Journal of Algebraic Systems
Vol. 8, No. 2, (2021), pp 141-154

ϕ-ALMOST DEDEKIND RINGS AND Φ-ALMOST DEDEKIND MODULES

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Abstract. The purpose of this paper is to introduce some new classes of rings and modules that are closely related to the classes of almost Dedekind domains and almost Dedekind modules. We introduce the concepts of ϕ-almost Dedekind rings and Φ-almost Dedekind modules and study some properties of this classes. In this paper we get some equivalent conditions for ϕ-almost Dedekind rings and Φ-almost Dedekind modules and obtain the relationship between ϕ-almost Dedekind rings and Φ-almost Dedekind modules.

1. Introduction

We assume throughout this paper all rings are commutative with $1 \neq 0$ and all modules are unitary. Let R be a ring with identity and $Nil(R)$ be the set of nilpotent elements of R. Recall from [19] and [11], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. Badawi in [9], [10], [11], [12], [13] and [14] investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $Nil(R)$ is a divided prime ideal of $R\}$. Anderson and Badawi in [6] and [7] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class $\mathcal{H}$. Also, Lucas and Badawi in [15] generalized the concept of Mori domains to the

DOI: 10.22044/jas.2019.8245.1402.
MSC(2010): Primary: 65F05; Secondary: 46L05, 11Y50.
Keywords: ϕ-almost Dedekind ring; ϕ-Dedekind ring; Φ-almost Dedekind module; Φ-Dedekind module.
Received: 31 March 2019, Accepted: 31 December 2019.
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context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ the set of zero divisors of R and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denoted the total quotient ring of R. We start by recalling some background material. A nonzero divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq Nil(R)$. If I is a nonnil ideal of $R \in \mathcal{H}$, then $Nil(R) \subset I$. In particular, it holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [6] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ given by $\phi(a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$ and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for every $x \in R$.

For a nonzero ideal I of R let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. It is obvious that $II^{-1} \subseteq R$. An ideal I of R is called invertible, if $II^{-1} = R$. An integral domain R is called a Dedekind domain if every nonzero ideal of R is invertible. Recall from [21] that a ring R is called a Dedekind ring if every regular ideal of R is invertible. An integral domain R is called almost Dedekind if for each nonzero prime ideal P of R, R_P is a Dedekind domain. We generaliz the concept of almost Dedekind domains to the context of commutative rings with zero divisors. A ring R is an almost Dedekind if for each regular prime ideal P of R, R_P is a Dedeking ring. Let $R \in \mathcal{H}$. Then a nonnil ideal I of R is called ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. Recall from [7] that R is called ϕ-Dedekind ring if every nonnil ideal of R is ϕ-invertible.

Let R be a ring and M be an R-module. Then M is a multiplication R-module if every submodule N of M has the form IM for some ideal I of R. If M be a multiplication R-module and N a submodule of M, then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [16]. Let M be a multiplication R-module, $N = IM$ and $L = JM$ be submodules of M for ideals I and J of R. Then, the product of N and L is denoted by $N.L$ or NL and is defined by IJM [5]. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [1]. By [24, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [17]. Let M be an R-module and set

$$T = \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\}$$

$$= (R \setminus Z(M)) \cap (R \setminus Z(R)).$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [17, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$ and $N_\nu = (N^{-1})^{-1}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. An R-module M is called a Dedekind module if every nonzero submodule of M is invertible, [23]. If N is an invertible submodule of a faithful multiplication module M over an integral domain R, then $(N :_R M)$ is invertible [3]. Let R be a ring and M an R-module. Then M is said to be an almost Dedekind module if for each prime ideal P of R, M_P is an R_P- Dedekind module. Clearly Dedekind modules are almost Dedekind, [4].
Let M be an R-module. An element $r \in R$ is said to be zero divisor on M if $rm = 0$ for some $0 \neq m \in M$. The set of zero divisors of M is denoted by $Z_R(M)$ (briefly, $Z(M)$). It is easy to see that $Z(M)$ is not necessarily an ideal of R, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule N of M is called a nilpotent submodule if $[N :_R M]^n N = 0$ for some positive integer n. An element $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M [2]. We let $Nil(M)$ to denote the set of all nilpotent elements of M; then $Nil(M)$ is a submodule of M provided that M is a faithful module, and if in addition M is multiplication, then $Nil(M) = Nil(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [2, Theorem 6]. If M contains no nonzero nilpotent elements, then M is called a reduced R-module. A submodule N of M is said to be a nonnil submodule if $N \nsubseteq Nil(M)$. Recall that a submodule N of M is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$. If N is a prime submodule of M, then $p := [N :_R M]$ is a prime ideal of R. In this case we say that N is a p-prime submodule of M. Let N be a submodule of multiplication R-module M, then N is a prime submodule of M if and only if $[N :_R M]$ is a prime ideal of R if and only if $N = pM$ for some prime ideal p of R with $[0 :_R M] \subseteq p$, [17, Corollary 2.11]. Recall from [4] that a prime submodule P of M is called a divided prime submodule if $P \subset Rm$ for every $m \in M \setminus P$; thus a divided prime submodule is comparable to every submodule of M.

Now assume that $T^{-1}(M) = \mathfrak{T}(M)$. Set

$$\mathbb{H} = \{M \mid Nil(M) \text{ is a divided prime submodule of } M\}.$$

For an R-module $M \in \mathbb{H}$, $Nil(M)$ is a prime submodule of M. So $P := [Nil(M) :_R M]$ is a prime ideal of R. If M is an R-module and $Nil(M)$ is a proper submodule of M, then $[Nil(M) :_R M] \subseteq Z(R)$. Consequently, $R \setminus Z(R) \subseteq R \setminus [Nil(M) :_R M]$. In particular, $T \subseteq R \setminus [Nil(M) :_R M]$ [25]. Recall from [25] that we can define a mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ which is clearly an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule N of M is said to be Φ-invertible if $\Phi(N)$ is an invertible submodule of $\Phi(M)$ [22]. An R-module M is called a Φ-Dedekind module if every nonnil submodule of M is Φ-invertible [22]. In this paper we introduce the concepts ϕ-almost Dedekind rings and Φ-almost Dedekind modules and get some properties of them.

2. ϕ-almost Dedekind rings

Definition 2.1. A ring R is said to be a ϕ-almost Dedekind ring if for each nonnil prime ideal P of R, R_P is a ϕ-Dedekind ring.

Theorem 2.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-almost Dedekind ring if and only if $\frac{R}{Nil(R)}$ is an almost Dedekind domain.*

Proof. Let R be a ϕ-almost Dedekind ring. Then for each nonnil prime ideal P of R, R_P is a ϕ-Dedekind ring. Then, by [7, Theorem 2.5], $(\frac{R}{Nil(R)})_{\frac{P}{Nil(R)}} = \frac{R_P}{Nil(R_P)}$ is a Dedekind domain. Therefore $\frac{R}{Nil(R)}$ is an almost Dedekind domain. Conversely, let $\frac{R}{Nil(R)}$ is an almost Dedekind domain. Then for each nonzero prime ideal $\frac{P}{Nil(R)}$, $(\frac{R}{Nil(R)})_{\frac{P}{Nil(R)}} = \frac{R_P}{Nil(R_P)}$ is a Dedekind domain. So, by [7, Theorem 2.5], R_P is a ϕ-Dedekind ring. Therefore, R is a ϕ-almost Dedekind ring. □

Corollary 2.3. *Let $R \in \mathcal{H}$. If R is a ϕ-Dedekind ring, then R is a ϕ-almost Dedekind ring.*

Proof. Suppose that R is a ϕ-Dedekind ring. Then, by [7, Theorem 2.5], $\frac{R}{Nil(R)}$ is a Dedekind domain and so $\frac{R}{Nil(R)}$ is an almost Dedekind domain. Therefore, by Theorem 2.2, R is a ϕ-almost Dedekind ring. □

Theorem 2.4. *Let $R \in \mathcal{H}$. Then R is a ϕ-almost Dedekind ring if and only if $\phi(R)$ is an almost Dedekind ring.*

Proof. Let R be a ϕ-almost Dedekind ring. Then for each nonnil prime ideal P of R, R_P is a ϕ-Dedekind ring. So, by [7, Corollary 2.2], $\phi(R_P) = \phi(R)_{\phi(P)}$ is a Dedekind ring for each regular prime ideal $\phi(P)$ of $\phi(R)$. Therefore, $\phi(R)$ is an almost Dedekind ring. Conversely, let

$\phi(R)$ is an almost Dedekind ring. Then for each regular prime ideal $\phi(P)$ of $\phi(R)$, $\phi(R_P) = \phi(R)_{\phi(P)}$ is a Dedekind ring. Hence, by [7, Corollary 2.2], R_P is a ϕ-Dedekind ring for each nonnil prime ideal P of R. Therefore R is a ϕ-almost Dedekind ring. □

In veiw of Theorem 2.2, Theorem 2.4 and [6, Lemma 2.5], we have the following result.

Corollary 2.5. *Let $R \in \mathcal{H}$. Then the following are equivalent:*
(1) *R is a ϕ-almost Dedekind ring;*
(2) *$\phi(R)$ is an almost Dedekind ring;*
(3) *$\frac{R}{Nil(R)}$ is an almost Dedekind domain;*
(4) *$\frac{\phi(R)}{Nil(\phi(R))}$ is an almost Dedekind domain.*

Our non-domain examples of ϕ-almost Dedekind rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in [21]. We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R \text{ and } b \in B\}$, and let (r, b) and (s, c) be to elements of $R(+)B$. Define
(1) $(r, b) = (s, c)$ if $r = s$ and $b = c$.
(2) $(r, b) + (s, c) = (r + s, b + c)$.
(3) $(r, b)(s, c) = (rs, bs + rc)$.
Under these definitions, $R(+)B$ becomes a commutative ring with identity.

Example 2.6. Let D be an almost Dedekind domain with quotient field L. Set $R = D(+)L$. Then $R \in \mathcal{H}$ and R is a ϕ-almost Dedekind ring which is not an almost Dedekind domain.

Proof. Since D is a domain, so $\{0\}$ is a prime ideal of D. Hence $Nil(R) = \{0\}(+)L$ is a divided prime ideal of R. Let $(a, x) \in R \backslash Nil(R)$ and $(0, y) \in Nil(R)$. Then $(0, y) = (a, x)(0, y/a)$ and hence $R \in \mathcal{H}$. Also, $R/Nil(R)$ is ring-isomorphic to D and D is a almost Dedekind domain, we conclude that R is a ϕ-almost Dedekind ring by Theorem 2.2. But R is not a domain. □

The following is an example of a ring $R \in \mathcal{H}$ which is an almost Dedekind ring but not a ϕ-almost Dedekind ring.

Example 2.7. Let D be an integral domain with quotient field L which is not an almost Dedekind domain and set $R = D(+)(L/D)$. Then $R \in \mathcal{H}$ is an almost Dedekind ring which is not a ϕ-almost Dedekind ring.

Proof. By previous Example, $Nil(R) = \{0\}(+)(L/D)$ is a divided prime ideal of R and thus $R \in \mathcal{H}$. Since every nonunit of R is zero divisor, we conclude that R is an almost Dedekind ring. Since $R/Nil(R)$ is ring-isomorphic to D and D is not an almost Dedekind domain, so R is not a ϕ-almost Dedekind ring by Theorem 2.2. □

Theorem 2.8. *Let $R \in \mathcal{H}$. If R is a ϕ-almost Dedekind ring, then R is an almost Dedekind ring.*

Proof. Suppose that R is a ϕ-almost Dedekind ring. Then for each nonnil prime ideal P of R, R_P is a ϕ-Dedekind ring. Hence, by [7, Theorem 2.12], R_P is a Dedekind ring. Therefore R is an almost Dedekind ring. □

Theorem 2.9. *Let $R \in \mathcal{H}$ such that $Nil(R) = Z(R)$. Then R is a ϕ-almost Dedeking ring if and only if R is an almost Dedekind ring.*

Proof. Suppose that R is an almost Dedekind ring. Then $\phi(R) = R$ is an almost Dedekind ring. Therefore, by Theorem 2.4, R is a ϕ-almost Dedekind ring. Conversely is clear by Theorem 2.8. □

Theorem 2.10. *Let $R \in \mathcal{H}$ be a ϕ-almost Dedekind ring and I be an ideal with $I \subset Nil(R)$. Then $\frac{R}{I}$ is a ϕ-almost Dedekind ring.*

Proof. Suppose that $I \subset Nil(R)$. Then $Nil(\frac{R}{I}) = \frac{Nil(R)}{I}$ is a divided prime ideal of $\frac{R}{I}$. Thus $\frac{R}{I} \in \mathcal{H}$. Since $\frac{\frac{R}{I}}{Nil(\frac{R}{I})}$ is ring-isomorphic to $\frac{R}{Nil(R)}$ and $\frac{R}{Nil(R)}$ is an almost Dedekind domain by Theorem 2.2, we conclude that $\frac{R}{I}$ is a ϕ-almost Dedekind ring. □

Theorem 2.11. *Let $R \in \mathcal{H}$. Then R is a ϕ-almost Dedekind ring if and only if*
(1) *nonnil prime ideals of R are nonnil maximal, and*
(2) *nonnil primary ideal of R are prime powers.*

Proof. Let R be a ϕ-almost Dedekind ring. Then, by Theorem 2.2, $\frac{R}{Nil(R)}$ is an almost Dedekind domain. Let P be a nonnil prime ideal of R. So $\frac{P}{Nil(R)}$ is a nonzero prime ideal of $\frac{R}{Nil(R)}$. Hence, by [20, Theorem 1], $\frac{P}{Nil(R)}$ is a maximal ideal of $\frac{R}{Nil(R)}$. Thus P is a nonnil maximal of R. For (2), suppose that Q is a nonnil primary ideal of R. Then $\frac{Q}{Nil(R)}$ a primary ideal of $\frac{R}{Nil(R)}$. So, by [20, Theorem 1], $\frac{Q}{Nil(R)} = \frac{P^n}{Nil(R)}$ for a positive integer n. Therefore $Q = P^n$. Conversely is clear by a same argument. □

Proposition 2.12. *Let $R \in \mathcal{H}$. If R is a ϕ-almost Dedekind ring and I a nonnil proper ideal of R, then $\bigcap_{n=1}^{\infty} I^n = (0)$.*

Proof. Suppose that R is a ϕ-almost Dedekind ring and I a nonnil proper ideal of R. Then, by Theorem 2.2, $\frac{R}{Nil(R)}$ is an almost Dedekind domain and $\frac{I}{Nil(R)}$ is a proper ideal of $\frac{R}{Nil(R)}$. So, by [20, Corollary 1], $\bigcap_{n=1}^{\infty}(\frac{I}{Nil(R)})^n = (0)$. Therefore $\bigcap_{n=1}^{\infty} I^n = (0)$. □

Theorem 2.13. *Let $R \in \mathcal{H}$ be a ϕ-almost Dedekind ring. Then R is a ϕ-Dedekind ring if and only if every nonnil proper ideal of R is contained in only finitely many nonnil maximal ideals.*

Proof. Let R be a ϕ-almost Dedekind ring. Then, by Theorem 2.2, $\frac{R}{Nil(R)}$ is an almost Dedekind domain.
Suppose that R is a ϕ-Dedekind ring. So, by [7, Theorem 2.5], $\frac{R}{Nil(R)}$ is a Dedekind domain. Hence, by [20, Theorem 3], every nonzero proper ideal of $\frac{R}{Nil(R)}$ is contained in only finitely many maximal ideals. Therefore every nonnil proper ideal of R is contained in only finitely many nonnil maximal ideals. Conversely, let every nonnil proper ideal of R be contained in only finitely many nonnil maximal ideals. Then every nonzero proper ideal of $\frac{R}{Nil(R)}$ is contained in only finitely many maximal ideals. So, by [20, Theorem 3], $\frac{R}{Nil(R)}$ is a Dedekind domain. Therefore, by [7, Theorem 2.5], R is a ϕ-Dedekind ring. □

3. Φ-almost Dedekind modules

Ali in [4] generalized the concept of almost Dedekind domains to faithful multiplication modules over a ring R as follow:

Definition 3.1. Let R be a ring and M an R-module. Then M is said to be an almost Dedekind module if for each prime ideal P of R, M_P is an R_P-module.

Recall from [4] that if R is an integral domain and M a faithful multiplication R-module, then R is an almost Dedekind domain if and only if M is an almost Dedekind module.

Theorem 3.2. *Let R be an integral domain and M a faithful multiplication R-module. Then M is an almost Dedekind module if and only if*
(1) *nonzero proper prime submodules of M are maximal, and*
(2) *primary submodules of M are prime powers.*

Proof. Let M be an almost Dedekind module. Then, by [4], R is an almost Dedekind domain. Let N be a nonzero proper prime submodule of M. So $(N :_R M)$ is a nonzero prime ideal of R. Thus, by [20, Theorem 1], $(N :_R M)$ is a maximal ideal of R. Hence $N = (N :_R M)M$ is a maximal submodule of M and therefore (1) holds. Now, let N be a primary submodule of M. Then, by [3, Lemma 4], $(N :_R M)$ is a primary ideal of R. Thus, by [20, Theorem 1], $(N :_R M) = P^n$ for some possitive integer n. Therefore $N = (N :_R M)M = P^n M$ and hence (2) is holds. By a same argument the converse is obvous. $\square$

Proposition 3.3. *Let R be an integral domain and M a faithful multiplication R-module. If M is an almost Dedekind module and N a proper submodule of M, then $\bigcap_{n=1}^{\infty} N^n = (0)$.*

Proof. Suppose that M is an almost Dedekind module and N a proper submodule of M. Then, by [4], R is an almost Dedekind domain and $(N :_R M)$ is a proper ideal of R. Thus, by [20, Proposition 1], $\bigcap_{n=1}^{\infty}(N :_R M)^n = (0)$. Since M is multiplication, $\bigcap_{n=1}^{\infty} N^n = (0)$. $\square$

Theorem 3.4. *Let R be an integral domain and M a faithful multiplication almost Dedekind R-module. Then M is a Dedekind module if and only if every nonzero proper submodule of M is contained in only finitely many maximal submodules.*

Proof. Since M is an almost Dedekind module, R is an almost Dedekind module, by [4]. Let M be a Dedekind module and so R is a Dedekind domain. Hence, by [20, Theorem 3], every nonzero proper ideal of R is contained in only finitely many maximal ideals. Since M is multiplication, every nonzero proper submodule of M is contained in only finitely many maximal submodules. By a similar argument the converse is clear. $\square$

Now we generalize the above properties to the class modules in $\mathbb{H}$.

Definition 3.5. Let R be a ring and M be an R-module. Then M is said to be a Φ-almost Dedekind module if for each prime ideal P of R, M_P is a Φ-Dedekind module.

Theorem 3.6. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. Then M is a Φ-almost Dedekind module if and only if $\frac{M}{Nil(M)}$ is an almost Dedekind module.*

Proof. Let M be a Φ-almost Dedekind module. Then for each prime ideal P of R, M_P is a Φ-Dedekind module. So, by [22, Theorem 2.10], $(\frac{M}{Nil(M)})_P = \frac{M_P}{Nil(M_P)}$ is a Dedekind module. Therefore $\frac{M}{Nil(M)}$ is an almost Dedekind module. Conversely, let $\frac{M}{Nil(M)}$ is an almost Dedekind module. Then for each prime ideal P of R, $(\frac{M}{Nil(M)})_P = \frac{M_P}{Nil(M_P)}$ is a Dedekind module. Thus, by [22, Theorem 2.10], M_P is a Φ-Dedekind module. Therefore M is a Φ-almost Dedekind module. $\square$

Theorem 3.7. ([22, Lemma 2.6]) *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then $\frac{M}{Nil(M)}$ is isomorphic to $\frac{\Phi(M)}{Nil(\Phi(M))}$ as R-module.*

Corollary 3.8. *Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then M is a Φ-almost Dedekind module if and only if $\frac{\Phi(M)}{Nil(\Phi(M))}$ is an almost Dedekind module.*

Corollary 3.9. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. If M is a Φ-Dedekind module, then M is a Φ-almost Dedekind module.*

Proof. Suppose that M is a Φ-Dedekind module. Then, by [22, Theorem 2.10], $\frac{M}{Nil(M)}$ is a Dedekind module and so $\frac{M}{Nil(M)}$ is an almost Dedekind module. Therefore, by Theorem 3.6, M is a Φ-almost Dedekind module. $\square$

Theorem 3.10. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. Let M be a Φ-almost Dedekind module and N a submodule of M such that $N \subset Nil(M)$. Then $\frac{M}{N}$ is a Φ-almost Dedekind module.*

Proof. Suppose that $N \subset Nil(M)$. Then $Nil(\frac{M}{N}) = \frac{Nil(M)}{N}$ is a divided prime submodule of $\frac{M}{N}$. Thus $\frac{M}{N} \in \mathbb{H}$. Since $\frac{\frac{M}{N}}{Nil(\frac{M}{N})}$ is module-isomorphic to $\frac{M}{Nil(M)}$ and $\frac{M}{Nil(M)}$ is an almost Dedekind module by Theorem 3.6, we conclude that $\frac{M}{N}$ is a Φ-almost Dedekind module. $\square$

Lemma 3.11. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are hold:*
(1) If $R \in \mathcal{H}$ is a ϕ-Dedekind ring, then M is a Φ-Dedekind module;
(2) If $M \in \mathbb{H}$ is a Φ-Dedekind module, then R is a ϕ-Dedekind ring.

Proof. Since $Nil(R) \subseteq Ann(\frac{M}{Nil(R)M}) = Ann(\frac{M}{Nil(M)})$, we have:
(1) Let $R \in \mathcal{H}$. Then, by [25, Proposition 3], $M \in \mathbb{H}$. If R is a ϕ-Dedekind ring, then [7, Theorem 2.5], $\frac{R}{Nil(R)}$ is a Dedekind domain. So, by [4], $\frac{M}{Nil(M)}$ is a Dedekind module. Therefore, by [22, Theorem 2.10], M is a Φ-Dedekind module.
(2) Let $M \in \mathbb{H}$. Then, by [25, Proposition 3], $R \in \mathcal{H}$. If M is a Φ-Dedekind module, then by [22, Theorem 2.10], $\frac{M}{Nil(M)}$ is a Dedekind module. So, by [4], $\frac{R}{Nil(R)}$ is a Dedekind domain. Therefore, by [7, Theorem 2.5], R is a ϕ-almost Dedekind ring. $\square$

Theorem 3.12. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. Then M is a Φ-almost Dedekind module if and only if $\Phi(M)$ is an almost Dedekind module.*

Proof. Let M be a Φ-almost Dedekins R-module. Then for each prime ideal P of R, M_P is a Φ-Dedekind R_P-module. So, by Lemma 3.11, R_P is a ϕ-Dedekind ring. Hence, by [7, Corollary 2.2], $\phi(R_P) = \phi(R)_{\phi(P)}$ is a Dedekind ring. Thus $\Phi(M)_{\phi(P)}$ is a Dedekind $\phi(R)$-module. Therefore, $\Phi(M)$ is an almost Dedekind $\phi(R)$-module. By a same argument conversely is clear. □

In veiw of Theorem 3.6, Corollary 3.8 and Theorem 3.12, we have the following result.

Corollary 3.13. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. The following are equivalent:*
(1) *M is a Φ-almost Dedekind module;*
(2) *$\Phi(M)$ is an almost Dedekind module;*
(3) *$\frac{M}{Nil(M)}$ is an almost Dedekind module;*
(4) *$\frac{\Phi(M)}{Nil(\Phi(M))}$ is an almost Dedekind module.*

Theorem 3.14. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are hold:*
(1) *If $R \in \mathcal{H}$ is a ϕ-almost Dedekind ring, then M is a Φ-almost Dedekind module;*
(2) *If $M \in \mathbb{H}$ is a Φ-almost Dedekind module, then R is a ϕ-almost Dedekind ring.*

Proof. Since $Nil(R) \subseteq Ann(\frac{M}{Nil(R)M}) = Ann(\frac{M}{Nil(M)})$, we have:
(1) Let $R \in \mathcal{H}$. Then, by [25, Proposition 3], $M \in \mathbb{H}$. If R is a ϕ-almost Dedekind ring, then Theorem 2.2, $\frac{R}{Nil(R)}$ is an almost Dedekind domain. So, by [4], $\frac{M}{Nil(M)}$ is an almost Dedekind module. Therefore, by Theorem 3.6, M is a Φ-almost Dedekind module.
(2) Let $M \in \mathbb{H}$. Then, by [25, Proposition 3], $R \in \mathcal{H}$. If M is a Φ-almost Dedekind module, then by Theorem 3.6, $\frac{M}{Nil(M)}$ is an almost Dedekind module. So, by [4], $\frac{R}{Nil(R)}$ is an almost Dedekind domain. Therefore, by Theorem 2.2, R is a ϕ-almost Dedekind ring. □

Lemma 3.15. *Let R be a ring, M a finitely generated faithful multiplication R-module and N a nonnil submodule of M. Then N is a Φ-invertible submodule of M if and only if $(N :_R M)$ is an invertible ideal of R.*

Proof. Suppose that N is a Φ-invertible submodule of M. Then $\Phi(N)$ is an invertible submodule of $\Phi(M)$. So $\Phi(NN^{-1}) = \Phi(N)\Phi(N^{-1}) = \Phi(M)$. Thus, by [26, Lemma 3.3], $NN^{-1} = M$. Hence, by [3, Lemma 1], $(N :_R M)(N :_R M)^{-1} = R$. Therefore $(N :_R M)$ is an invertible ideal of R. Conversely, suppose that $(N :_R M)$ is an invertible ideal of R. Then $(N :_R M)(N :_R M)^{-1} = R$. So, by [3, Lemma 1], $NN^{-1} = M$. Thus $\Phi(NN^{-1}) = \Phi(N)\Phi(N^{-1}) = \Phi(M)$. Hence $\Phi(N)$ is an invertible submodule of $\Phi(M)$. Therefore N is a Φ-invertible submodule of M. □

Lemma 3.16. *Let R be an integral domain and $M \in \mathbb{H}$ a faithful multiplication R-module. Then M is Φ-Dedekind module if and only if R is a Dedekind domain.*

Proof. Let M be a Φ-Dedekind module. Then every nonnil submodule N of M is Φ-invertible. So, by Lemma 3.15, $(N :_R M)$ is an invertible ideal of R. Therefore R is a Dedekind domain. By a similar argument the converse is clear. □

Proposition 3.17. *Let R be an integral domain and $M \in \mathbb{H}$ a faithful multiplication R-module. Then M is a Φ-almost Dedekind module if and only if R is an almost Dedekind domain.*

Proof. Suppose that M is Φ-almost Dedekind module. Then for each prime ideal P of R, M_P is a Φ-Dedekind module. Hence, by Lemma 3.16, R_P is a Dedekind domain. Therefore R is an almost Dedekind domain. Conversely, let R is an almost Dedekind domain. Then for each nonzero prime ideal P of R, R_P is a Dedekind domain. So, by Lemma 3.16, M_P is a Φ-Dedekind module for each prime ideal P of R. Therefore M is a Φ-almost Dedekind module. □

Theorem 3.18. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. Then M is an Φ-almost Dedekind module if and only if*
(1) *nonnil prime submodules of M are nonnil maximal, and*
(2) *nonnil primary submodules of M are prime powers.*

Proof. Let M be a Φ-almost Dedekind module. Then, by Theorem 3.6, $\frac{M}{Nil(M)}$ is an almost Dedekind module. So, by [4], R is an almost Dedekind domain. Suppose that N is a nonnil prime submodule of M. Then $\frac{N}{Nil(M)}$ is a prime submodule of $\frac{M}{Nil(M)}$ and so $(\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)}$ is a prime ideal of R. Hence, by [20, Theorem 1], $(\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)}$ is a maximal ideal of R. Thus $\frac{N}{Nil(M)}$ is a maximal submodule of $\frac{M}{Nil(M)}$. Therefore N is a nonnil maximal submodule of M. For (2), let N be a nonnil primary submodule of M. Then $\frac{N}{Nil(M)}$ is a primary submodule of $\frac{M}{Nil(M)}$ and so $(\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)})$ is a primary ideal of R, by [3, Lemma 4]. So, by [20, Theorem 1], $(\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)}) = P^n$ for a positive integer n. Hence $\frac{N}{Nil(M)} = \frac{P^n M}{Nil(M)}$. Thus $N = P^n M$. By a similar argument, the converse is clear. □

Proposition 3.19. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $Nil(M) = Z(R)M$. If M is an Φ-almost Dedekind module and N a nonnil proper submodule of M, then $\bigcap_{n=1}^{\infty} N^n = (0)$.*

Proof. Let M be a Φ-almost Dedekind module and N a nonnil proper submodule of M. Then, by Theorem 3.6, $\frac{M}{Nil(M)}$ is an almost Dedekind module and $\frac{N}{Nil(M)}$ is a proper submodule of $\frac{M}{Nil(M)}$. So, by [4], R is an almost Dedekind domain and $(\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)})$ is a proper ideal of R. So, by [20, Corollary 1], $\bigcap_{n=1}^{\infty}((\frac{N}{Nil(M)} :_R \frac{M}{Nil(M)}))^n = (0)$. Since M is multiplication, $\bigcap_{n=1}^{\infty}(\frac{N}{Nil(M)})^n = (0)$. Therefore $\bigcap_{n=1}^{\infty} N^n = (0)$. □

Theorem 3.20. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication Φ-almost Dedekind R-module with $Nil(M) = Z(R)M$. Then M is a Φ-Dedekind module if and only if every nonnil proper submodule of M is contained in only finitely many nonnil maximal submodules.*

Proof. Suppose that M is a Φ-almost Dedekind module. Then, by Theorem 3.6, $\frac{M}{Nil(M)}$ is an almost Dedekind module. So, by [4], R is an almost Dedekind domain. Now, if M is a Φ-Dedekind module, then, by [22, Theorem 2.10], $\frac{M}{Nil(M)}$ is a Dedekind module and so R is a Dedekind domain. Hence, by [20, Theorem 1], every nonzero proper ideal of R is contained in only finitely many maximal ideals. Since M is multiplication, every nonzero proper submodule of $\frac{M}{Nil(M)}$ is contained in only finitely many maximal submodules. Therefore, every nonnil proper submodule of M is contained in only finitely many nonnil maximal submodules. The converse part can be handled similarly.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and insightful suggestions, which leads to an improvements of the paper.

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Journal of Algebraic Systems

ϕ-ALMOST DEDEKIND RINGS AND Φ-ALMOST DEDEKIND MODULES

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حلقه‌های ϕ- تقریباً ددکیند و مدول‌های Φ- تقریباً ددکیند

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هدف از این مقاله معرفی یک رده جدید از حلقه‌ها و مدول‌ها که ارتباط نزدیکی با دامنه‌های تقریباً ددکیند و مدول‌های تقریباً ددکیند دارند، می‌باشد. در این مقاله، مفاهیم حلقه‌های ϕ- تقریباً ددکیند و مدول‌های Φ- تقریباً ددکیند را معرفی می‌کنیم و بعضی از ویژگی‌های این رده از حلقه‌ها و مدول‌ها را مطالعه می‌کنیم. شرایط معادلی برای حلقه‌های ϕ- تقریباً ددکیند و مدول‌های Φ- تقریباً ددکیند به‌دست می‌آوریم و روابط بین حلقه‌های ϕ- تقریباً ددکیند و مدول‌های Φ- تقریباً ددکیند را بررسی می‌کنیم.

کلمات کلیدی: حلقه ϕ- تقریباً ددکیند، حلقه ϕ- ددکیند، مدول Φ- تقریباً ددکیند، مدول Φ- ددکیند.