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ANNIHILATOR OF LOCAL COHOMOLOGY MODULES UNDER THE RING EXTENSION $R \subset R[X]$

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ABSTRACT. Let R be a commutative Noetherian ring, I an ideal of R and M a non-zero R-module. In this paper, we calculate the extension of annihilator of local cohomology modules $H^t_I(M)$, $t \geq 0$, under the ring extension $R \subset R[X]$ (resp., $R \subset R[[X]]$). By using this extension we will present some of the faithfulness conditions of local cohomology modules, and show that if the Lynch's conjecture [11] holds in $R[[X]]$, then it will holds in R.

1. Introduction

Throughout this paper, R denotes a commutative Noetherian ring (with identity) and I is an ideal of R. The local cohomology modules $H^i_I(M)$, $i = 0, 1, 2, \ldots$, of an R-module M with respect to I were introduced by Grothendieck [9]. They arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism

$$H^i_I(M) \cong \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [9] or [5] for more details about local cohomology.

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Recall that, for an R-module M, the *cohomological dimension* of M with respect to I, denoted by $\operatorname{cd}(I, M)$, is defined as

$$\operatorname{cd}(I, M) := \sup\{i \in \mathbb{Z} : H_I^i(M) \neq 0\}.$$

One of the important problems in commutative algebra is determining the annihilator of local cohomology modules. This problem has been studied by several authors; see, for example, [1, 2, 3, 10, 11, 13]. In this paper, we will calculate the extension of annihilator of local cohomology modules, under the ring extension $R \subset R[X]$ (resp., $R \subset R[[X]]$), as the first main result.

Lynch in [11] conjectured the following:

For every Noetherian local ring $(R, \mathfrak{m})$ and any ideal I of R, if $\operatorname{cd}(I, R) = t > 0$ then $\dim R/\operatorname{Ann}_R H_I^t(R) = \dim R/\Gamma_I(R)$.

An another aim of this paper is to find a relation between the Lynch's conjecture in R and $R[[X]]$.

2. Main results

The following lemma will be quite useful in this section.

Lemma 2.1. *Let R be a Noetherian ring and M be a non-zero R-module. Let X be an indeterminate over R. Then for every monic polynomial $f \in R[X]$ of positive degree, the following statements hold:*

(i) $\Gamma_{fR[X]}(M \otimes_R R[X]) = 0$.

(ii) *For every positive integer n,*

$$(0 :_{H^1_{fR[X]}(M \otimes_R R[X])} f^n) \cong M[X]/f^n M[X].$$

In particular, $H^1_{fR[X]}(M \otimes_R R[X]) \neq 0$.

(iii) $\operatorname{cd}(IR[X] + fR[X], R[X]) = \operatorname{cd}(I, R) + 1$.

Proof. See [4, Lemma 2.9 and Theorem 2.10]. □

The next theorem is the first main result of this paper.

Theorem 2.2. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R-module. Let $H_I^t(M) \neq 0$, for integer $t \geq 0$. If $J := \operatorname{Ann}_R H_I^t(M)$, then*

$$\operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) = \operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) = JR[X],$$

where X is an indeterminate over R and $f \in R[X]$ is a monic polynomial of positive degree.

Proof. Since $R[X]$ is a faithfully flat R-algebra, it follows from [5, Theorem 4.3.2] that $H^t_I(M) \longrightarrow H^t_{IR[X]}(M[X])$ is injective. Therefore,

$$J = \operatorname{Ann}_R H^t_I(M) = \operatorname{Ann}_R H^t_{IR[X]}(M[X]) = \operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) \cap R.$$

So $JR[X] \subseteq \operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X])$.

On the other hand, using Lemma 2.1 and [14, Corollary 1.4] yield the following isomorphism

$$H^{t+1}_{IR[X]+fR[X]}(M[X]) \cong H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \neq 0.$$

So

$$\operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) \subseteq \operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X])),$$

and hence

$$JR[X] \subseteq \operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]).$$

Now, we claim that $\operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \subseteq JR[X]$. Let $g = a_0 + a_1X + \cdots + a_nX^n$ be a non-zero polynomial with $a_j \in R$, for all $0 \leq j \leq n$, and let $g \notin JR[X]$ but $g \in \operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X]))$. Since $g \notin JR[X]$, it follows that there exists $a_{j'} \in R \setminus J$, for $0 \leq j' \leq n$, such that $a_{j'}H^t_I(M) \neq 0$ and hence $a_{j'}b \neq 0$ for some $b \in H^t_I(M)$. It is clear that

$$g \notin \operatorname{Ann}_{R[X]}((H^t_I(M))[X]) = \operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]).$$

On the other hand, using Lemma 2.1 yields the following exact sequence

$$0 \longrightarrow (H^t_I(M))[X] \longrightarrow ((H^t_I(M))[X])_f \longrightarrow H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \longrightarrow 0,$$

which implies that $((H^t_I(M))[X])_f/(H^t_I(M))[X] \cong H^1_{fR[X]}(H^t_{IR[X]}(M[X]))$. Thus

$$g((H^t_I(M))[X])_f \subseteq (H^t_I(M))[X].$$

In fact, $g(((H^t_I(M))[X])[1/f]) \subseteq (H^t_I(M))[X]$. Let $m > n$ be an integer, and set $h := b/f^m \in ((H^t_I(M))[X])[1/f]$. Since f is a monic polynomial of positive degree, it follows that f^m is a monic polynomial of degree at least m. Since $gh \in (H^t_I(M))[X]$, it follows that $gb \in f^m(H^t_I(M))[X]$. But, $0 \neq gb = a_0b + a_1bX + \cdots + a_nbX^n$ and $n < m$, which is a contradiction. Therefore,

$$\begin{aligned} JR[X] &\subseteq \operatorname{Ann}_{R[X]}((H^t_I(M))[X]) \\ &= \operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) \\ &\subseteq \operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \\ &\subseteq JR[X]. \end{aligned}$$

Thus, we obtain that

$$\operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) = \operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) = JR[X].$$

□

With a similar argument, we have the following corollary.

Corollary 2.3. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R-module. Let $H^t_I(M) \neq 0$, for integer $t \geq 0$. If $J := \operatorname{Ann}_R H^t_I(M)$, then*

$$\operatorname{Ann}_{R[[X]]} H^t_{IR[[X]]}(M[[X]]) = \operatorname{Ann}_{R[[X]]} H^{t+1}_{IR[[X]]+XR[[X]]}(M[[X]]) = JR[[X]].$$

□

Corollary 2.4. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R-module. Let X be an indeterminate over R and f be a monic polynomial of positive degree. For an integer $t \geq 0$, $\operatorname{Ann}_R H^t_I(M) = 0$ if and only if $\operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) = 0$ (resp., $\operatorname{Ann}_{R[[X]]} H^{t+1}_{IR[[X]]+XR[[X]]}(M[[X]]) = 0$).*

Proof. The assertion follows from Theorem 2.2 (resp., Corollary 2.3). □

The following theorem will be useful in the proof of Corollary 2.6.

Theorem 2.5. *Let R be a (not necessarily local) Noetherian ring, I an ideal of R and M a finitely generated R-module such that $\operatorname{cd}(I, M) = t > 0$. Then $H^t_I(M)$ is not finitely generated.*

Proof. First, it is clear that for every $\mathfrak{p} \in \operatorname{Supp} H^t_I(M)$, $\operatorname{cd}(IR_\mathfrak{p}, M_\mathfrak{p}) = \operatorname{cd}(I, M) = t$. So, without loss of generality, we may assume that $(R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module. Since $\operatorname{Supp} M = \operatorname{Supp} R/\operatorname{Ann} M$, it follows from [7, Theorem 2.2] and [5, Theorem 4.2.1] (the Independence Theorem) that

$$\operatorname{cd}(I, M) = \operatorname{cd}(I, R/\operatorname{Ann} M) = \operatorname{cd}(I(R/\operatorname{Ann} M), R/\operatorname{Ann} M).$$

Since $H^t_{I(R/\operatorname{Ann} M)}(-)$ is a right exact functor and M is an $R/\operatorname{Ann} M$-module, it follows from [5, Exercise 6.1.8] that

$$\begin{aligned} H^t_I(M)/\mathfrak{m} H^t_I(M) &\cong H^t_I(M) \otimes_R R/\mathfrak{m} \\ &\cong (H^t_{I(R/\operatorname{Ann} M)}(R/\operatorname{Ann} M) \otimes_{R/\operatorname{Ann} M} M) \otimes_R R/\mathfrak{m} \\ &\cong H^t_{I(R/\operatorname{Ann} M)}(R/\operatorname{Ann} M) \otimes_{R/\operatorname{Ann} M} M/\mathfrak{m} M \\ &\cong H^t_{I(R/\operatorname{Ann} M)}(M/\mathfrak{m} M) \cong H^t_I(M/\mathfrak{m} M) = 0. \end{aligned}$$

Therefore, $H^t_I(M) = \mathfrak{m} H^t_I(M)$ and hence by Nakayama's lemma we can deduce that the R-module $H^t_I(M)$ is not finitely generated. □

Corollary 2.6. *Let R be a (not necessarily local) Noetherian ring, I an ideal of R and M a finitely generated R-module. If $\operatorname{cd}(I, M) = 1$, then $\operatorname{Ann}_R H^1_I(M) \subseteq Z_R(M)$.*

Proof. Let $\operatorname{Ann}_R H^1_I(M) \nsubseteq Z_R(M)$. Hence there exists $x \in \operatorname{Ann}_R H^1_I(M)$, such that $x \notin Z_R(M)$. An exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces the exact sequence

$$\Gamma_I(M/xM) \xrightarrow{f} H^1_I(M) \xrightarrow{x} H^1_I(M),$$

which implies that $H^1_I(M)$ is a finitely generated R-module. But, in view of Theorem 2.5, this is a contradiction. □

For the next result we need the following lemma.

Lemma 2.7. *Let R be a commutative Noetherian ring and X be an indeterminate over R. Then every associated prime ideal of $R[X]$ is extended, and hence*

$$\operatorname{Ass}_{R[X]}(R[X]) = \{\mathfrak{p}\, R[X] : \mathfrak{p} \in \operatorname{Ass}_R(R)\}.$$

Proof. See [8, Theorem]. □

Theorem 2.8. *Let R be a Noetherian ring, I an ideal of R and M a non-zero R-module. Let X be an indeterminate over R and $f \in R[X]$ be a monic polynomial of positive degree. Then for an integer $t \geq 0$, $\operatorname{Ann}_R H^t_I(M) \subseteq Z_R(R)$ (the set of all zero-divisors of R) if and only if $\operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) \subseteq Z_{R[X]}(R[X])$.*

Proof. The assertion follows from Theorem 2.2, [12, Theorem 7.5] and Lemma 2.7. □

Corollary 2.9. *Let R be a Noetherian ring and I an ideal of R with $\operatorname{cd}(I, R) = 1$. Let X be an indeterminate over R and $f \in R[X]$ be a monic polynomial of positive degree. Then*

$$\operatorname{Ann}_{R[X]} H^2_{IR[X]+fR[X]}(R[X]) \subseteq Z_{R[X]}(R[X]).$$

In particular, if R is an integral domain, then

$$\operatorname{Ann}_{R[X]} H^2_{IR[X]+fR[X]}(R[X]) = \operatorname{Ann}_{R[X]} H^1_{IR[X]}(R[X]) = 0.$$

Proof. The first assertion follows from Corollary 2.6 and Theorem 2.8. Moreover, the last assertion is immediate from Corollary 2.4. □

Remark 2.10. Let R be a Noetherian ring, I an ideal of R and M an R-module. If $H^t_I(M) \neq 0$, for integer t, then it follows from Lemma 2.1 that $H^t_{IR[X]}(M[X])$ is not an injective $R[X]$-module. So, $\operatorname{injdim}_{R[X]} H^t_{IR[X]}(M[X]) > 0$.

Corollary 2.11. *Let R be a Noetherian ring, I an ideal of R and M an R-module, such that $H^t_I(M) \neq 0$, for integer $t \geq 0$. If*

$$\operatorname{injdim}_{R[X]} H^t_{IR[X]}(M[X]) = 1,$$

then

$$\operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) = \operatorname{Ann}_{R[X]} H^t_{IR[X]}(M[X]) \subseteq Z_{R[X]}(R[X]),$$

where f is a monic polynomial of positive degree.
In particular, $\operatorname{Ann}_R H^t_I(M) \subseteq Z_R(R)$.

Proof. Since $\operatorname{injdim}_{R[X]} H^t_{IR[X]}(M[X]) = 1$, there is an exact sequence

$$0 \longrightarrow H^t_{IR[X]}(M[X]) \longrightarrow \mathbb{E}_0 \longrightarrow \mathbb{E}_1 \longrightarrow 0,$$

as an injective resolution of $H^t_{IR[X]}(M[X])$, which induces the following exact sequence

$$\Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \longrightarrow 0.$$

If

$$\begin{aligned}\operatorname{Ann}_{R[X]} H^{t+1}_{IR[X]+fR[X]}(M[X]) &= \operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \\ &\nsubseteq Z_{R[X]}(R[X]),\end{aligned}$$

then there exists $g \in \operatorname{Ann}_{R[X]} H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \backslash Z_{R[X]}(R[X])$. So, we have the following exact sequence

$$\Gamma_{fR[X]}(\mathbb{E}_1)/g\Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \longrightarrow 0.$$

Since $H^1_{fR[X]}(H^t_{IR[X]}(M[X])) \neq 0$, it follows that $\Gamma_{fR[X]}(\mathbb{E}_1) \neq g\Gamma_{fR[X]}(\mathbb{E}_1)$. On the other hand, $\Gamma_{fR[X]}(\mathbb{E}_1)$ is an injective $R[X]$-module. Thus the exact sequence

$$0 \longrightarrow R[X] \xrightarrow{g} R[X] \longrightarrow R[X]/gR[X] \longrightarrow 0,$$

induces the exact sequence $\Gamma_{fR[X]}(\mathbb{E}_1) \xrightarrow{g} \Gamma_{fR[X]}(\mathbb{E}_1) \longrightarrow 0$. Therefore, $\Gamma_{fR[X]}(\mathbb{E}_1) = g\Gamma_{fR[X]}(\mathbb{E}_1)$, which is a contradiction. □

We need the following notation in the proof of Corollary 2.12.
Notation. [6, Theorem A.11] Let $(R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings. If M is a finitely generated R-module and N is an R-flat finitely generated S-module, then

$$\dim_S(M \otimes_R N) = \dim_R M + \dim_S N/\mathfrak{m}N.$$

Corollary 2.12. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, I an ideal of R and i be a non-negative integer. Let X be an indeterminate over R, and set $S := R[[X]]$. Then, the following statements are equivalent:*

(i) $\dim_R R/\operatorname{Ann}_R H^i_I(R) = \dim_R R/\Gamma_I(R)$.
(ii) $\dim_S S/\operatorname{Ann}_S H^i_{IS}(S) = \dim_S S/\Gamma_{IS}(S)$.

Proof. Since $R \longrightarrow S$ is a faithfully flat ring homomorphism, it follows from Corollary 2.3 that

$$R/\operatorname{Ann}_R H^i_I(R) \otimes_R S \cong S/\operatorname{Ann}_S H^i_{IS}(S).$$

Also using [5, Theorem 4.3.2] yields the following isomorphism

$$R/\Gamma_I(R) \otimes_R S \cong S/\Gamma_{IS}(S).$$

Now, the assertion follows from the Notation. □

Now, we are ready to state and prove the second main result of this paper.

Corollary 2.13. *Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring and let I be an ideal of R with $\operatorname{cd}(I, R) = t > 0$. If the Lynch's conjecture holds in $R[[X]]$, then it holds in R.*

Proof. By the assumption we have

$$\dim_{R[[X]]} R[[X]]/\operatorname{Ann}_{R[[X]]} H^{\operatorname{cd}(\mathbb{J},R[[X]])}_{\mathbb{J}}(R[[X]]) = \dim_{R[[X]]} R[[X]]/\Gamma_{\mathbb{J}}(R[[X]]),$$

for any ideal $\mathbb{J}$ of $R[[X]]$ with $\operatorname{cd}(\mathbb{J}, R[[X]]) > 0$. Since $R[[X]]$ is a faithfully flat R-algebra, it follows easily that

$$\operatorname{cd}(IR[[X]], R[[X]]) = \operatorname{cd}(I, R) = t > 0.$$

So, we have

$$\dim_{R[[X]]} R[[X]]/\operatorname{Ann}_{R[[X]]} H^t_{IR[[X]]}(R[[X]]) = \dim_{R[[X]]} R[[X]]/\Gamma_{IR[[X]]}(R[[X]]).$$

Now using Corollary 2.12 yields the assertion. □

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References

1. K. Bahmanpour, Annihilators of local cohomology modules, *Comm. Algebra*, **43** (2015), 2509–2515.
2. K. Bahmanpour, A note on Lynch's conjecture, *Comm. Algebra*, **45** (2017), 2738–2745.
3. K. Bahmanpour, J. A'zami and G. Ghasemi, On the annihilators of local cohomology modules, *J. Algebra*, **363** (2012), 8–13.
4. K. Bahmanpour and M. Seidali Samani, On the cohomological dimension of finitely generated modules, *Bull. Korean Math. Soc.*, **55** (2018), 311–317.
5. M. P. Brodmann and R. Y. Sharp, *Local Cohomology; An Algebraic Introduction with Geometric Applications*, Cambridge University Press, Cambridge, 1998.

6. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge University Press, Cambridge, 1998.
7. K. Divaani-Aazar, R. Naghipour and M. Tousi, Cohomological dimension of certain algebraic varieties, *Proc. Amer. Math. Soc.*, **130** (2002), 3537–3544.
8. C. Faith, Associated primes in commutative polynomial rings, *Comm. Algebra*, **28** (2000), 3983–3986.
9. A. Grothendieck, *Local cohomology*, Notes by R. Hartshorne, Lecture Notes in Math., Vol. 862, Springer, New York, 1966.
10. C. Huneke and J. Koh, Cofiniteness and vanishing of local cohomology modules, *Math. Proc. Cambridge Philos. Soc.*, **110** (1991), 421–429.
11. L. R. Lynch, Annihilators of top local cohomology, *Comm. Algebra*, **40** (2012), 542–551.
12. H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1986.
13. P. Schenzel, Cohomological annihilators, *Math. Proc. Cambridge Philos. Soc.*, **91** (1982), 345–350.
14. P. Schenzel, *On the use of local cohomology in algebra and geometry*, Six lectures on commutative algebra, Bellaterra 1996, 241–292.

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ANNIHILATOR OF LOCAL COHOMOLOGY MODULES UNDER THE RING EXTENSION $R \subset R[X]$

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پوچساز مدول‌های کوهمولوژی موضعی تحت توسیع حلقه‌ای $R \subset R[X]$

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فرض کنید R حلقه‌ای جابجایی و نوتری، I ایده‌آلی از R و M یک R-مدول غیرصفر باشد. ما در این مقاله توسیع پوچساز مدول‌های کوهمولوژی موضعی $H_I^t(M)$، $t \geq \circ$، را تحت توسیع حلقه‌ای $R \subset R[X]$ (به‌ترتیب $R \subset R[[X]]$) محاسبه می‌کنیم. با استفاده از این توسیع، برخی از شرایط وفاداری مدول‌های کوهمولوژی موضعی را ارائه داده، و نشان می‌دهیم که اگر حدس لینچ [۱۱]، در $R[[X]]$ برقرار باشد، آنگاه در R نیز برقرار خواهد بود.

کلمات کلیدی: پوچ‌ساز، بعد کوهمولوژیکی، یکدست وفادار، کوهمولوژی موضعی، مقسوم‌علیه صفر.