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PRIMARY ZARISKI TOPOLOGY ON THE PRIMARY SPECTRUM OF A MODULE

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Abstract. Let R be a commutative ring with identity and let M be an R-module. We define the primary spectrum of M, denoted by $\mathcal{PS}(M)$, to be the set of all primary submodules Q of M such that $(\operatorname{rad} Q : M) = \sqrt{(Q : M)}$. In this paper, we topologize $\mathcal{PS}(M)$ with a topology having the Zariski topology on the prime spectrum $\operatorname{Spec}(M)$ as a subspace topology. We investigate compactness and irreducibility of this topological space and provide some conditions under which $\mathcal{PS}(M)$ is a spectral space.

1. Introduction

Throughout this paper, R denotes a commutative ring with nonzero identity. We always denote by $\operatorname{Spec}(R)$ the set of all prime ideals of R. It is well know that $\operatorname{Spec}(R)$ is a topological space whose closed sets are $V(I) = \{p \in \operatorname{Spec}(R) \mid p \supseteq I\}$ for each ideal I of R (see, for example, [4, 7, 10]). Over the past twenty years or so, there have appeared in the literature, several papers giving many different generalizations of the Zariski topology over the spectrum of certain modules. Most of these considerations concern generalizations of prime ideals from rings to modules or ideals (see, for example, [1, 3, 5, 6, 8, 12, 16]). Let M be an R-module. A proper submodule P of M is called a prime (or $\mathfrak{p}$-prime) submodule of M, if for every $r \in R$ and $x \in M$, $rx \in P$ implies that $r \in \mathfrak{p} = (P : M) = \{r \in R \mid rM \subseteq P\}$ or

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$x \in P$. It is easily seen that $\mathfrak{p} = (P : M)$ is a prime ideal of R. The prime spectrum of M, denoted by $\mathrm{Spec}(M)$, is the set of all prime submodules of M. Lu [12] introduced and studied a topology over $\mathrm{Spec}(M)$, called the Zariski topology, which is a generalization of the usual Zariski topology over $\mathrm{Spec}(R)$. In that topological space, the closed sets are $V(N) = \{P \in \mathrm{Spec}(M) | (P : M) \supseteq (N : M)\}$ for all R-modules N. Here, this motivates us to introduce a new generalization of the Zariski topology from rings to modules, which inherits most of properties of the Zariski topology over $\mathrm{Spec}(M)$. In particular, for an R-module M, this topological space contains $\mathrm{Spec}(M)$ with the Zariski topology as a topological subspace.

For a proper submodule N of an R-module M, the radical of N, denoted by $\mathrm{rad}\, N$, is the intersection of all prime submodules of M containing N or, in case there are no such prime submodules, $\mathrm{rad}\, N$ is M (see, for example, [5, 8, 10, 9, 11, 13]). For an ideal I of a ring R, we assume throughout that $\sqrt{I}$ denotes the radical of I. A proper submodule Q of M is called a primary (or $\mathfrak{p}$-primary) submodule of M, if for $r \in R$ and $x \in M$, $rx \in Q$ implies that $r \in \mathfrak{p} = \sqrt{(Q : M)}$ or $x \in Q$. We call the set of all primary submodules Q of M satisfying the condition $(\mathrm{rad}\, Q : M) = \sqrt{(Q : M)}$ the *primary spectrum* of M and denote it by $\mathcal{PS}(M)$. Clearly $\mathrm{Spec}(M) \subseteq \mathcal{PS}(M)$. The inclusion is not strict in general. For example, for a vector space $\mathbb{V}$ over a field F, $\mathrm{Spec}(\mathbb{V}) = \mathcal{PS}(\mathbb{V}) =$ the set of all proper subspaces of $\mathbb{V}$, while for the ring of integers $\mathbb{Z}$ as a $\mathbb{Z}$-module, $\mathrm{Spec}(\mathbb{Z}) \subset \mathcal{PS}(\mathbb{Z})$. It should be noted that $\mathrm{rad}\, Q \neq M$ for all $Q \in \mathcal{PS}(M)$, since $\sqrt{(Q : M)}$ is a prime ideal of R.

In [12], it is shown that for an R-module M, the natural map $\psi : \mathrm{Spec}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(P) = (P : M)/\mathrm{Ann}(M)$ is continuous with respect to the Zariski topology, and so $\mathrm{Spec}(M)$ is a connected space. For a submodule N of M, we set $\nu(N) = \{Q \in \mathcal{PS}(M) | (\mathrm{rad}\, Q : M) \supseteq (N : M)\}$. It is shown that the collection of these sets satisfies the axioms for closed sets in $\mathcal{PS}(M)$ (Theorem 2.1). We call this topology, the *primary Zariski topology*, or $\mathcal{PZ}$-topology for short. A topological space X is a spectral space if X is homeomorphic to $\mathrm{Spec}(S)$ with the Zariski topology for some ring S. Equivalently, the topological space (X, τ) is spectral if and only if X is a quasi-compact T_0-space, the quasi-compact open subsets of X are closed under finite intersection and form an open basis for τ, and every nonempty irreducible closed subset of X is the closure of a singleton set with respect to τ; see [11]. The topological space $\mathrm{Spec}(M)$ equipped with the Zariski topology has been studied from the point of view of spectral spaces in

[2, 3, 11, 12, 18]. For example, it has been proved that $\mathrm{Spec}(M)$ is a spectral space if and only if $\mathrm{Spec}(M)$ is a finite set or M is a nontorsion module; see [3, Theorem 3.4].

In this work, we study $\mathcal{PS}(M)$ equipped with the $\mathcal{PZ}$-topology from the viewpoint of being a spectral space. In Section 2, for an R-module M, we introduce the map

$$\phi : \mathcal{PS}(M) \to \mathrm{Spec}(R/\operatorname{Ann}(M))$$

given by $\phi(Q) = (\operatorname{rad} Q : M)/\operatorname{Ann}(M)$. It should be noted that $(\operatorname{rad} Q : M)$ is a prime ideal of R, since Q is a primary submodule of R and $(\operatorname{rad} Q : M) = \sqrt{(Q : M)}$. We investigate some conditions under which ϕ is injective, surjective, open, and closed.

In Section 3, we find a base for the $\mathcal{PZ}$-topology whose elements are quasi-compact subsets of $\mathcal{PS}(M)$ and deduce that the $\mathcal{PZ}$-topology is quasi-compact (see Theorem 3.3 and Corollary 3.4).

In Section 4, we investigate the irreducibility of $\mathcal{PS}(M)$ with respect to the $\mathcal{PZ}$-topology. Especially, it is shown that there is a bijection between irreducible components of $\mathcal{PS}(M)$ and minimal prime ideals of $\mathrm{Spec}(R/\mathrm{Ann}(M))$ (see Corollary 4.5). We also provide a connection between the irreducible decomposition of a submodule N of M and the irreducible decomposition of the closed set $\nu(N)$ of $\mathcal{PS}(M)$.

Finally, in Section 5, we collect together some observations and results that concern some conditions under which $PS(M)$ is a spectral space. In fact, we show that, if ϕ is a surjective map, then $PS(M)$ is a spectral space if and only if it is a T_0-space if and only if ϕ is a homeomorphism (see Theorem 5.3). In particular, if M is a finitely generated multiplication module, then the mapping $Q \mapsto \operatorname{rad} Q$ is a homeomorphism from $\mathcal{PS}(M)$ to $\mathrm{Spec}(M)$ (Corollary 5.4).

2. Primary Zariski topology

In this section, we first introduce the primary Zariski topology over the primary spectrum $\mathcal{PS}(M)$ and then investigate relationships between $\mathcal{PS}(M)$ and $\mathrm{Spec}(R/\mathrm{Ann}(M))$.

Theorem 2.1. *Suppose that N and N' are submodules of an R-module M and that $(N_i)_{i\in\mathbb{N}}$ is a family of submodules of M, indexed by the set I. Then*

(1) $\mathcal{PS}(M) = \nu(0)$;
(2) $\emptyset = \nu(M)$;
(3) $\bigcap_{i\in I} \nu(N_i) = \nu(\sum_{i\in I}(N_i : M)M)$;
(4) $\nu(N) \cup \nu(N') = \nu(N \cap N')$.

Proof. (1) and (2) are clear.

(3) Suppose that $Q \in \bigcap_{i\in I} \nu(N_i)$. Then $(\operatorname{rad} Q : M) \supseteq (N_i : M)$, for all $i \in I$, which implies that $\operatorname{rad} Q \supseteq (\operatorname{rad} Q : M)M \supseteq \sum_{i\in I}(N_i : M)M$. Thus we have $(\operatorname{rad} Q : M) \supseteq (\sum_{i\in I}(N_i : M)M : M)$, and so $Q \in \nu(\sum_{i\in I}(N_i : M)M)$.

To establish the reverse inclusion, let $Q \in \nu(\sum_{i\in I}(N_i : M)M)$ such that

$$(\operatorname{rad} Q : M) \supseteq (\sum_{i\in I}(N_i : M)M : M).$$

Then, for each $i \in I$, $(\operatorname{rad} Q : M) \supseteq ((N_i : M)M : M) = (N_i : M)$. Hence $Q \in \bigcap_{i\in I} \nu(N_i)$.

(4) It is clear that $\nu(N) \cup \nu(N') \subseteq \nu(N \cap N')$. Now, assume that $Q \in \nu(N \cap N')$. Then

$$(\operatorname{rad} Q : M) \supseteq (N \cap N' : M) = (N : M) \cap (N' : M).$$

Since $(\operatorname{rad} Q : M)$ is a prime ideal, $(\operatorname{rad} Q : M) \supseteq (N : M)$ or $(\operatorname{rad} Q : M) \supseteq (N' : M)$, which means that $Q \in \nu(N) \cup \nu(N')$. $\square$

In view of Theorem 2.1, the collection $\{\nu(N)|\ N \text{ is a submodule of } M\}$ satisfies the axioms for closed sets of a topology on $\mathcal{PS}(M)$, which is called the *primary Zariski topology*, or simply $\mathcal{PZ}$-topology. It can easily be checked that $V(N) = \nu(N) \bigcap \operatorname{Spec}(M)$, and so $\operatorname{Spec}(M)$ with the Zariski topology is a topological subspace of $\mathcal{PS}(M)$ equipped with the $\mathcal{PZ}$-topology.

Consider ϕ and ψ as described in the Introduction. Also, for $\mathfrak{p} \in \operatorname{Spec}(R)$, we set $\mathcal{PS}_{\mathfrak{p}}(M) = \{Q \in \mathcal{PS}(M)|(\operatorname{rad} Q : M) = \mathfrak{p}\}$. Recall that an R-module M is multiplication if each submodule of M has the form IM for some ideal I of R; see [9]. In this case, we can take $I = (N : M)$. It is easy to verify that if M is a multiplication module, then ψ is injective.

Proposition 2.2. *Let M be an R-module and let $Q, Q' \in \mathcal{PS}(M)$. Consider the following statements.*

(1) *If $\nu(Q) = \nu(Q')$, then $Q = Q'$.*
(2) *$|\mathcal{PS}_{\mathfrak{p}}(M)| \leq 1$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$.*
(3) *ϕ is injective.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). Moreover, if M is multiplication, then (4) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Suppose that $Q, Q' \in \mathcal{PS}_{\mathfrak{p}}(M)$. Then $(\operatorname{rad} Q : M) = (\operatorname{rad} Q' : M) = \mathfrak{p}$, and so $\nu(Q) = \nu(Q')$. Thus, by the assumption (1), $Q = Q'$.

(2) $\Rightarrow$ (3) Suppose that $Q, Q' \in \mathcal{PS}(M)$ and $\phi(Q) = \phi(Q')$. Then $(\operatorname{rad} Q : M) = (\operatorname{rad} Q' : M) = \mathfrak{p}$, and so $Q, Q' \in \mathcal{PS}_{\mathfrak{p}}(M)$. Thus the assumption (2) implies that $Q = Q'$.

(3) $\Rightarrow$ (1) It is clear. □

Corollary 2.3. *If $|\mathcal{PS}_{\mathfrak{p}}(M)| = 1$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$, then ϕ is a bijective map.*

Proof. By Proposition 2.2, it is easily proved. □

Proposition 2.4. *Let M be a nonzero R-module. Consider the following statements.*

(1) *M is finitely generated.*
(2) *ψ is surjective.*
(3) *ϕ is surjective and $S_{\mathfrak{p}}(\mathfrak{p}M)$ is a $\mathfrak{p}$-prime submodule of M for every $\mathfrak{p} \in V(\operatorname{Ann}(M))$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3). Moreover, if M is multiplication, then all of the above statements are equivalent.

Proof. (1) $\Rightarrow$ (2) This is immediate from [14, Theorem 2.1].

(2) $\Leftrightarrow$ (3) Clearly ϕ is surjective. The second statement follows from [14, Propositions 4.4 and 4.5].

(2) $\Rightarrow$ (1) This is immediate from [14, Proposition 3.8]. □

Throughout the rest of the paper, for an R-module M, the symbol $\overline{R}$ denotes the ring $R/\operatorname{Ann}(M)$, and $\overline{I}$ denotes the ideal $I/\operatorname{Ann}(M)$ of $\overline{R}$.

Proposition 2.5. *Let M be an R-module. Then $\phi^{-1}(V(\overline{I})) = \nu(IM)$, for every ideal $I \in V(\operatorname{Ann}(M))$, and therefore ϕ is continuous.*

Proof. Suppose that $Q \in \phi^{-1}(V(\overline{I}))$. Then $\phi(Q) \in V(\overline{I})$, and so $(\operatorname{rad} Q : M) \supseteq I$. It follows that $\operatorname{rad} Q \supseteq (\operatorname{rad} Q : M)M \supseteq IM$. Hence $Q \in \nu(IM)$. Therefore $\phi^{-1}(V(\overline{I})) \subseteq \nu(IM)$. For the reverse inclusion, let $Q \in \nu(IM)$. Then

$$\phi(Q) = \overline{(\operatorname{rad} Q : M)} \supseteq \overline{(IM : M)} \supseteq \overline{I}.$$

Hence $Q \in \phi^{-1}(V(\overline{I}))$.

□

It is immediate from [12, Theorem 3.6] that if ψ is surjective, then $\psi(V(N)) = V(\overline{N : M})$ and $\psi(\operatorname{Spec}(M) - V(N)) = \operatorname{Spec}(\overline{R}) - V(\overline{(N : M)})$ for every submodule N of M, which means that ψ is both closed and open. Thus ψ is a bijection if and only if ψ is a homeomorphism. In the following theorem, we obtain similar results for ϕ.

Theorem 2.6. *Let M be an R-module. If ϕ is a surjection, then $\phi(\nu(N)) = V(\overline{(N : M)})$ and $\phi(\mathcal{PS}(M) - \nu(N)) = \mathrm{Spec}(\overline{R}) - V(\overline{(N : M)})$ for every submodule N of M, which means that the map ϕ is both closed and open.*

Proof. In view of Proposition 2.5, for every submodule N of M, we have

$$\phi^{-1}(V(\overline{(N : M)})) = \nu((N : M)M) = \nu(N).$$

This implies that $\phi(\nu(N)) = \phi(\phi^{-1}(V(\overline{(N : M)}))) = V(\overline{(N : M)})$. For the last part, consider the following equality:

$$\begin{aligned}\phi(\mathcal{PS}(M) - \nu(N)) =& \phi(\phi^{-1}(\mathrm{Spec}(\overline{R})) - \phi^{-1}(V(\overline{(N : M)})))\\ =& \mathrm{Spec}(\overline{R}) - V(\overline{(N : M)}).\end{aligned}$$

□

Corollary 2.7. *Let M be an R-module. Then ϕ is a bijection if and only if ϕ is a homeomorphism.*

Theorem 2.8. *Let M be an R-module and let ϕ be a surjective map. Then the following statements are equivalent:*

(1) *$\overline{R}$ is an indecomposable ring;*
(2) *$\mathcal{PS}(M)$ with the $\mathcal{PZ}$-topology is a connected space;*
(3) *$\mathrm{Spec}(M)$ with the Zariski topology is a connected space.*

Proof. (1) ⇒ (2) Let $\overline{R}$ be an indecomposable ring. We suppose that $\mathcal{PS}(M)$ is not connected with respect to the $\mathcal{PZ}$-topology and seek a contradiction. Then there exist two nonempty disjoint open sets $\nu(N_1)^c$ and $\nu(N_2)^c$ such that $\mathcal{PS}(M) = \nu(N_1)^c \cup \nu(N_2)^c$. Since ϕ is surjective, by Theorem 2.6, we have

$$\mathrm{Spec}(\overline{R}) = \phi(\nu(N_1)^c) \cup \phi(\nu(N_2)^c) = V(\overline{(N_1 : M)})^c \cup V(\overline{(N_2 : M)})^c.$$

It is easy to see that $V(\overline{(N_i : M)})^c \neq \emptyset$, since $\nu(N_i)^c \neq \emptyset$ for $i = 1, 2$. Also, we have $\nu(N_1 \cap N_2)^c = (\nu(N_1) \cup \nu(N_2))^c = \nu(N_1)^c \cap \nu(N_2)^c = \emptyset$, and thus

$$\begin{aligned}\mathrm{Spec}(\overline{R}) &= \phi(\mathcal{PS}(M)) = \phi(\nu(N_1 \cap N_2))\\ &= V(\overline{(N_1 \cap N_2 : M)})\\ &= V(\overline{(N_1 : M)} \cap \overline{(N_2 : M)}).\end{aligned}$$

It follows that

$$\begin{aligned} V(\overline{(N_1:M)})^c \cap V(\overline{(N_2:M)})^c &= (V(\overline{(N_1:M)}) \cup V(\overline{(N_2:M)}))^c \\ &= (V(\overline{(N_1:M) \cap (N_2:M)})^c \\ &= V(\overline{(N_1 \cap N_2:M)})^c = \emptyset. \end{aligned}$$

Thus $\mathrm{Spec}(\overline{R})$ is not a connected space, in contradiction to [7, Corollary 2, p 104].

(2) $\Rightarrow$ (3) Assume by way of contradiction that $\mathrm{Spec}(M)$ is a disconnected space. Thus $\mathrm{Spec}(M) = V(N_1)^c \cup V(N_2)^c$ with $V(N_1)^c \cap V(N_2)^c = \emptyset$, for some nonempty closed subsets $V(N_1)$ and $V(N_2)$ of $\mathrm{Spec}(M)$. We show that $\mathcal{PS}(M) = \nu(N_1)^c \bigcup \nu(N_2)^c$. Suppose that $Q \in \mathcal{PS}(M)$. Since $\mathrm{rad}\, Q \neq M$, there is $P \in \mathrm{Spec}(M)$ with $P \supseteq Q$. Now since $(P:M) \not\supseteq (N_i:M)$ for $i=1$ or $i=2$, we have $(\mathrm{rad}\, Q:M) \not\supseteq (N_i:M)$ for $i=1$ or $i=2$. Thus $Q \in \nu(N_1)^c \cup \nu(N_2)^c$. Moreover, $\nu(N_1)^c \cap \nu(N_2)^c = \emptyset$, because if $Q \in \nu(N_1)^c \cap \nu(N_2)^c$, then $(\mathrm{rad}\, Q:M) \not\supseteq (N_i:M)$ for $i=1,2$. It follows that there is $P \in \mathcal{PS}(M)$ such that $(P:M) \not\supseteq (\mathrm{rad}\, N_i:M)$ for $i=1,2$. This implies that $P \in V(N_1)^c \cap V(N_2)^c$, a contradiction. Therefore $\mathcal{PS}(M)$ is not a connected space.

(3)$\Rightarrow$(1) Since $\mathrm{Spec}(M)$ is a connected space, by [12, Corollary 3. 8], $\overline{R}$ has no idempotent element other than $\overline{0}$ and $\overline{1}$. Now, it is clear that $\overline{R}$ is indecomposable. $\square$

3. A base for primary Zariski topology on $\mathcal{PS}(M)$

For each $r \in R$, we set $\mathcal{B}_r := \mathcal{PS}(M) - \nu(rM)$. Our first result in this section shows that $\mathcal{B} = \{\mathcal{B}_r | r \in R\}$ is a base for the primary Zariski topology on $\mathcal{PS}(M)$. Next, we prove that $\mathcal{B}_r$'s are compact, and so $\mathcal{PS}(M)$ is a compact space.

Lemma 3.1. *With the above notations, $\mathcal{B} = \{\mathcal{B}_r | r \in R\}$ forms a base for the $\mathcal{PZ}$-topology over $\mathcal{PS}(M)$.*

Proof. Let U be an open set in $\mathcal{PS}(M)$. Then $U = (\nu(N))^c$ for some submodule N of M. Hence we have

$$\begin{aligned} U &= \mathcal{PS}(M) - \nu(N) \\ &= \mathcal{PS}(M) - \nu((N:M)M) \\ &= \mathcal{PS}(M) - \nu(\sum_{r_i \in (N:M)} r_i M) \\ &= \mathcal{PS}(M) - \cup_{r_i \in (N:M)} \nu(r_i M) \\ &= \cap_{r_i \in (N:M)} \mathcal{B}_{r_i}. \end{aligned}$$

This completes the proof. $\square$

Let R be a ring and let $\mathcal{D}_r = \mathrm{Spec}(R) - V(rR)$. It is well known that $\{\mathcal{D}_r | r \in R\}$ forms a base for the Zariski topology on $\mathrm{Spec}(R)$. It is also proved that for each $r \in R$, $\mathcal{D}_r$ and thus $\mathrm{Spec}(R)$ is a compact space with respect to the Zariski topology. This assertion has been proved for the $\mathrm{Spec}(M)$ with respect to the Zariski topology (see [12, Corollary 4.2]). Now, we show that the similar statement holds for $\mathcal{PS}(M)$ with the $\mathcal{PZ}$-topology. We begin with the following lemma.

Lemma 3.2. *Let ϕ be as in Theorem 2.5. Then, for each $r \in R$, it follows that*

(1) $\phi^{-1}(\mathcal{D}_{\bar{r}}) = \mathcal{B}_r$;
(2) $\phi(\mathcal{B}_r) \subseteq \mathcal{D}_{\bar{r}}$; *the equality holds if ϕ is surjective.*

Proof. (1) First, assume that $Q \in \mathcal{PS}(M) \setminus \phi^{-1}(\mathcal{D}_{\bar{r}})$. Thus $r \in (\mathrm{rad}\, Q : M)$. It follows that $rM \subseteq \mathrm{rad}\, Q)$, and thus $(rM : M) \subseteq (\mathrm{rad}\, Q : M)$. Hence $Q \in \mathcal{PS}(M) \setminus \mathcal{B}_r$. Therefore $\mathcal{B}_r \subseteq \phi^{-1}(\mathcal{D}_{\bar{r}})$. For the reverse inclusion, let $Q \in \phi^{-1}(\mathcal{D}_{\bar{r}})$. Then $\bar{\mathfrak{p}} = \phi(Q) = \overline{\sqrt{(Q : M)}} \in \mathcal{D}_{\bar{r}}$. Thus $r \notin \mathfrak{p}$. It follows that $(rM : M) \nsubseteq (\mathrm{rad}\, Q : M)$. Therefore $Q \notin \nu(M)$, and so $\phi^{-1}(\mathcal{D}_{\bar{r}}) \subseteq \mathcal{B}_r$, as required.
(2) Let $r \in R$, $Q \in \mathcal{PS}(M)$, and $\sqrt{(Q : M)} = \mathfrak{p}$. If $\bar{\mathfrak{p}} \in V(\bar{r}R)$, then

$$(\mathrm{rad}\, Q : M) = \sqrt{(Q : M)} = \mathfrak{p} \supseteq rR,$$

and so $rM \subseteq \mathrm{rad}\, Q$. It follows that $(rM : M) \subseteq (\mathrm{rad}\, Q : M)$, and hence $Q \in \nu(rM)$. This means that if $Q \in \mathcal{B}_r$, then $\phi(Q) = \bar{\mathfrak{p}} \in \mathcal{D}_{\bar{r}}$. Thus, we have $\phi(\mathcal{B}_r) \subseteq \mathcal{D}_{\bar{r}}$. To prove the reverse inclusion, let $\mathfrak{p} \in \mathcal{D}_{\bar{r}}$. Thus $r \in R \setminus \mathfrak{p}$. Since ϕ is surjective, there exists $Q \in \mathcal{PS}(M)$ such that $(\mathrm{rad}\, Q : M) = \mathfrak{p}$. Thus $(rM : M) \nsubseteq (\mathrm{rad}\, Q : M)$, and hence $Q \in \mathcal{B}_r$. Hence $\mathcal{D}_{\bar{r}} \subseteq \phi(\mathcal{B}_r)$.

$\square$

Theorem 3.3. *Let M be an R-module. If ϕ is surjective, then for each $r \in R$, $\mathcal{B}_r$ is compact in $\mathcal{PS}(M)$.*

Proof. Assume that $\mathcal{B}_r$ is covered by an open covering in $\mathcal{PS}(M)$. Since $\mathcal{B} = \{\mathcal{B}_r | r \in R\}$ is a base for the $\mathcal{PZ}$-topology, $\mathcal{B}_r \subseteq \bigcup_{\lambda\in\Lambda} \mathcal{B}_{r_\lambda}$ for some open cover $\{\mathcal{B}_{r_\lambda} | \lambda \in \Lambda\} \subseteq \mathcal{B}$. Thus, by Lemma 3.2, $\mathcal{D}_{\bar{r}} = \phi(\mathcal{B}_r) \subseteq \bigcup_{\lambda\in\Lambda} \phi(\mathcal{B}_{r_\lambda}) = \bigcup_{\lambda\in\Lambda} \mathcal{D}_{\bar{r}_\lambda}$. Now since, by [7, Proposition 12], $\mathrm{Spec}(\bar{R})$ is a compact space with respect to the Zariski topology, we have $\mathcal{D}_{\bar{r}} = \phi(\mathcal{B}_r) \subseteq \bigcup_{i=1}^{k} \mathcal{D}_{\bar{r}_{\lambda_i}}$ for some $\lambda_i \in \Lambda$ $(1 \leq i \leq k)$. Hence, in view of Lemma 3.2, we have $\mathcal{B}_r \subseteq \phi^{-1}(\phi(\mathcal{B}_r)) = \bigcup_{i=1}^{k} \phi^{-1}(\mathcal{D}_{\bar{r}_{\lambda_i}}) = \bigcup_{i=1}^{k} \mathcal{B}_{r_{\lambda_i}}$, as desired. $\square$

Corollary 3.4. *Let M be an R-module. If ϕ is surjective, then $\mathcal{PS}(M)$ is a compact space with respect to the $\mathcal{PZ}$-topology.*

Proof. This is an immediate consequence of Theorem 3.3 and the fact that $\mathcal{PS}(M) = \mathcal{B}_1$, where 1 is the unit element of R. □

Corollary 3.5. *Let M be an R-module and let ϕ be surjective. Then the compact open sets of $\mathcal{PS}(M)$ are closed under finite intersections and form an open base.*

Proof. First, it follows from Lemma 3.2, that

$$\mathcal{B}_{rs} = \phi^{-1}(\mathcal{D}_{\overline{rs}}) = \phi^{-1}(\mathcal{D}_{\bar{r}} \cap \mathcal{D}_{\bar{s}}) = \phi^{-1}(\mathcal{D}_{\bar{r}}) \cap \phi^{-1}(\mathcal{D}_{\bar{s}}) = \mathcal{B}_r \cap \mathcal{B}_s,$$

for some $\mathcal{B}_r, \mathcal{B}_s \in \mathcal{B}$. Thus, by Lemma 3.1, every open covering of any intersection of compact sets is a finite union of the elements $\mathcal{B}$. Now the result follows from Theorem 3.3. □

4. Irreducibility in $\mathcal{PS}(M)$

Recall that a topological space X is irreducible, if for any decomposition $X = C_1 \bigcup C_2$ with closed subsets C_i of X for $i = 1, 2$, we have $C_1 = X$ or $C_2 = X$. A subset X' of X is irreducible if it is irreducible as a subspace of X. An irreducible component of a topological space X is a maximal irreducible subset of X.

Let M be an R-module. In this section, we investigate the irreducibility in $\mathcal{PS}(M)$; in particular, we show that, for a submodule N of M, irreducible components in a primary decomposition of N into primary submodules of M relate to irreducible components of an irreducible decomposition of the closed set $\nu(N)$ in $\mathcal{PS}(M)$.

Throughout the rest of this section, assume that $cl(Y)$ is the closure of a subset Y in $\mathcal{PS}(M)$ with respect to the $\mathcal{PZ}$-topology and that $\eta(Y)$ is the intersection $\bigcap_{Q \in Y} \operatorname{rad} Q$.

Lemma 4.1. *Let M be an R-module and let Y be a subset of $\mathcal{PS}(M)$. Then $cl(Y) = \nu(\eta(Y))$. In particular, if $(0) \in Y$, then Y is dense in $\mathcal{PS}(M)$, that is, $cl(Y) = \mathcal{PS}(M)$.*

Proof. Suppose that $Q \in Y$. Then $(\operatorname{rad} Q : M) \supseteq (\eta(Y) : M)$, and so $Q \in \nu(\eta(Y))$. Hence $Y \subseteq \nu(\eta(Y))$. Therefore $cl(Y) \subseteq \nu(\eta(Y))$. For the reverse inclusion, suppose that N is a submodule of M such that $Y \subseteq \nu(N)$. Then, for each $Q \in \nu(N)$, $(\operatorname{rad} Q : M) \supseteq (N : M)$. Let

$Q' \in \nu(\eta(Y))$. Then

$$(\operatorname{rad} Q' : M) \supseteq (\eta(Y) : M) = (\bigcap_{Q \in \nu(N)} \operatorname{rad} Q : M)$$
$$= \bigcap_{Q \in \nu(N)} (\operatorname{rad} Q : M) \supseteq (N : M).$$

Hence $Q' \in \nu(N)$. This implies that $\nu(\eta(Y))$ is the smallest closed subset of $\mathcal{PS}(M)$ containing Y. Therefore $\nu(\eta(Y)) = cl(Y)$.

Now suppose for the remainder of the proof that $(0) \in Y$. Thus $cl(Y) = \nu(\eta(Y)) = \nu(\operatorname{rad}(0)) = \mathcal{PS}(M)$. □

Theorem 4.2. *Let M be an R-module. Then, for each $Q \in \mathcal{PS}(M)$, the closed set $\nu(Q)$ is irreducible in $\mathcal{PS}(M)$.*

Proof. It follows from Lemma 4.1 and the fact that the closure of each irreducible subset of a topological space is irreducible. □

Theorem 4.3. *Let M be an R-module and let $Y \subseteq \mathcal{PS}(M)$. If $\eta(Y)$ is a primary submodule of M, then Y is irreducible. Conversely if Y is irreducible, then $\Upsilon = \{(\operatorname{rad} Q : M) \mid Q \in Y\}$ is an irreducible subset of* $\operatorname{Spec}(R)$*, that is, $\eta(\Upsilon) = (\eta(Y) : M)$ is a prime ideal of R.*

Proof. Suppose that $\eta(Y)$ is a primary submodule of M. By Lemma 4.1 and Theorem 4.2, $cl(Y) = \nu(\eta(Y))$ is irreducible, and so Y is irreducible.

Conversely, if Y is irreducible, then by Theorem 2.5, $\{\overline{(\operatorname{rad} Q : M)} \mid Q \in Y\}$ is irreducible, and thus, by[15, p. 129, Proposition 14], we have that

$$\overline{\bigcap_{Q \in Y} (\operatorname{rad} Q : M)} = \overline{(\bigcap_{Q \in Y} \operatorname{rad} Q : M)} = \overline{(\eta(Y) : M)}$$

is a prime ideal of $\overline{R}$. Therefore

$$\eta(\Upsilon) = (\bigcap_{Q \in Y} \operatorname{rad} Q : M) = (\eta(Y) : M)$$

is a prime ideal of R, and hence, by the first part, Υ is an irreducible submodule of $\operatorname{Spec}(R)$. □

Theorem 4.4. *Let M be an R-module, let $Y \subseteq \mathcal{PS}(M)$, and let ϕ be surjective. Then Y is an irreducible closed subset of $\mathcal{PS}(M)$ if and only if $Y = \nu(Q)$ for some $Q \in \mathcal{PS}(M)$.*

Proof. Suppose that Y is an irreducible closed subset of $\mathcal{PS}(M)$. Then $Y = \nu(N)$ for some submodule N of M. Thus, by Theorem 4.3,

$$(\eta(\nu(N)) : M) = (\eta(Y) : M) = \mathfrak{p}$$

is a prime ideal of R. Since ϕ is surjective, there exists $Q \in \mathcal{PS}(M)$ such that $(\operatorname{rad} Q : M) = \mathfrak{p} = (\eta(\nu(N)) : M)$. This implies that

$$Y = \nu(N) = \nu(\eta(\nu(N))) = \nu(Q).$$

$\square$

Corollary 4.5. *Let M be an R-module and let ϕ be surjective. Then the correspondence $\nu(Q) \longmapsto \overline{(\operatorname{rad} Q : M)}$ provides a bijection from the set of irreducible components of $\mathcal{PS}(M)$ to the set of minimal prime ideals of $\overline{R}$.*

Proof. First we show that the given correspondence is well-defined. For this, let $\nu(Q) = \nu(Q')$ for some $Q, Q' \in \mathcal{PS}(M)$. Then $(\operatorname{rad} Q : M) \supseteq (Q' : M)$ and $(\operatorname{rad} Q' : M) \supseteq (Q : M)$. Now by taking radical of both side of these inclusions, we have $(\operatorname{rad} Q : M) = (\operatorname{rad} Q : M)$. Moreover, if $\nu(Q)$ is an irreducible component, $\mathfrak{p} = (\operatorname{rad} Q : M)$, and $\mathfrak{p}' \subseteq \mathfrak{p}$ for some $\mathfrak{p}' \in \operatorname{Spec}(R)$, then by the surjectivity of ϕ, there is $Q' \in \mathcal{PS}(M)$ such that $(\operatorname{rad} Q' : M) = \mathfrak{p}'$. It follows that $\nu(Q) \subseteq \nu(Q')$. Since, by Theorem 4.2, $\nu(Q')$ is irreducible, we have $\nu(Q) = \nu(Q')$, which implies that $\mathfrak{p} = \mathfrak{p}'$. Thus $\overline{\mathfrak{p}}$ is a minimal prime ideal of $\overline{R}$.
For the surjectivity of the correspondence, assume that $\bar{\mathfrak{p}}$ is a minimal prime ideal of $\bar{R}$. Then, since ϕ is surjective, there exists $Q \in \mathcal{PS}(M)$ such that $\overline{(\operatorname{rad} Q : M)} = \bar{\mathfrak{p}}$. Moreover, by Theorem 4.2, $\nu(Q)$ is irreducible. Now let $\nu(Q) \subseteq Y$ for some irreducible subset Y of $\mathcal{PS}(M)$. Without loss of generality, we may assume that Y is closed, since the closure of an irreducible subset of $\mathcal{PS}(M)$ is irreducible. Thus, by Theorem 4.4, there exists $Q' \in \mathcal{PS}(M)$ such that $Y = \nu(Q')$. It follows that $\mathfrak{p} = (\operatorname{rad} Q : M) \supseteq (\,radQ' : M)$, and so, by the minimality of $\mathfrak{p}$, we have $\mathfrak{p} = (\operatorname{rad} Q : M) = (\operatorname{rad} Q' : M)$. Hence $\nu(Q) = \nu(Q') = Y$. This means that $\nu(Q)$ is an irreducible component of $\mathcal{PS}(M)$. $\square$

Recall that a submodule N of an R-module M has a primary decomposition if it is a finite intersection of primary submodules Q_i $(1 \leq i \leq n)$ of M. This primary decomposition is called irreducible, if $Q_i \not\supseteq \bigcap_{i \neq j} Q_j$ and $\sqrt{(Q_i : M)}$'s are all distinct.

In contrast, we say that $\nu(N) = \bigcup_{i=1}^{n} \nu(Q_i)$, for some $Q_i \in \mathcal{PS}(M)$ $(1 \leq i \leq n)$, is an irreducible decomposition for $\nu(N)$, if $\nu(Q_i) \not\subseteq \bigcup_{i \neq j} \nu(Q_j)$ and $\nu(Q_i)$ $(1 \leq i \leq n)$ are all distinct irreducible subsets.

Theorem 4.6. *Let M be a finitely generated module over a Noetherian ring R. Then, for each submodule N of M, $\nu(N)$ has an irreducible decomposition.*

Proof. Suppose that R is Noetherian, and let N be a submodule of a finitely generated R-module M. By [7, Theorem 18.20], N has an irreducible decomposition $\bigcap_{i=1}^{n} Q_i$. Then, in view of Theorem 2.1, we have $\nu(N) = \bigcup_{i=1}^{n} \nu(Q_i)$ and $\nu(Q_j) \nsubseteq \bigcup_{i \neq j} \nu(Q_i)$. Also by Theorem 4.2, every closed set $\nu(Q_i)(1 \leq i \leq n)$ is irreducible. Observe that $\nu(Q_i)$'s are all distinct, because $\sqrt{(Q_i : M)}$'s are all distinct. □

Theorem 4.7. *Let R be a ring and let F be a free R-module. If $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is an irreducible decomposition for an ideal I of R, then $\nu(IF) = \bigcup_{i=1}^{n} \nu(\mathfrak{q}_i F)$ is an irreducible decomposition for $\nu(IF)$.*

Proof. Suppose that $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a primary decomposition for I. By [17, Lemma 1.1], $IF = \bigcap_{i=1}^{n} \mathfrak{q}_i F$ is an irreducible primary decomposition for IF. Thus $\nu(IF) = \bigcup_{i=1}^{n} \nu(\mathfrak{q}_i F)$, where $\nu(\mathfrak{q}_i F) \nsubseteq \bigcup_{i \neq j} \nu(\mathfrak{q}_j F)$. Note that, in view of Theorem 4.2, $\nu(\mathfrak{q}_i F)$'s are irreducible subsets of $\mathcal{PS}(F)$. Since the ideals $\sqrt{(\mathfrak{q}_i F : F)} = \sqrt{\mathfrak{q}_i}$ $(1 \leq i \leq n)$ are distinct in R, it follows that $\nu(\mathfrak{q}_i F)$'s are all distinct. □

If $\mathfrak{m}$ is a maximal ideal of R, then we set $T_{\mathfrak{m}}(M) = \{m \in M : (1-p)m = 0 \text{ for some } p \in \mathfrak{m}\}$. We say that M is $\mathfrak{m}$-cyclic provided there exist $x \in \mathfrak{m}$ and $m \in M$ such that $(1-x)M \subseteq Rm$.

Theorem 4.8. *Let $\mathfrak{q}$ be a primary ideal of R and let M be a faithful multiplication R-module. Let $r \in R, x \in M$ be such that $rx \in \mathfrak{q}M$. Then $r \in \sqrt{q}$ or $x \in \mathfrak{q}M$. In particular, if $qM \neq M$, then qM is a primary submodule of M.*

Proof. Let $r \notin \sqrt{\mathfrak{q}}$, and set $K := \{r \in R : rx \in \mathfrak{q}M\}$. Suppose that $K \neq R$, and seek a contradiction. Then there exists a maximal ideal $\mathfrak{m}$ of R such that $K \subseteq \mathfrak{m}$. Clearly $x \notin T_{\mathfrak{m}}(M)$. If $x \in T_{\mathfrak{m}}(M)$, then $(1-p)x = 0$ for some $p \in \mathfrak{m}$. Thus $0 = (1-p)x \in \mathfrak{q}M$, and so $1-p \in K \subseteq \mathfrak{m}$. Hence $1 \in \mathfrak{m}$, a contradiction. By [9, Theorem1.2], M is $\mathfrak{m}$-cyclic; therefore there exist $m \in M$ and $p \in \mathfrak{m}$ such that $(1-p)M \subseteq Rm$. In particular, $(1-p)x = sm$ and $(1-p)rx = rsm = tm$ for some $s \in R$ and $t \in \mathfrak{q}$. Thus $(rs-t)m = 0$. Since $(1-p)M \subseteq Rm$, it follows that $(1-p)\operatorname{Ann}(m)M \subseteq R\operatorname{Ann}(m)m = 0$, and so $(1-p)\operatorname{Ann}(m)M = 0$. Now $[(1-p)\operatorname{Ann}(m)]M = 0$ implies $(1-p)\operatorname{Ann}(m) = 0$, because M is faithful and hence $(1-p)rs = (1-p)t \in \mathfrak{q}$. Indeed, $rs-t \in \operatorname{Ann}(m)$, and so $(1-p)(rs-t) = 0$, $(1-p)rs = (1-p)t$. Now since $(1-p), r \in \sqrt{\mathfrak{q}}$ and $\mathfrak{q}$ is primary, we have $s \in \mathfrak{q}$. It follows that $(1-p)x = sm \in \mathfrak{q}M$ and hence $1-p \in K \subseteq \mathfrak{m}$, a contradiction. It follows that $K = R$ and $x \in \mathfrak{q}M$, as required. □

Corollary 4.9. *Let M be a faithful multiplication module over a Noetherian ring R. If $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is an irreducible decomposition for*

I, then $IM = \bigcap_{i=1}^{n} \mathfrak{q}_i M$ is an irreducible decomposition for IM, and therefore $\nu(IM) = \bigcup_{i=1}^{n} \nu(\mathfrak{q}_i M)$ is an irreducible decomposition for $\nu(IM)$.

Proof. Since M is a multiplication R-module and R is Noetherian, it follows that M is Noetherian. By [9, Theorem 1.6], $IM = (\bigcap_{i=1}^{n} \mathfrak{q}_i)M = \bigcap_{i=1}^{n}(\mathfrak{q}_i M)$. Also, [9, Theorem 1.6(i) and Theorem 3.1(ii)] implies that $q_i M \not\supseteq \bigcap_{i=1}^{n} q_j M$ and $\sqrt{(q_i M : M)} = \sqrt{q_i}$, which are distinct. Observe that, by Theorem 4.8, the submodules $\mathfrak{q}_i M$ $(1 \leq i \leq n)$ are primary submodules of M. Now, by Theorem 4.6, $\nu(IM) = \bigcup_{i=1}^{n} \nu(q_i M)$ is an irreducible decomposition for $\nu(IM)$. $\square$

5. $\mathcal{PS}(M)$ as a spectral space

In this section, we investigate $\mathcal{PS}(M)$ with the $\mathcal{PZ}$-topology from the view point of being a spectral space.

A topological space X is a T_0-space if, for every pair of distinct points of X, at least one of them has a neighborhood not containing the other. It is well known that X is a T_0-space if and only if the closure of distinct points are distinct. Clearly, for a ring R, $\mathrm{Spec}(R)$ is a T_0-space with the Zariski topology. However, if M is a vector space with $\dim(M) > 1$, then $\mathcal{PS}(M)$ is the set of all proper subspaces of M and the $\mathcal{PZ}$-topology on $\mathcal{PS}(M)$ is $\{\mathcal{PS}(M), \emptyset\}$. Thus $\mathcal{PS}(M)$ is not a T_0-space.

A topological space X is called a spectral space if it is homeomorphic with the spectrum of a commutative ring R equipped with the Zariski topology. In [11, Proposition 4], spectral spaces have been characterized by Hochster as the topological spaces X that satisfy the following conditions:

(1) X is compact;
(2) the compact open subsets of X are closed under finite intersection and form an open base;
(3) each irreducible closed subset of X is the closure of a singleton set;
(4) X is a T_0-space.

Now, if ϕ is a surjective map, we showed that $\mathcal{PS}(M)$ together with the $\mathcal{PZ}$-topology satisfies conditions (1), (2) and (3) (see Corollary 3.4, Corollary 3.5 and Theorem 4.4). Thus $\mathcal{PS}(M)$ is a spectral space if and only if it is a T_0-space.

Theorem 5.1. *Let M be an R-module and let ϕ be a surjective map. Then $\mathcal{PS}(M)$ is a T_0-space if and only if $|\mathcal{PS}_{\mathfrak{p}}(M)| \leq 1$, for every $\mathfrak{p} \in \mathrm{Spec}(R)$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{PS}(M)$ is a T_0-space and Q_1, Q_2 are two distinct $\mathfrak{p}$-primary submodules of M. Since $(\text{rad}(Q_1) : M) = (\text{rad}(Q_2) : M) = \mathfrak{p}$, it is easy to see that $\nu(Q_1) = \nu(Q_2)$. By Lemma 4.1, this contradicts the fact that the closure of distinct points are distinct. Thus we have at most only one $\mathfrak{p}$-primary submodule.

($\Leftarrow$) Let Q_1 and Q_2 be two distinct primary submodules of M. By the assumption that the ideals $(\text{rad}(Q_1) : M)$ and $(\text{rad}(Q_2) : M)$ are distinct, we have that $Q_2 \notin \nu(Q_1)$ and $Q_1 \notin \nu(Q_2)$. Hence, by Lemma 4.1, $cl(\{Q_1\}) \neq cl(\{Q_2\})$. $\square$

A topological space X is a T_1-space if, for every pair of distinct points, each of them has a neighborhood not containing the other. It can easily be checked that X is a T_1-space if and only if every singleton subset is closed.

Theorem 5.2. *Let M be a finitely generated R-module. Then $\mathcal{PS}(M)$ is a T_1-space if and only if $\mathcal{PS}(M) = \mathcal{MAX}(M)$, where $\mathcal{MAX}(M)$ is the set of all maximal submodules of M. In this case, $\mathcal{PS}(M) = \text{Spec}(M) = \mathcal{MAX}(M)$.*

Proof. Suppose that $\mathcal{PS}(M)$ is a T_1-space. Then every singleton subset of $\mathcal{PS}(M)$ is closed. Assume that $Q \in \mathcal{PS}(M)$. Hence, by Lemma 4.1, $\nu(Q) = cl(\{Q\}) = \{Q\}$. Since M is finitely generated, there exists $N \in \mathcal{MAX}(M)$ such that $Q \subseteq N$. It follows that $(Q : M) \subseteq (N : M)$ and thus $N \in \nu(Q) = \{Q\}$, since N is a prime submodule of M. Hence $N = Q$, and so $Q \in \mathcal{MAX}(M)$. Therefore $\mathcal{PS}(M) \subseteq \mathcal{MAX}(M)$. The reverse inclusion is clear.

Conversely, suppose that $\{Q\}$ is a singleton subset of $\mathcal{PS}(M)$. If $Q' \in \nu(Q)$, then $\sqrt{(Q' : M)} \supseteq \sqrt{(Q : M)}$. Since $(Q : M)$ and $(Q' : M)$ are maximal ideals of R, $(Q : M) = (Q' : M)$. It follows that $Q \bigcap Q' \in \mathcal{PS}(M)$, and so $Q \bigcap Q' \in \mathcal{MAX}(M)$. Hence $Q = Q'$, and so $\nu(Q) = \{Q\}$. Therefore $\mathcal{PS}(M)$ is a T_1-space.
The final claim now follows from the fact that $\mathcal{MAX}(M) \subseteq \text{Spec}(M) \subseteq \mathcal{PS}(M)$. $\square$

Theorem 5.3. *Let M be an R-module and let ϕ be a surjective map. Then the following statements are equivalent.*

(1) *$\mathcal{PS}(M)$ is a spectral space;*
(2) *$\mathcal{PS}(M)$ is a T_0-space;*
(3) *ϕ is injective;*
(4) *ϕ is a homeomorphism.*

Proof. (1)$\Rightarrow$(2) This is immediate from Hochster's characterization.

(2)$\Rightarrow$(3) Let $\phi(Q) = \phi(Q')$ for $Q, Q' \in \mathcal{PS}(M)$. It follows that $(\text{rad}\, Q : M) = (\text{rad}\, Q' : M)$. Thus $Q, Q' \in \mathcal{PS}_{\mathfrak{p}}(M)$, where $\mathfrak{p} =$

$(\operatorname{rad} Q : M)$. Now by the assumption and Theorem 5.1, we have $Q = Q'$.

(3)⇒(4) follows from Corollary 2.7.

(4)⇒(1) It is clear. □

The following corollary is immediate from Theorems 2.4 and 5.3.

Corollary 5.4. *Let M be a finitely generated multiplication R-module. Then the mapping $Q \mapsto \operatorname{rad} Q$ is a homeomorphism from $\mathcal{PS}(M)$ to* $\operatorname{Spec}(M)$.

Proof. By [12, Corollary 6.6], it is easily obtained. □

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PRIMARY ZARISKI TOPOLOGY ON THE PRIMARY SPECTRUM OF A MODULE

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توپولوژی زاریسکی اولیه روی طیف اولیه‌ی یک مدول

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فرض کنید R یک حلقه جابجایی و یکدار و M یک R- مدول باشد. منظور از طیف اولیه‌ی یک مدول که با نماد $\mathcal{PS}(M)$ نشان داده می‌شود، مجموعه تمام زیرمدول‌های اولیه‌ی Q از M است به گونه‌ای که $(rad(Q) : M) = \sqrt{(Q : M)}$. در این مقاله $\mathcal{PS}(M)$ را به یک توپولوژی مجهز می‌کنیم که توپولوژی زاریسکی روی طیف اول $Spec(M)$ توپولوژی زیرفضایی آن است. ما فشردگی و تحویل‌ناپذیری این فضای توپولوژیک را بررسی کرده و شرایطی را فراهم می‌آوریم که $\mathcal{PS}(M)$ یک فضای طیفی شود.

کلمات کلیدی: طیف اولیه، توپولوژی زاریسکی اولیه، زیرمدول اولیه، ایده آل اول.