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A NEW CHARACTERIZATION OF SIMPLE GROUP $G_2(q)$ WHERE $q \leqslant 11$

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ABSTRACT. In this paper, we prove that every finite group G with the same order and largest element order as $G_2(q)$, where $q \leq 11$ is necessarily isomorphic to the group $G_2(q)$.

1. INTRODUCTION

Let G be a finite group and $\pi_e(G)$ denote the set of element orders of G. In 1987, Shi [15] posed the following conjecture:

Conjecture. If G is a finite group and M is a finite simple group. Then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$.

Mazurov et al. [16] proved that this conjecture is valid for all finite simple groups. Some other researchers studied the characterization of finite simple groups by using fewer conditions. For example, He and Chen [2, 6, 8] proved that the simple K_3-groups, sporadic simple groups and $L_2(q)$ with $q \leq 125$ are determined by their orders and the largest, the second largest and the third largest element orders. They also characterized in [7, 9] some simple K_4-groups, $G_2(3)$, $G_2(4)$ and $G_2(5)$ by using the group orders and the largest element orders. In the following, it is proved that the simple K_4-groups of type $L_2(q)$, the simple K_5-groups of type $L_3(p)$ with $(3, p-1) = 1$, Suzuki groups $Sz(q)$ where $q-1$ or $q \pm \sqrt{2q}+1$ is a prime number and $L_2(p)$ such that $p \neq 7$

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is a prime number can be uniquely determined by their orders and the largest element orders [5, 12, 13, 18]. In this paper, our main aim is to prove the following theorem:

Theorem 1.1. *The simple groups $G_2(7)$, $G_2(8)$, $G_2(9)$ and $G_2(11)$ are recognizable by their order and the largest element orders.*

From this, the following corollary is derived.

Corollary 1.2. *The simple groups $G_2(q)$, with $q \leq 11$ are recognizable by their order and the largest element orders.*

Throughout this paper, we use the following definitions and notations: The *prime graph* $\Gamma(G)$ of a group G is a simple graph whose vertices are the primes dividing the group order of G and two vertices p and q are joined by an edge if and only if $pq \in \pi_e(G)$. Denote by $T(G) = \{\pi_i(G) | 1 \leq i \leq t(G)\}$ the set of all connected components of the graph $\Gamma(G)$, where $t(G)$ is the number of connected components of $\Gamma(G)$. If the order of G is even, we assume that $2 \in \pi_1(G)$. The *socle* of G is the subgroup generated by the set of all minimal normal subgroup of G; it is denoted by $\mathrm{S}oc(G)$. For $p \in \pi(G)$, we denote by $\mathrm{S}yl_p(G)$ and G_p the set of all Sylow $p-$subgroups of G and a Sylow $p-$subgroup of G, respectively. Also, we denote the highest power of p dividing the order of G by $e_p(G)$.

2. Preliminaries

In this section, we consider some results which will be needed for our further investigations.

The set $\pi_e(G)$ is closed and partially ordered by the divisibility relation; therefore, it is determined uniquely from the subset $\mu(G)$ of all maximal elements of $\pi_e(G)$ with respect to divisibility.

Lemma 2.1. [4, 11] *Let q be a power of a prime p. Then*

(a) $\mu(G_2(q)) \subseteq \{8, 12, 2, 2(q \pm 1), q^2 - 1, q^2 \pm q + 1\} \subseteq \pi_e(G_2(q))$ *for* $p = 2$;
(b) $\mu(G_2(q)) = \{p^2, p(q \pm 1), q^2 - 1, q^2 \pm q + 1\}$ *for* $p = 3, 5$;
(c) $\mu(G_2(9)) = \{p(q \pm 1), q^2 - 1, q^2 \pm q + 1\}$ *for* $p > 5$;

As an immediate consequence of Lemma 2.1, we have the following corollary.

Corollary 2.2. *The following statements hold:*

(a) $\mu(G_2(7)) = \{42, 43, 48, 56, 57\}$;
(b) $\mu(G_2(8)) \subseteq \{8, 12, 14, 18, 57, 63, 73\}$;
(c) $\mu(G_2(9)) = \{72, 73, 80, 81, 90, 91\}$;

(d) $\mu(G_2(11)) = \{110, 111, 120, 132, 133\}$.

The following lemma is useful in dealing with a Frobenius group.

Lemma 2.3. [10] *Let G be a Frobenius group with kernel K and complement H. Then*

(a) *K is a nilpotent group;*
(b) *$|H|$ divide $|K| - 1$;*
(c) *$t(G) = 2$ and the prime graph component of G are $\pi(H)$ and $\pi(K)$.*
(d) *Every non-identity element of H induces by conjugation an automorphism of K which is fixed-point-free.*

Definition 2.4. A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and $\frac{G}{H}$ are Frobenius groups with kernels H and $\frac{K}{H}$, respectively.

Lemma 2.5. [1] *Let G be a 2-Frobenius group of even order. Then $t(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi\left(\frac{K}{H}\right) = \pi_2$, $\pi(H) \cup \pi\left(\frac{G}{K}\right) = \pi_1$ and $\left|\frac{G}{K}\right|$ divides $\left|\mathrm{Aut}(\frac{K}{H})\right|$. Moreover, H is a nilpotent group and G is a solvable group.*

The structure of finite groups with non-connected prime graph is described in the following lemma.

Lemma 2.6. [17] *Let G be a finite group with $t(G) \geq 2$. Then one of the following statments hold:*

(a) *G is a Frobenius or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ where H is a nilpotent π_1-group, $\frac{K}{H}$ is a non-abelian simple group and $\frac{G}{K}$ is a π_1-group such that $|\frac{G}{K}|$ divides $\left|\mathrm{Out}(\frac{K}{H})\right|$. Moreover, each odd order components of G is also an odd order component of $\frac{K}{H}$.*

Lemma 2.7. [14] *Let $R = R_1 \times R_2 \times \cdots \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, where H_i and H_j are not isomorphic if $i \neq j$. Then $\mathrm{Aut}(R) \cong \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times \cdots \times \mathrm{Aut}(R_k)$ and $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$ where in this wreath product $\mathrm{Aut}(H_i)$ appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times \cdots \times \mathrm{Out}(R_k)$ and $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$.*

Lemma 2.8. [3] *Let G be a group and N be a normal subgroup of G with order p^n, $n \geq 1$. If $(r, |\mathrm{Aut}(N)|) = 1$, where $r \in \pi(G)$, then G has an element of order pr.*

TABLE 1.

S	$\|S\|$	$\|\mathrm{Out}(S)\|$	S	$\|S\|$	$\|\mathrm{Out}(S)\|$
$\mathbb{A}_5$	$2^2 \cdot 3 \cdot 5$	2	$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	2
$\mathbb{A}_6$	$2^3 \cdot 3^2 \cdot 5$	2^2	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	1
$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2	$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	2
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_2(27)$	$2^2 \cdot 3^2 \cdot 19 \cdot 37$	6
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_2(11^3)$	$2^2 \cdot 3^2 \cdot 5.7 \cdot 11^3 \cdot 19.37$	6
$U_3(3)$	$2^5 \cdot 3^3.7$	2	$G_2(11)$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 7.11^6.19.37$	1
$\mathbb{A}_7$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(7^3)$	$2^3 \cdot 3^2 \cdot 7^3 \cdot 19.43$	6
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	2^2	$G_2(7)$	$2^8 \cdot 3^3 \cdot 7^6 \cdot 19.43$	1
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$U_3(7)$	$2^7 \cdot 3 \cdot 7^3.43$	1
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$L_3(8)$	$2^9 \cdot 3^2 \cdot 7^2.73$	6
$\mathbb{A}_8$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(2^9)$	$2^9 \cdot 3^3 \cdot 7.19.73$	9
$\mathbb{A}_9$	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$G_2(8)$	$2^{18} \cdot 3^5 \cdot 7^2.19.37$	3
$\mathbb{A}_{10}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2.73$	2
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	${}^3D_4(3)$	$2^6 \cdot 3^{12} \cdot 7^2.13^2.73$	1
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$L_2(3^7)$	$2^3 \cdot 3^6 \cdot 5.7 \cdot 13 \cdot 73$	14
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5.7.13.73$	12
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	$S_4(27)$	$2^6 \cdot 3^{12} \cdot 5 \cdot 7^2.13^2.73$	6
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	3	$G_2(9)$	$2^8 \cdot 3^{12} \cdot 5^2.7.13.73$	4
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1			
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2			

3. MAIN RESULTS

In this section, we study the characterization problem of the simple groups $G_2(q)$ for $q \in \{7, 8, 9, 11\}$ by their orders and the largest element orders. We denote the largest element order of G by $m(G)$.

Proposition 3.1. *If G is a finite group such that $m(G) = m(G_2(7))$ and $|G| = |G_2(7)|$, then $G \cong G_2(7)$.*

Proof. According to Corollary 2.2, $m(G_2(7)) = 57$. Since $|G| = |G_2(7)| = 2^8.3^3.7^6.19.43$ and $m(G) = m(G_2(7)) = 57$, it follows that 43 is an isolated vertex of $\Gamma(G)$, and therefore $t(G) \geq 2$. Now, we show that G is neither Frobenius group nor 2-Frobenius group.

Assume that $G = KH$ is a Frobenius group with kernel K and complement H. By Lemma 2.3(c), $T(G) = \{\pi(H), \pi(K)\}$. Since $|H|$ divides $|K| - 1$ by Lemma 2.3(b), it follows that $|H| = 43$ and $|K| = 2^8.3^3.7^6.19$. Let $K_{19} \in Syl_{19}(K)$, then by nilpotency of K we have $K_{19} \trianglelefteq G$. Hence, H acts on K_{19} by conjugation. This action is fixed-point-free on K_{19}, by Lemma 2.3(d), and so $K_{19}H$ is a Frobenius group. Therefore by Lemma 2.3(b), $|H| \big| |K_{19}| - 1$ which implies that $43 \big| 19 - 1$, a contradiction.

Suppose that G is a 2-Frobenius group. By Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi\left(\frac{G}{K}\right) = \pi_1$, $\pi\left(\frac{K}{H}\right) = \pi_2$ and $\left|\frac{G}{K}\right| \big| \left|\mathrm{Aut}(\frac{K}{H})\right|$. As 43 is an isolated vertex of $\Gamma(G)$, it follows that $\pi(H) \cup \pi\left(\frac{G}{K}\right) = \{2, 3, 7, 19\}$ and $\left|\frac{K}{H}\right| = 43$. Since $\left|\frac{G}{K}\right| \big| |\mathrm{Aut}(\frac{K}{H})| = 42$, we conclude that $19 \in \pi(H)$. Let $H_{19} \in Syl_{19}(H)$, then H_{19} is a normal Sylow 19-subgroup of G by nilpotency of H. Because of $(43, |\mathrm{Aut}(H_{19})|) = 1$, Lemma 2.8 implies that $19.43 \in \pi_e(G)$, a contradiction.

Hence Lemma 2.6(b) implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where $\frac{K}{H}$ is a non-abelian simple group and $\frac{G}{K}$ is a π_1-group such that $\left|\frac{G}{K}\right| \mid \left|\mathrm{Out}(\frac{K}{H})\right|$. Moreover, each odd order component of G is an odd order component of $\frac{K}{H}$. Therefore 43 is an isolated vertex of prime graph of $\frac{K}{H}$. Now, according to the results collected in Table 1, we deduce that $\frac{K}{H}$ is isomorphic to one of the following groups: $L_2(7^3)$ or $G_2(7)$.

If $\frac{K}{H}$ is isomorphic to $L_2(7^3)$, then $(\left|\frac{G}{K}\right|, 19) = 1$ by $\left|\mathrm{Out}(\frac{K}{H})\right| = 6$ and so the Sylow 19$-$subgroup of H is of order 19 and is normal in G. Since $(43, |\mathrm{Aut}(H_{19})|) = 1$, it follows that G has an element of order 19.43 by Lemma 2.8, which is a contradiction.

Therefore, $\frac{K}{H}$ is isomorphic to $G_2(7)$ and since $|G| = |G_2(7)|$, we obtain $|H| = 1$ and $G \cong G_2(7)$. □

Proposition 3.2. *If G is a finite group such that $m(G) = m(G_2(8))$ and $|G| = |G_2(8)|$, then $G \cong G_2(8)$.*

Proof. By Corollary 2.2, $m(G_2(8)) = 73$. As $|G| = |G_2(8)| = 2^{18}.3^5.7^2.19$ $.73$ and $m(G) = m(G_2(8)) = 73$, it follows that 73 is an isolated vertex of $\Gamma(G)$ and $t(G) \geq 2$.

Suppose that G is a Frobenius group with kernel K and complement H. Then by Lemma 2.3(b), $|H|$ divides $|K| - 1$ and so $|H| < |K|$, moreover $T(G) = \{\pi(H), \pi(K)\}$. Therefore, we have $|H| = 73$ and $19 \in \pi(K)$. Now, by using the same technique as in the proof of Proposition 3.1, we get that HK_{19} is a Frobenius group. Hence $|H|$ divides $|K_{19}| - 1$, namely, $73 \big| 19 - 1$, a contradiction.

Assume that G is a 2-Frobenius group. By Lemma 2.5, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi\left(\frac{G}{K}\right) = \{2, 3, 7, 19\}$ and $\left|\frac{K}{H}\right| = 73$. Since $\left|\frac{G}{K}\right| \big| |\mathrm{Aut}(\frac{K}{H})| = 72$, it follows that $19 \in \pi(H)$. Let $H_{19} \in Sly_{19}(H)$, then by nilpotency of H we have $H_{19} \trianglelefteq G$ and so by Lemma 2.8, $19.73 \in \pi_e(G)$ since $(73, |\mathrm{Aut}(H_{19})|) = 1$, a contradiction.

Therefore by Lemma 2.6(b), it follows that $\frac{K}{H}$ is a non-abelian simple group and $\frac{G}{K}$ is a π_1-group such that $\left|\frac{G}{K}\right| \mid \left|\mathrm{Out}(\frac{K}{H})\right|$. In addition, each odd-order component of G is also an odd order component of $\frac{K}{H}$. So 73 is an isolated vertex in $\Gamma(\frac{K}{H})$. Now, Table 1 shows us that $\frac{K}{H}$ is isomorphic to $L_2(2^9)$ or $G_2(8)$.

If $\frac{K}{H} \cong L_2(2^9)$, then $19 \mid |H|$ because $\left|\frac{G}{K}\right| \mid \left|\mathrm{Out}(\frac{K}{H})\right| = 9$. Moreover, as $(73, |\mathrm{Aut}(H_{19})|) = 1$, it follows that $19.73 \in \pi_e(G)$ by Lemma 2.8, which is a contradiction.

Therefore, we have $\frac{K}{H} \cong G_2(8)$. Because $|G| = |G_2(8)|$, we can get that $|H| = 1$, and thus $G \cong G_2(8)$. □

Proposition 3.3. *If G is a finite group such that $m(G) = m(G_2(9))$ and $|G| = |G_2(9)|$, then $G \cong G_2(9)$.*

Proof. In this case, we have $|G| = |G_2(9)| = 2^8.3^{12}.5^2.7.13.73$ and $m(G) = m(G_2(9)) = 91$. Hence 73 is an isolated vertex in the prime graph of G and $t(G) \geq 2$.

By similar argument as in the proof of Propositions 3.1 and 3.2, one can show that G is not a Frobenius group and 2-Frobenius group. So it follows by Lemma 2.6 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where $\frac{K}{H}$ is a non-abelian simple group and $\frac{G}{K}$ is a π_1-group such that $\left|\frac{G}{K}\right| \mid \left|\mathrm{Out}(\frac{K}{H})\right|$. Thus 73 is an isolated vertex of the prime graph of G. Now, according to the results in Table 2, it follows that $\frac{K}{H} \cong L_2(3^6)$ or $G_2(9)$.

If $\frac{K}{H} \cong L_2(3^6)$, then $13 \in \pi(H)$ by $\left|\mathrm{Out}(\frac{K}{H})\right| = 12$. Moreover, since $(73, |\mathrm{Aut}(H_{13})|) = 1$, Lemma 2.8 implies that $13.73 \in \pi_e(G)$, which is impossible.

Thus $\frac{K}{H} \cong G_2(9)$. Since $|G| = |G_2(9)|$, we deduce that $|H| = 1$ and $G \cong G_2(9)$. □

Proposition 3.4. *If G is a finite group such that $m(G) = m(G_2(11))$ and $|G| = |G_2(11)|$, then $G \cong G_2(11)$.*

Proof. Since $|G| = |G_2(11)| = 2^6.3^3.5^2.7.11^6.19.37$ and also $m(G) = m(G_2(11)) = 133$, we have $5.37 \notin \pi_e(G)$ and $19.37 \notin \pi_e(G)$. Now, we divide the proof into two steps:

Step 1. *Let K be the maximal normal solvable subgroup of G, Then K is a $\{5, 19, 37\}'$-group. In particular, G is non-solvable.*

Assume first that $\{p, q, r\} = \{5, 19, 37\}$ and $\{p, q, r\} \subseteq \pi(K)$. Since K is solvable, it includes the solvable Hall $\{19, 37\}$-subgroup, which is a cyclic subgroup of order 19.37. Hence $19.37 \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction.

Next, we assume that $\{p, q\} \subseteq \pi(K)$ and $r \notin \pi(K)$. Let T be a $\{p, q\}$-Hall subgroup of K of order $p^i q$, where $i = 1$ or 2. By calculating the number of Sylow subgroups of T, we get that T is a nilpotent subgroup of G.

If $\{p, q\} \neq \{5, 19\}$, then $p.q \in \pi_e(K) \subseteq \pi_e(G)$, a contradiction.

If $\{p,q\}=\{5,19\}$, then K is a $\{2,3,7,11,p\}$-group. Let K_p be a Sylow $p-$subgroup of K. By Frattini argument, we have $G=KN_G(K_p)$. Since $37\notin\pi(K)$, 37 must divide $|N_G(K_p)|$ and so $N_G(K_p)$ contains an element x of order 37. Now, it is seen that $\langle x\rangle K_p$ is a nilpotent subgroup of order $p^i.37$, where $i=1$ or 2 and so $p.37\in\pi_e(K)\subseteq\pi_e(G)$, a contradiction.

Finally, assume that $\{p,q\}\cap\pi(K)=\emptyset$ and $r\in\pi(K)$. In this case, K is a $\{2,3,7,11,r\}$-group and we consider a Sylow r-subgroup K_r of K. Again using the Frattini argument, we have $G=KN_G(K_r)$. Since $\{p,q\}\cap\pi(K)=\emptyset$, it follows that p and q must divide $|N_G(K_r)|$ and thus $N_G(K_r)$ contains two elements of orders p and q, say x and y, respectively. Obviously, $\langle x\rangle K_r$ and $\langle y\rangle K_r$ are nilpotent subgroups of orders $p.r^i$ and $q.r^i$, where $i=1$ or 2, which implies that $\{p.r,q.r\}\subseteq\pi_e(G)$, a contradiction. Therefore, K is a $\{5,19,37\}'$-group. In addition, since $G\neq K$ hence G is non-solvable.

Step 2. *The quotient* $\frac{G}{K}$ *is an almost simple group. In fact, we have* $S\leq\frac{G}{K}\lesssim\mathrm{Aut}(S)$*, where* S *is a finite non-abelian simple group.*

Let $\bar{G}=\frac{G}{K}$ and $S=Soc(\bar{G})$. Since G is non-solvable group, it follows that $S=P_1\times P_2\times\cdots\times P_m$ where P_i's are finite non-abelian simple groups and $S\trianglelefteq\bar{G}\lesssim\mathrm{Aut}(S)$. Since $\pi(P_i)\subseteq\pi(G)=\{2,3,5,7,11,19,37\}$, from Table 1 it follows that the simple group P_i is isomorphic to one of tha following simple groups:

$$A_5, A_6, L_2(7), L_2(8), U_3(3), A_7, L_3(4), A_8, L_2(11), M_{11}, M_{12}, L_2(19),$$
$$J_1, L_3(11), L_2(37), U_3(11), L_2(11^3), G_2(11)$$

It is clear that $\{5,19,37\}\subseteq\pi(\bar{G})\subseteq\pi(\mathrm{Aut}(S))$, because K is a $\{5,19,37\}'$-group. Now, we claim that $\{p,q,r\}=\{5,19,37\}\subseteq\pi(S)$. Assume to the contrary that $r\notin\pi(S)$. Then $r\in\pi(\mathrm{Out}(S))$ because $r\big||\mathrm{Aut}(S)|$ and $r\nmid|\mathrm{Inn}(S)|$. By Lemma 2.7, $\mathrm{Out}(S)=\mathrm{Out}(S_1)\times\mathrm{Out}(S_2)\times\cdots\times\mathrm{Out}(S_k)$, where each S_j is a direct product of isomorphic $P_i's$ such that $S\cong S_1\times S_2\times\cdots\times S_k$. Therefore, $r\big||\mathrm{Out}(S_j)|$ for some j, where S_j is a direct product of t isomorphic simple groups P_i. By Lemma 2.7, we obtain $|\mathrm{Out}(S)|=|\mathrm{Out}(P_i)|^t.t!$. Since r does not divide $|\mathrm{Out}(P_i)|$ by Table 1, it follows that $r|t!$. Therefore, $t\geq r\geq5$ and hence 2^{10} must divides the order of G, which is a contradiction.

Now, using the facts that $\{5,19,37\}\subseteq\pi(S)$ and order consideration, it is easily checked from Table 1, that $S\cong L_2(11^3)$ or $G_2(11)$.

If $S\cong L_2(11^3)$, then we have $e_5(\mathrm{Aut}(S))=1$ while $e_5(G)=2$, and this forces $5\in\pi(K)$, which is a contradiction. Therefore $S\cong G_2(11)$

and so $G_2(11) \leq \frac{G}{K} \lesssim \mathrm{Aut}(G_2(11))$. Now, by the fact that $|G| = |G_2(11)|$, it follows that $K = 1$ and $G \cong G_2(11)$. $\square$

References

1. G. Y. Chen, On structure of frobenius groups and 2-frobenius groups, *J. Southwest China Normal Univ.*, **20** (1995), 485–487.
2. G. Y. Chen, L. G. He and J. H. Xu, A new characterization of sporadic simple groups, *Ital. J. Pure Appl. Math.*, **30** (2013), 373–392.
3. Y. H. Chen, G. Y. Chen and J. B. Li, Recognizing simple K_4-groups by few special conjugacy class sizes, *Bull. Malays. Math. Sci. Soc.*, **38** (2015), 51–72.
4. D. Deriziotis, Conjugacy class and centralizers of semisimple elements in finite groups of lie type, Vorlesungen Fachbereich Math. Univ. Essen 1984.
5. B. Ebrahimzadeh, A. Iranmanesh, A. Tehranian and H. Parvizi Mosaed, A characterization of Suzuki groups by order and the largest element order, *J. Sci. I. R. Iran.* **27** (2016), 353–355.
6. L. G. He and G. Y. Chen, A new characterization of $L_2(q)$ where $q = p^n$ <125, *Ital. J. Pure Appl. Math.*, **28** (2011), 127–136.
7. L. G. He and G. Y. Chen, A new characterization of some simple groups, *J. Sichuan Normal Univ. (Natur. Sci).*, **35** (2012), 590–594.
8. L. G. He and G. Y. Chen, A new characterization of simple K_3-groups, *Comm. Algebra*, **40** (2012), 3903–3911.
9. L. G. He and G. Y. Chen, A new characterization of simple K_4-groups, *J. Math. Research Appl.* **35** (2015), 400–406.
10. D. Gorenstein, *Finite Groups*, Harper and Row, New York, 1968.
11. W. M. Kantor and A. Seress, Prime power graphs for groups of Lie type, *J. Algebra*, **247** (2002), 370–434.
12. J. B. Li, W. J. Shi and D. P. Yu, A characterization of some $PGL(2, q)$ by maximum element orders, *Bull. Korean Math. Soc.*, **52** (2015), 2025–2034.
13. J. B. Li, D. P. Yu, G. Y. Chen and W. J. Shi, A new characterization of simple K_4-groups of type $L_2(q)$ and their automorphism groups, *Bull. Iran. Math. Soc.*, **43** (2017), 501–514.
14. J. S. Robinson Derek, *A Course in the Theory of Groups*, second edition, Springer Verlag, New York, 2003.
15. W. J. Shi, A new characterization of the sporadic simple groups, *Group theory, Walter de Gruytern, Berlin-New York.* (1989), 531– 540.
16. A. V. Vaseliev, M. A. Grechkoseeva and V. D. Mazurov, Characterization of finite simple groups by spectrume and order, *Algebra Logic*, 48 (2009), 385-409.
17. J. S. Williams, Prime graph componeents of finite group, *J. Algebra*, **69** (1981), 487–513.
18. D. P. Yu, G. Y. Chen, L.G. Zhang and W. J. Shi, A new characterization of simple K_5-groups of type $L_3(p)$, *Bull. Iran. Math. Soc.*, **45** (2019), 771–781.

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A NEW CHARACTERIZATION OF SIMPLE GROUP $G_2(q)$ WHERE $q \leq 11$

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فرض کنید G یک گروه متناهی و $m(G)$ بزرگترین مرتبه‌ی عضوهای گروه G باشد. دراین صورت گوییم گروه G توسط $m(G)$ و مرتبه‌اش r- تشخیص‌پذیر است، هرگاه r گروه غیریکریخت مانند H وجود داشته باشد به‌طوری‌که $|G| = |H|$ و $m(G) = m(H)$. حال اگر $r = 1$، آنگاه گوییم G تشخیص‌پذیر است. در این مقاله نشان می‌دهیم که گروه ساده‌ی $G_2(q)$ به ازای $q \leq 11$ تشخیص‌پذیر است.

کلمات کلیدی: تشخیص پذیری، بزرگترین مرتبه‌ی عضوهای گروه، گروه ساده.