TOP LOCAL COHOMOLOGY AND TOP FORMAL LOCAL COHOMOLOGY MODULES WITH SPECIFIED ATTACHED PRIMES

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ABSTRACT. Let \((R, \mathfrak{m})\) be a Noetherian local ring, \(M\) be a finitely generated \(R\)-module of dimension \(n\) and \(\mathfrak{a}\) be an ideal of \(R\). In this paper, generalizing the main results of Dibaei and Jafari [3] and Rezaei [8], we will show that if \(T\) is a subset of \(\text{Ass}_R M\), then there exists an ideal \(\mathfrak{a}\) of \(R\) such that \(\text{Att}_R H^n_{\mathfrak{a}}(M) = T\). As an application, we give some relationships between top local cohomology modules and top formal local cohomology modules.

1. Introduction

Throughout this paper, let \((R, \mathfrak{m})\) be a commutative Noetherian local ring, \(\mathfrak{a}\) be an ideal of \(R\) and \(M\) be a finitely generated \(R\)-module of dimension \(n\). For an \(R\)-module \(M\), the \(i\)-th local cohomology module of \(M\) with respect to \(\mathfrak{a}\) is defined as

\[
H^i_{\mathfrak{a}}(M) = \lim_{n \geq i} \text{Ext}^i_R(R/\mathfrak{a}^n, M).
\]

For the basic properties of local cohomology the reader can refer to [2]. Also, for each \(i \geq 0\); \(\delta^i_{\mathfrak{a}}(M) := \lim_{\mathfrak{T}} H^i_{\mathfrak{m}}(M/\mathfrak{a}^t M)\) is called the \(i\)-th formal local cohomology module of \(M\) with respect to \(\mathfrak{a}\). The formal local cohomology modules have been studied by several authors; see
for example [1], [5] and [9]. Let $M$ be a finitely generated $R$-module of dimension $n$, then $\operatorname{Max}\{i \in \mathbb{Z} : H^i_a(M) \neq 0\} \leq n$ by [2, Theorem 6.1.2] and $\operatorname{Max}\{i \in \mathbb{Z} : \mathfrak{F}^i_a(M) \neq 0\} \leq n$ by [9, Theorem 4.5]. Recall that the module $H^n_a(M)$ is called a top local cohomology module if $\operatorname{Max}\{i \in \mathbb{Z} : H^i_a(M) \neq 0\} = n$ and the module $\mathfrak{F}^n_a(M)$ is called a top formal local cohomology module if $\operatorname{Max}\{i \in \mathbb{Z} : \mathfrak{F}^i_a(M) \neq 0\} = n$. For each Artinian $R$-module $A$, we denote by $\operatorname{Att}_R A$ the set of all attached prime ideals of $A$.

In section 2, we show that any subset $T$ of $\operatorname{Assh}_R M$, where $\operatorname{Assh}_R M = \{p \in \operatorname{Ass}_R M : \dim(R/p) = \dim M\}$, can be expressed as the set of attached primes of the top local cohomology module $H^n_a(M)$ for some ideal $a$ of $R$. This generalizes a result of Dibaei and Jafari [3] to Noetherian local rings that are not necessarily complete.

We say that the top local cohomology module $H^n_a(M)$ satisfies the property $(\ast)$, if $\operatorname{Att}_R H^n_a(M) = \{p \in \operatorname{Ass}_R M : \dim(R/p) = n \text{ and } \sqrt{a + p} = m\}$. Rezaei in [8], showed that if $(R, m)$ is a complete Noetherian local ring and $M$ is a finitely generated $R$-module of dimension $n$ then for each ideal $a$ of $R$ there exists an ideal $b$ such that $H^n_a(M) \cong \mathfrak{F}^n_a(M)$ and there exists an ideal $c$ such that $\mathfrak{F}^n_a(M) \cong H^n_c(M)$. In section 3, we generalize this result. In fact, we show that over Noetherian local rings that are not necessarily complete, there exists an ideal $c$ such that $\mathfrak{F}^n_a(M) \cong H^n_c(M)$ and if $H^n_a(M)$ satisfies the property $(\ast)$ then there exists an ideal $b$ such that $H^n_a(M) \cong \mathfrak{F}^n_b(M)$.

For any ideal $a$ of $R$, the radical of $a$, denoted by $\sqrt{a}$, is defined to be the set $\{x \in R : x^n \in a \text{ for some } n \in \mathbb{N}\}$. Also, we denote $\{p \in \operatorname{Spec} R : p \supseteq a\}$ by $V(a)$ and $\operatorname{Min} V(a)$ by $\operatorname{Min}(a)$. For an $R$-module $M$, we show the set of minimal members of associated primes of $M$ by $\operatorname{mAss}_R(M)$. For any unexplained notation and terminology, we refer the reader to [2] and [6].

2. TOP LOCAL Cohomology MODULES WITH SPECIFIED ATTACHED PRIMES

In this section, we study the set of attached primes of top local cohomology modules.

**Notation 2.1.** Let $a$ be an ideal of $R$ and $M$ be a finitely generated $R$-module of dimension $n$. Let $0 = \bigcap_{p \in \operatorname{Ass}_R M} N(p)$ be a reduced primary
decomposition of the submodule 0 of $M$. Following [7], we set

$$\text{Ass}_R(\mathfrak{a}, M) = \{ \mathfrak{p} \in \text{Ass}_R M : \dim(R/\mathfrak{p}) = n \text{ and } \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m} \}. $$

Set $N^\mathfrak{a} = \bigcap_{\mathfrak{p} \in \text{Ass}_R(\mathfrak{a}, M)} N(\mathfrak{p})$. Note that $N^\mathfrak{a}$ does not depend on the choice of the reduced primary decomposition of zero because

$$\text{Ass}_R(\mathfrak{a}, M) \subseteq m\text{Ass}_R M. $$

It is clear that $\text{Ass}_R(\mathfrak{a}, M) = \text{Ass}_R(M/N^\mathfrak{a})$ and

$$\text{Ass}_R N^\mathfrak{a} = \text{Ass}_R M \setminus \text{Ass}_R(\mathfrak{a}, M). $$

For each integer $l \geq 0$ and any subset $S$ of $\text{Spec} R$ we define

$$S_l := \{ \mathfrak{p} \in S : \dim(R/\mathfrak{p}) = l \}. $$

**Lemma 2.2.** Let $N^\mathfrak{a}$ be defined as above. Then the following statements are equivalent:

(i) $H^n_R(N^\mathfrak{a}) = 0$;

(ii) $H^n_R(M) \cong H^n_R(M/N^\mathfrak{a})$;

(iii) $\text{Att}_R H^n_R(M) = \text{Att}_R H^n_R(M/N^\mathfrak{a}) = \text{Ass}_R(\mathfrak{a}, M)$.

**Proof.** By the exact sequence

$$H^n_R(N^\mathfrak{a}) \to H^n_R(M) \to H^n_R(M/N^\mathfrak{a}) \to 0$$

it is enough for us to prove (iii)⇒(i). Suppose, on the contrary, that $H^n_R(N^\mathfrak{a}) \neq 0$. Then there exists $\mathfrak{p} \in \text{Att}_R H^n_R(N^\mathfrak{a})$. By [4, Theorem A], $\mathfrak{p} \in \text{Ass}_R N^\mathfrak{a}$ and $\text{cd}(\mathfrak{a}, R/\mathfrak{p}) = n$ and so $\mathfrak{p} \in \text{Att}_R H^n_R(M) = \text{Att}_R H^n_R(M/N^\mathfrak{a})$. But by Notation 2.1, $\text{Att}_R H^n_R(M/N^\mathfrak{a}) = \text{Ass}_R(\mathfrak{a}, M)$, that means $\mathfrak{p} \in \text{Ass}_R(\mathfrak{a}, M) = \text{Ass}_R(M/N^\mathfrak{a})$, a contradiction. \qed

**Definition 2.3.** Let $\mathfrak{a}$ be an ideal of $R$, $M$ be a finitely generated $R$-module of dimension $n$ and $N^\mathfrak{a}$ be defined as in Notation 2.1. We say $H^n_R(M)$ satisfies the property $(\ast)$, if one of the equivalent conditions of Lemma 2.2 holds.

**Proposition 2.4.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of $R$ such that $H^n_R(M)$ satisfies the property $(\ast)$. If $\text{Att}_R H^n_R(\mathfrak{b}, M) \subseteq \text{Att}_R H^n_R(\mathfrak{a}, M)$, then there exists an epimorphism $H^n_R(M) \to H^n_R(M)$. 

**Proof.** Since $H^n_R(M)$ satisfies the property $(\ast)$, we have

$$H^n_R(M) \cong H^n_R(M/N^\mathfrak{a}) \cong H^n_R(M/N^\mathfrak{a})$$

and

$$\text{Att}_R H^n_R(\mathfrak{a}, M) = \text{Att}_R H^n_R(M/N^\mathfrak{a}) = \text{Ass}_R(\mathfrak{a}, M) = \text{Ass}_R(M/N^\mathfrak{a})$$

where, $N^\mathfrak{a} = \bigcap_{\mathfrak{p} \in \text{Ass}_R(\mathfrak{a}, M)} N(\mathfrak{p})$. Now we show that $H^n_R(N^\mathfrak{a}) = 0$. Suppose, on the contrary, that $H^n_R(N^\mathfrak{a}) \neq 0$. Then there exists a prime ideal
\( p \in \text{Att}_R \mathcal{H}^\alpha(M) \) and therefore for this prime ideal, by [4, Theorem A] we have, \( p \in \text{Ass}_R N^\alpha \) and \( \text{cd}(b, R/p) = n \). Since \( \text{Ass}_R N^\alpha \subseteq \text{Ass}_R M \), we have \( p \in \text{Att}_R \mathcal{H}^\alpha(M) \) and therefore \( p \in \text{Att}_R \mathcal{H}^\alpha(M) \) that is a contradiction by Notation 2.1. So, \( \mathcal{H}^\alpha(M) \cong \mathcal{H}^\alpha(M/N^\alpha) \). By [2, Proposition 8.1.2], for each \( x \in m \setminus b \), there is a long exact sequence

\[
\cdots \rightarrow \mathcal{H}^\alpha_{b + Rx}(M/N^\alpha) \rightarrow \mathcal{H}^\alpha_b(M/N^\alpha) \rightarrow \mathcal{H}^\alpha_b((M/N^\alpha)_x) \rightarrow \cdots
\]

where \( (M/N^\alpha)_x \) is the localization of \( M/N^\alpha \) at \( \{x^i : i \geq 0\} \). Note that \( \mathcal{H}^\alpha_b(M/N^\alpha) \) is Artinian and \( \mathcal{H}^\alpha_b((M/N^\alpha)_x) \cong (\mathcal{H}^\alpha_b(M/N^\alpha))_x \). It follows that \( \mathcal{H}^\alpha_b((M/N^\alpha)_x) = 0 \) and so there exists an epimorphism \( \mathcal{H}^\alpha_{b + Rx}(M/N^\alpha) \rightarrow \mathcal{H}^\alpha_b(M/N^\alpha) \). Repeating the argument with \( b + Rx \) in place of \( b \) and continuing gives an epimorphism \( \mathcal{H}^\alpha_m(M/N^\alpha) \rightarrow \mathcal{H}^\alpha_b(M/N^\alpha) \) and so we have the epimorphism \( \mathcal{H}^\alpha_a(M) \rightarrow \mathcal{H}^\alpha_b(M) \).

\begin{corollary}
Let \( a \) and \( b \) be two ideals of \( R \) such that \( \mathcal{H}^\alpha_a(M) \) and \( \mathcal{H}^\alpha_b(M) \) satisfy the property \((*)\). If \( \text{Att}_R \mathcal{H}^\alpha_a(M) = \text{Att}_R \mathcal{H}^\alpha_b(M) \), then \( \mathcal{H}^\alpha_a(M) \cong \mathcal{H}^\alpha_b(M) \).
\end{corollary}

\begin{proof}
As in the proof of Proposition 2.4, since

\[
\text{Att}_R \mathcal{H}^\alpha_a(M) = \text{Att}_R \mathcal{H}^\alpha_b(M),
\]

we have \( N^\alpha = N^b \) and so

\[
\mathcal{H}^\alpha_a(M) \cong \mathcal{H}^\alpha_m(M/N^\alpha) \cong \mathcal{H}^\alpha_m(M/N^b) \cong \mathcal{H}^\alpha_b(M).
\]

\end{proof}

Dibaei and Jafari in [3], have shown that if \( R \) is a complete Noetherian local ring and \( M \) is a finitely generated \( R \)-module of dimension \( n \), then any subset \( T \) of \( \text{Assh}_R M \) can be expressed as the set of attached primes of the top local cohomology module \( \mathcal{H}^\alpha_a(M) \) for some ideal \( a \) of \( R \) (see [3, Theorem 2.8]). In the next theorem, we generalize this result to Noetherian local rings that are not necessarily complete.

\begin{theorem}
Let \( M \) be a finitely generated \( R \)-module of dimension \( n \) and \( T \) be a subset of \( \text{Assh}_R(M) \), then there exists an ideal \( a \) of \( R \) such that \( \text{Att}_R \mathcal{H}^\alpha_a(M) = T \).
\end{theorem}

\begin{proof}
Let \( \text{Assh}_R M = \{p_1, \ldots, p_k\} \) and \( T = \{p_1, \ldots, p_r\} \), where \( r \leq k \). When \( r = k \), the result is immediate from [2, Theorem 7.3.2], just take \( a = m \). We therefore assume henceforth in this proof that \( r < k \). So \( \text{Assh}_R M \setminus T = \{p_{r+1}, \ldots, p_k\} \). Since, for each \( 1 \leq i \leq k \), \( p_i \) is a minimal associated prime of \( M \), we have \( \cap_{i=r+1}^k p_i \notin \bigcup_{i=1}^r p_i \). So we can choose an element \( y \in \cap_{i=r+1}^k p_i \setminus \bigcup_{i=1}^r p_i \). Set \( M = \overline{\{(\cap_{i=1}^r p_i)M\}} \), then \( \text{Assh}_R M = T \) and \( \dim(M) = n \). Since \( y \notin \bigcup_{i=1}^r p_i \), there are
elements $x_1, \ldots, x_{n-1}$ such that $y, x_1, \ldots, x_{n-1}$ forms a system of parameters for $R$-module $\overline{M}$. Set $\mathfrak{a} = \langle y, x_1, \ldots, x_{n-1} \rangle$. It follows from [2, Independence Theorem 4.2.1 and Exercise 6.1.9] that

$$H^n_\mathfrak{a}(M) \otimes_{\overline{M}} \frac{R}{\bigcap_{i=1}^n \mathfrak{p}_i} \cong H^n_\mathfrak{a}(M \otimes_{\overline{M}} \frac{R}{\bigcap_{i=1}^n \mathfrak{p}_i}) \cong H^n_{\mathfrak{a}}(\overline{M}) \cong H^n_{\mathfrak{m}}(\overline{M}) \neq 0.$$  

We can now use [2, Theorem 7.3.2 and Exercise 7.2.6] to deduce that

$$T = \text{Att}_R H^n_\mathfrak{a}(\overline{M}) \subseteq \text{Att}_R H^n_\mathfrak{a}(M).$$  

On the other hand, if $\mathfrak{p}_i \in \text{Assh}_R M \setminus T$, then

$$H^n_\mathfrak{a}(\frac{R}{\mathfrak{p}_i}) = H^n_{\langle y, x_1, \ldots, x_{n-1} \rangle}(\frac{R}{\mathfrak{p}_i}) \cong H^n_{\langle x_1, \ldots, x_{n-1} \rangle}(\frac{R}{\mathfrak{p}_i}) \text{ by [2, Theorem 3.3.1]} = 0.$$  

It follows from this observation and [4, Theorem A] that $\text{Att}_R H^n_\mathfrak{a}(M) = T$. Hence $\text{Att}_R H^n_\mathfrak{a}(M) = T$ and this completes the proof. \(\square\)

**Remark 2.7.** Let $M$ be a finitely generated $R$-module of dimension $n$ and $T$ be a subset of $\text{Assh}_R M$. By Theorem 2.6, there exists an ideal $\mathfrak{a}$ of $R$ such that $\text{Att}_R H^n_\mathfrak{a}(M) = T$. By the choice of this ideal in the proof of Theorem 2.6, one can see that, for each $\mathfrak{p} \in T$, $\sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m}$. Therefore

$$\text{Att}_R H^n_\mathfrak{a}(M) = \{ \mathfrak{p} \in \text{Assh}_R M : \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m} \} = \text{Assh}_R(\mathfrak{a}, M)$$  

and so $\text{Att}_R H^n_\mathfrak{a}(M) = \text{Att}_R H^n_\mathfrak{a}(M/N^\mathfrak{a})$. Hence $H^n_\mathfrak{a}(M)$ satisfies the property ($\ast$).

3. Some Results on Top Formal Local Cohomology

In [8], Rezaei proved that if $(R, \mathfrak{m})$ is a complete Noetherian local ring and $M$ is a finitely generated $R$-module of dimension $n$, then for each ideal $\mathfrak{a}$ of $R$ there exists an ideal $\mathfrak{b}$ such that $H^n_\mathfrak{a}(M) \cong H^n_\mathfrak{b}(M)$ and there exists an ideal $\mathfrak{c}$ such that $H^n_\mathfrak{a}(M) = H^n_\mathfrak{c}(M)$. In this section we give a generalization of this result.

**Lemma 3.1.** (See [8, Theorem 2.2].) Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ be a finitely generated $R$-module of dimension $n$. If $T$ is a proper subset of $\text{Assh}_R M$, then $\text{Att}_R H^n_\mathfrak{a}(M) = T$ where $\mathfrak{a} := \bigcap_{\mathfrak{p}_i \in T} \mathfrak{p}_i$ is an ideal of $R$.

**Lemma 3.2.** (See [8, Lemma 2.4].) Let $\mathfrak{a}$ be an ideal of a Noetherian local ring $R$ and $M$ be a finitely generated $R$-module. If $M$ is an $\mathfrak{a}$-torsion module, then $\widehat{H}_\mathfrak{a}^i(M) \cong H^i_\mathfrak{m}(M)$ for all $i \geq 0$. 

By [5, Proposition 2.1], if \( \mathfrak{a} \) is an ideal of \( R \) and \( M \) is a finitely generated \( R \)-module of dimension \( n \), then \( \mathfrak{F}_b^\alpha(M) \) is an Artinian \( R \)-module and there exists an integer \( n_0 \) such that \( \mathfrak{F}_b^\alpha(M) \cong \frac{H_a^\alpha(M)}{a^{n_0}H_a^\alpha(M)} \).

Now we can reduce the completeness assumption in [8, Theorem 2.5] to the assumption that \( H_a^\alpha(M) \) satisfies the property (*)

**Theorem 3.3.** Let \( \mathfrak{a} \) and \( \mathfrak{b} \) be two ideals of a Noetherian local ring \((R, \mathfrak{m})\) and \( M \) be a finitely generated \( R \)-module of dimension \( n \) such that \( H_a^\alpha(M) \) satisfies the property (*). If \( \text{Att}_R H_a^\alpha(M) = \text{Att}_R \mathfrak{F}_b^\alpha(M) \), then \( H_a^\alpha(M) \cong \mathfrak{F}_b^\alpha(M) \).

**Proof.** Since \( H_a^\alpha(M) \) satisfies the property (*), by Notation 2.1 and Definition 2.3 we have \( H_a^\alpha(N^0) = 0 \) and

\[
\text{Att}_R \mathfrak{F}_b^\alpha(M) = \text{Att}_R H_a^\alpha(M) = \{ \mathfrak{p} \in \text{Ass}_R M : \dim(R/\mathfrak{p}) = n \text{ and } \sqrt{\mathfrak{a} + \mathfrak{p}} = \mathfrak{m} \} = \text{Ass}_R(\mathfrak{a}, M).
\]

Now we show that the Artinian module \( \mathfrak{F}_b^\alpha(N^0) \) is zero. Suppose, on the contrary, that \( \mathfrak{F}_b^\alpha(N^0) \neq 0 \). Therefore there exists a prime ideal \( \mathfrak{p} \in \text{Att}_R \mathfrak{F}_b^\alpha(N^0) \). By [5, Proposition 2.1], \( \mathfrak{p} \in \text{Ass}_R N^0 \), \( \dim(R/\mathfrak{p}) = n \) and \( \mathfrak{b} \subseteq \mathfrak{p} \). Therefore \( \mathfrak{p} \in \text{Att}_R \mathfrak{F}_b^\alpha(M) = \text{Att}_R H_a^\alpha(M) = \text{Ass}_R(\mathfrak{a}, M) \), a contradiction. Therefore \( \mathfrak{F}_b^\alpha(N^0) = 0 \) and \( \mathfrak{F}_b^\alpha(M) \cong \mathfrak{F}_b^\alpha(M/N^0) \).

On the other hand, since \( \text{Att}_R \mathfrak{F}_b^\alpha(M) = \text{Ass}_R(M/N^0) \), we have \( \mathfrak{b} \subseteq \bigcap_{\mathfrak{p} \in \text{Ass}_R(M/N^0)} \mathfrak{p} \). Therefore \( M/N^0 \) is a \( \mathfrak{b} \)-torsion \( R \)-module and by Lemma 3.2, we have \( \mathfrak{F}_b^\alpha(M/N^0) \cong H_m^\alpha(M/N^0) \cong H_a^\alpha(M/N^0) \cong H_a^\alpha(M) \).

\( \square \)

**Corollary 3.4.** Let \( \mathfrak{a} \) be an ideal of a Noetherian local ring \((R, \mathfrak{m})\) such that \( H_a^\alpha(M) \) satisfies the property (*). Then \( H_a^\alpha(M) \cong \mathfrak{F}_b^\alpha(M) \), where \( \mathfrak{b} = \text{Ann}_R H_a^\alpha(M) \).

**Proof.** Let \( \mathfrak{b} = \text{Ann}_R H_a^\alpha(M) \), then \( \sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{p} \in \text{Ass}_R H_a^\alpha(M)} \mathfrak{p} \). Since

\[
\text{Att}_R H_a^\alpha(M) \subseteq \text{Assh}_R M,
\]

it follows from Lemma 3.1 that \( \text{Att}_R H_a^\alpha(M) = \text{Att}_R \mathfrak{F}_b^\alpha(M) \) and so by Theorem 3.3, we have \( H_a^\alpha(M) \cong \mathfrak{F}_b^\alpha(M) \).

\( \square \)

Now we can generalize [8, Theorem 2.6 (ii)] and [8, Corollary 2.7] to Noetherian local rings that are not necessarily complete.

**Theorem 3.5.** Let \( \mathfrak{a} \) be an ideal of a Noetherian local ring \((R, \mathfrak{m})\) and \( M \) be a finitely generated \( R \)-module of dimension \( n \). Then there exists an ideal \( \mathfrak{c} \) of \( R \) such that \( \mathfrak{F}_a^\alpha(M) \cong H_c^\alpha(M) \).
Proof. Since $\Att_R \mathfrak{F}^a_n(M) \subseteq \Assh_R M$, it follows from Theorem 2.6 that there exists an ideal $\mathfrak{c}$ of $R$ such that $\Att_R H^a_n(M) = \Att_R \mathfrak{F}^a_n(M)$. By Remark 2.7, $H^a_{\mathfrak{c}}(N^c) = 0$, where $N^c$ is defined as in Notation 2.1. Therefore $H^a_{\mathfrak{c}}(M)$ satisfies the property (*). Now by Theorem 3.3, $\mathfrak{F}^a_n(M) \cong H^a_{\mathfrak{c}}(M)$.

\textbf{Corollary 3.6.} Let $a$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and $M$ be a finitely generated $R$-module of dimension $n$. Then $\mathfrak{F}^a_n(M) \cong \mathfrak{F}_{Ann_R \mathfrak{F}^a_n(M)}(M)$.

\textbf{Proof.} By Theorem 3.5, there exists an ideal $\mathfrak{c}$ of $R$ such that $\mathfrak{F}^a_n(M) \cong H^a_{\mathfrak{c}}(M)$. As $H^a_{\mathfrak{c}}(M)$ satisfies the property (*), we have $H^a_{\mathfrak{c}}(M) \cong \mathfrak{F}_{Ann_R H^a_{\mathfrak{c}}(M)}(M)$ by Corollary 3.4, and so $\mathfrak{F}^a_n(M) \cong \mathfrak{F}_{Ann_R \mathfrak{F}^a_n(M)}(M)$, as required.

\textbf{Theorem 3.7.} Let $a$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and $M$ be a finitely generated $R$-module of dimension $n$ such that $H^a_n(M)$ satisfies the property (*). Then $H^a_n(M) \cong \frac{H^a_m(M)}{(Ann_R H^a_n(M)) H^a_m(M)}$.

\textbf{Proof.} By Corollary 3.4, we have $H^a_n(M) \cong \mathfrak{F}_{Ann_R H^a_n(M)}(M)$ and by [5, Proposition 2.1], there exists an integer $t_0$ such that

$$\mathfrak{F}_{Ann_R H^a_n(M)}(M) = \frac{H^a_m(M)}{(Ann_R H^a_n(M))^t H^a_m(M)}$$ for all $t \geq t_0$.

Hence $H^a_n(M) \cong \frac{H^a_m(M)}{(Ann_R H^a_n(M))^t H^a_m(M)}$ for all $t \geq t_0$ and so

$$Ann_R H^a_n(M) = Ann_R(\frac{H^a_m(M)}{(Ann_R H^a_n(M))^t H^a_m(M)})$$ for all $t \geq t_0$.

It follows that

$$(Ann_R H^a_n(M)) H^a_m(M) \subseteq (Ann_R H^a_n(M))^t H^a_m(M)$$ for all $t \geq t_0$.

Hence $(Ann_R H^a_n(M))^t H^a_m(M) = Ann_R H^a_n(M) H^a_m(M)$ for all $t \geq t_0$ and therefore

$$H^a_n(M) \cong \frac{H^a_m(M)}{(Ann_R \mathfrak{F}^a_n(M)) H^a_m(M)}.$$

\textbf{Theorem 3.8.} Let $a$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and $M$ be a finitely generated $R$-module of dimension $n$. Then $\mathfrak{F}^a_n(M) \cong \frac{H^a_m(M)}{(Ann_R \mathfrak{F}^a_n(M)) H^a_m(M)}$. 
Corollary 3.9. Let \( a \) and \( b \) be two ideals of a Noetherian local ring \((R, m)\) and \( M \) be a finitely generated \( R \)-module of dimension \( n \).

(i) If \( \text{Ann}_R \mathfrak{F}_a^n(M) = \text{Ann}_R \mathfrak{F}_b^n(M) \), then \( \mathfrak{F}_a^n(M) \cong \mathfrak{F}_b^n(M) \);
(ii) If \( \text{H}_a^n(M) \) satisfies the property (\( * \)) and
\[
\text{Ann}_R \text{H}_a^n(M) = \text{Ann}_R \mathfrak{F}_b^n(M),
\]
then \( \text{H}_a^n(M) \cong \mathfrak{F}_b^n(M) \);
(iii) If both \( \text{H}_a^n(M) \) and \( \text{H}_b^n(M) \) satisfy the property (\( * \)) and
\[
\text{Ann}_R \text{H}_a^n(M) = \text{Ann}_R \text{H}_b^n(M),
\]
then \( \text{H}_a^n(M) \cong \text{H}_b^n(M) \).

Proof. All items are clear by Theorem 3.7 and Theorem 3.8. □

Theorem 3.10. Let \( a \) be an ideal of a Noetherian local ring \((R, m)\) and \( M \) be a finitely generated \( R \)-module of dimension \( n \). Then

(i) We have the equalities
\[
\text{Att}_R \mathfrak{F}_a^n(M) = \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)) \cap \text{Assh}_R M
= \text{Min} \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)).
\]
(ii) If \( \text{H}_a^n(M) \) satisfies the property (\( * \)), then
\[
\text{Att}_R \text{H}_a^n(M) = \text{V}(\text{Ann}_R \text{H}_a^n(M)) \cap \text{Assh}_R M = \text{Min} \text{V}(\text{Ann}_R \text{H}_a^n(M)).
\]

Proof. (i) Since for each Artinian \( R \)-module \( A \),
\[
\text{Att}_R (A/aA) = \text{Att}_R A \cap \text{V}(a),
\]
by Theorem 3.8, we have
\[
\text{Att}_R \mathfrak{F}_a^n(M) = \text{Att}_R \text{H}_m^n(M) \cap \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M))
= \text{Assh}_R M \cap \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M))
\subseteq \text{Min} \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)).
\]
On the other hand
\[
\text{Min} \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)) = \text{Min} \text{Att}_R \mathfrak{F}_a^n(M)
\subseteq \text{Assh}_R M \cap \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)).
\]
Therefore
\[
\text{Att}_R \mathfrak{F}_a^n(M) = \text{Assh}_R M \cap \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)) = \text{Min} \text{V}(\text{Ann}_R \mathfrak{F}_a^n(M)).
\]
(ii) The proof is similar to the proof (i). □
Corollary 3.11. Let $a$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ and $M$ be a finitely generated $R$-module of dimension $n$. Then

(i) $\text{Att}_R \mathfrak{F}_a^n(M) = \text{Ass}_R \left( \frac{R}{\text{Ann}_R \mathfrak{F}_a^n(M)} \right)$.

(ii) If $H_a^n(M)$ satisfies the property $(*),$ then $\text{Att}_R H_a^n(M) = \text{Ass}_R \left( \frac{R}{\text{Ann}_R H_a^n(M)} \right)$.

Proof. (i) Since $\mathfrak{F}_a^n(M)$ is Artinian, it follows from [10, Theorem 3.1 and Theorem 3.3 (b)] that $\text{Ass}_R (R/\text{Ann}_R \mathfrak{F}_a^n(M)) \subseteq \text{Att}_R (\mathfrak{F}_a^n(M))$. But the sets $V(\text{Ann}_R \mathfrak{F}_a^n(M))$ and $\text{Ass}_R (R/\text{Ann}_R \mathfrak{F}_a^n(M))$ have the same minimal elements, by [10, Theorem 3.3 (c)]. Thus, by Theorem 3.10, $\text{Att}_R (\mathfrak{F}_a^n(M)) \subseteq \text{Ass}_R (R/\text{Ann}_R \mathfrak{F}_a^n(M))$. Therefore

$\text{Att}_R (\mathfrak{F}_a^n(M)) = \text{Ass}_R (R/\text{Ann}_R \mathfrak{F}_a^n(M))$.

(ii) The proof is similar to the proof (i). \hfill \Box

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TOP LOCAL COHOMOLOGY AND TOP FORMAL LOCAL COHOMOLOGY MODULES WITH SPECIFIED ATTACHED PRIMES

A. NAZARI AND F. RASTGOO

پایان‌مطابق دو مدل‌های کوهمولوژی و کوهومولوژی موضوعی، با ایده‌آل‌های اول چسبیده مشخص

علیرضا نظری و فهیمه راستگو

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فرض کنید (R, m) یک حلقه موضعی نوتری باشد. در این مقاله نشان می‌دهیم که با اضافه‌سازی هر زیر مجموعهی a ایده‌آل R از R، تاریخچه ایده‌آل 

\[ \text{Ass}_{R} M \]

از T باشد. با استفاده از این مطلب برخی ارتباطات بین 

\[ \text{Att}_{R} H_{n}^{a}(M) = T \]

با این مدل‌های کوهومولوژی موضوعی، مدل‌های کوهومولوژی موضوعی، مدل‌های کوهومولوژی موضوعی

کلمات کلیدی: ایده‌آل‌های اول چسبیده، مدل‌های کوهومولوژی موضوعی، مدل‌های کوهومولوژی موضوعی

\[ \text{Ass}_{R} M \]

روش‌های نوتری موضوعی

\[ \text{Att}_{R} H_{n}^{a}(M) = T \]

را تعمیم می‌دهد.

\[ \text{Ass}_{R} M \]


\[ \text{Att}_{R} H_{n}^{a}(M) = T \]