-poset $M_S = F_S/\rho$, every finitely generated S-poset N_S, every regular monomorphism $\iota : N_S \longrightarrow M_S$, and every S-morphism $f : N_S \longrightarrow A_S$ there exists an S-morphism $g : M_S \longrightarrow A_S$ such that $g\iota = f$.*

Proof. (i)$\Rightarrow$ (ii). Suppose that M_S, N_S, $\iota : N_S \longrightarrow M_S$, and $f : N_S \longrightarrow A_S$ are as stated in the assumption of part (ii). Consider A_S as a sub S-poset of $E(A_S)$, we have $f : N_S \longrightarrow E(A_S)$. Regular injectivity of $E(A_S)$ implies the existence of $h : M_S \longrightarrow E(A_S)$ such that $h\iota = f$. Assume that N_S is generated by $\{b_1, \ldots, b_n\}$. So $h(b_i) \in A_S$ for each $1 \leq i \leq n$. Now, applying Proposition 2.10, we get $g : M_S \longrightarrow A_S$ such that $g(b_i) = h(b_i)$ for each $1 \leq i \leq n$. Hence $g\iota = f$ and we have done.

(i)$\Rightarrow$ (ii). It suffices to show that A_S is pure in $E(A_S)$. Using Proposition 2.10, suppose that $C_S = F_S/\rho$ where F_S is a finitely generated free S-poset and ρ is a finitely generated congruence on F_S, $\varphi : C_S \longrightarrow E(A_S)$ is an S-morphism and $\{c_1, \ldots, c_n | \ \varphi(c_i) \in A\} \subseteq C_S$. Let N_S be generated by $\{c_1, \ldots, c_n\}$. Then $f = \varphi|_N : N_S \longrightarrow A_S$ and by assumption there exists an S-morphism $g : C_S \longrightarrow A_S$ such that $g\iota = f$. Thus $g(c_i) = f(c_i) = \varphi(c_i)$ for $i = 1, \ldots, n$, and the result follows. $\square$

Theorem 2.12. *The following statements are equivalent for any S-poset A_S:*

(i) *A_S is absolutely po-pure;*
(ii) *for every finitely presented S-poset M_S, every finitely generated sub S-poset $N_S \subseteq M_S$ and every S-morphism $f : N_S \longrightarrow E(A_S)$ such that $\mathrm{Im(f)} \subseteq \{\mathrm{c} | \ \mathrm{c} \nparallel \mathrm{a} \in \mathrm{A}\}$ there exists an S-morphism $g : M_S \longrightarrow A_S$ such that for each $b \in N$ we have $g(b) \nparallel a \nparallel f(b)$ for some $a \in A_S$.*

Proof. (i) $\Rightarrow$ (ii). Let M_S be a finitely presented S-poset, N_S be its finitely generated sub S-poset and $f : N_S \longrightarrow E(A_S)$ an S-morphism such that Im(f) $\subseteq$ {c| c $\nparallel$ a $\in$ A}. Regular injectivity of $E(A_S)$ implies the existence of $h : M_S \longrightarrow E(A_S)$ such that $h|_N = f$. Let $L = \{b_1, \ldots, b_n\}$ be a finite set of generating elements of N_S. Now $h(b_i) \nparallel a_i \in A_S$ and Proposition 2.9 implies the existence of an S-morphism $g : M_S \longrightarrow A_S$ with $g(b_i) \nparallel a_i$ for any $1 \leq i \leq n$. So for each $b_i s \in N$ we have $g(b_i s) \nparallel a_i s \nparallel f(b_i)s$.

(ii)$\Rightarrow$ (i). By assumption and using Proposition 2.9, A_S is po-pure in $E(A_S)$, and so A_S is absolutely po-pure. □

We conclude this section by considering the relationship between regular injectivity and absolute po-purity. In [9], the authors gave another characterization of regular injective S-posets.

Proposition 2.13. [9, Theorem 2.5] *An S-poset is regular injective if and only if any consistent system of inequations with constants from A_S has a solution in A_S.*

In view of the previous proposition we deduce that every regular injective S-poset is absolutely po-pure. Recall from [10] that a pomonoid S is called *right (po-)Noetherian* if it satisfies the ascending chain condition on right (po)ideals. Equivalently, all right (po)ideals of S are finitely generated.

In [9] it is shown that if every absolutely po-pure S-poset is weakly regular injective, then the pomonoid S is right po-Noetherian.

Proposition 2.14. *Every absolutely 1-po-pure S-poset over a right po-Noetherian pomonid is regular injective.*

Proof. Let S be a po-Noetherian pomonid and A_S be absolutely po-pure. To reach the contrary, suppose that $b \in E(A_S) \setminus A_S$. Let $I = \{s \in S | (\exists a \in A)(\ bs \leq a)\}$. If $I = \emptyset$, then $\leq_{\rho_B} |_A = \leq |_A$ where $B = [bS]$, is the convex ideal generated by b, and ρ_B is a Rees congruence on B, which is contradiction to Corollary 1.1. Now, suppose that $I \neq \emptyset$. Clearly, I is a poideal of S. Since S is po-Noetherian, we may assume that I is generated by the set $\{s_1, \ldots, s_n\}$. Now, consider the finite system $\Sigma = \{xs_i \leq bs_i | \ 1 \leq i \leq n\}$ of inequations which has a solution b in $E(A_S)$. So the system Σ has a solution $a \in A$. Take $\sigma = \nu(a, b)$. Let $a_1, a_2 \in A$ such that $a_1 \leq_\sigma a_2$. Then

$$a \leq at_1 \ bt_1 \leq at_2 \ bt_2 \leq at_3 \ \ldots \ bt_m \leq a_2,$$

where $t_i \in S$ for $1 \leq i \leq m$. It is obvious that $t_i \in I$ which implies that $at_i \leq bt_i$, and so $a_1 \leq a_2$. Thus $\leq_\sigma |_A = \leq |_A$, which is again a contradiction by Corollary 1.1. Therefore, $A_S = E(A_S)$ is regular injective. □

In [9, Corollary 2.5], it is shown that absolute 1-purity implies fg-weakly regular injectivity.

The following examples illustrate that weak regular injectivity does not imply absolute 1-po-purity and also absolute po-purity does not imply weak regular injectivity.

Example 2.15. Weak regular injectivity does not imply absolute 1-po-purity. Similar to [6, Example 3.6.17], let $S = T^1$, where $T = \{x, y\}$ is the two-element right zero semigroup with trivial order, then S is weakly regular injective. But since S does not have any local left zeros, S_S cannot be absolutely 1-po-pure.

Example 2.16. Absolute po-purity does not imply weak regular injectivity. Indeed, let $S = (N, min)\dot{\cup}\varepsilon$, where ε denotes the externally adjoined identity with the order $1 < 2 < 3 < \cdots < \varepsilon$. Then $K_S = S \setminus \{\varepsilon\}$ is a right ideal of S which is absolutely po-pure, but K_S is not weakly regular injective.

The following relations exist between absolute purity properties and regular injectivity of S-posets.

$$\begin{array}{ccccc} regular\ injective \Rightarrow & abs.\ po-pure & \Rightarrow & abs.\ 1-po-pure & \\ & \Downarrow & & \Downarrow & \\ & abs.\ pure & \Rightarrow & abs.\ 1-pure & \\ & & & \Downarrow & \\ & & & fg-w.\ regular\ injective & \end{array}$$

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A NEW CHARACTERIZATION OF ABSOLUTELY PO-PURE AND ABSOLUTELY PURE S-POSETS

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یک توصیف جدید از S-مجموعه‌های مرتب خالص مطلق و po-خالص مطلق

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در این مقاله، ما مفهوم po-خلوص با استفاده از S-مجموعه‌های مرتب دارای نمایش متناهی را تحقیق می‌کنیم و به بیان شرایط معادلی که تحت آن یک S-مجموعه‌ی مرتب، po-خالص مطلق می‌باشد می‌پردازیم. همچنین S-مجموعه‌های مرتب قویاً دارای نمایش متناهی را معرفی می‌کنیم تا به وسیله‌ی آن S-مجموعه‌های مرتب خالص مطلق را توصیف کنیم. مشابه با رسته‌ی سیستم‌ها، هر S-مجموعه‌ی مرتب دوری دارای نمایش متناهی با تصویری از یک تکواره‌ی مرتب بر روی یک رابطه‌ی هم‌نهشتی به طور متناهی تولید شده یکریخت می‌باشد. در پایان، روابط بین ویژگی انژکتیو منظم بودن و po-خلوص مطلق را مورد بررسی قرار می‌دهیم.

کلمات کلیدی: S-مجموعه‌های مرتب، تکواره‌های مرتب، po-خلوص مطلق، ۱-po-خالص، انژکتیو منظم.