CONTINUOUS FUNCTIONS ON LG-SPACES

A. R. ALIABAD* AND H. ZAREPOUR

ABSTRACT. By an $l$-generalized topological space, briefly an LG-space, we mean the ordered pair $(F, \tau)$ in which $F$ is a frame and $\tau$ is a subframe of $F$. This notion has been first introduced by A.R. Aliabad and A. Sheykhmiri in [LG-topology, Bull. Iran. Math. Soc., 41 (1), (2015), 239-258]. In this article, we define continuous functions on LG-spaces and determine conditions under which the continuous image of a compact element of an LG-space is compact. Moreover, we introduce the concept of connectedness for LG-spaces and determine conditions under which the continuous image of a connected element of an LG-space is connected. In fact, we show that LG-spaces are models for topological spaces as well as frames are models for topologies.

1. Introduction

A complete lattice $L$ is a lattice in which every subset has a supremum. Clearly, a complete lattice is a bounded lattice, i.e., it has the largest element 1 and the smallest element 0. A frame $F$ is a complete lattice in which the distributive law $a \land (\bigvee S) = \bigvee_{s \in S} (a \land s)$ holds for every $a \in F$ and $S \subseteq F$. A pseudocomplement of an element $a$ of a bounded lattice $L$ is defined by $\max\{x \in L : x \land a = 0\}$, if it exists, and denoted by $a^\ast$. Obviously, if $F$ is a frame, then $a^\ast = \bigvee\{x \in L : x \land a = 0\}$. Let $F$ be a frame. Then a subset $G$ of $F$ which is closed under finite meets and arbitrary joins is called a subframe of $F$. Let $(X, \tau)$ be any topological space. Then it is

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*Corresponding author.
clear that $\tau$ is a frame, and if $U \subseteq \tau$, then $\bigvee_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} U$ and $\bigwedge_{U \in \mathcal{U}} U = \text{int}_X(\bigcap_{U \in \mathcal{U}} U)$. In fact, this example is a basic model that inspires topologists to study a frame as a pointfree topology.

In [8] and later in [9], Guo-Jun Wang constructed a model for topological spaces on a completely distributive lattice. In [1], this view was followed and generalized to topological spaces known as LG-spaces. This article, in fact, is a continuation of the paper [1]. There are two viewpoints for introducing continuous functions on this structure. In [3], continuous functions are introduced from viewpoint of the locale. In the present paper, in frame viewpoint, we define continuous functions in the context of LG-spaces with a little difference from what is defined in [9]. So, in the following, we will recall some of the material from [1] that are needed to understand the issue better. The reader is referred to [6] and [5] for more details concerning frames. Also, see [4], [7] and [2] for more information about general lattice theory and general topology, respectively.

Note that a topological space $(X, \tau)$ can be considered as $(P(X), \tau)$. Clearly, $\tau$ is a subframe of $P(X)$ and via this viewpoint, the following definition is natural.

Throughout this paper, all lattices considered to be frames, unless otherwise stated explicitly.

The following definition has been proposed for the first time in [1] and for bounded pseudocomplemented distributive lattices, but in this article we will present it according to our purpose for the frames.

**Definition 1.1.** Let $F$ be a frame and $\tau$ be a subframe of $F$. Then $\tau$ is called an $l$-generalized topology on $F$ and $(F, \tau)$ (briefly, $F$) is called an $l$-generalized topological space. Every element of $\tau$ is said to be open and any element of $\tau^* = \{t^* : t \in \tau\}$ is said to be a closed element. Clearly, the set of closed elements is a $\land$-structure, since $\bigvee_{\lambda \in \Lambda} t_{\lambda}^* = \bigwedge_{\lambda \in \Lambda} t_{\lambda}^*$. Furthermore, if $\tau^*$ is a sublattice of $F$, then we say $\tau$ is an $l$-topology on $F$ and $(F, \tau)$ (briefly, $F$) is an $l$-topological space; for convenience, we denote an $l$-generalized topological space (resp., $l$-topological space) by LG-space (resp., $L$-space). Assuming that $\tau$ is an $l$-generalized topology on a frame $F$ and $a \in F$, we define $a^v = \bigvee\{t \in \tau : t \leq a\}$ and $\overline{a} = \bigwedge\{x \in \tau^* : a \leq x\}$. Sometimes, we use $\text{int}_\tau a$ and $\text{cl}_\tau a$ instead of $a^v$ and $\overline{a}$, respectively.

Obviously, in Definition 1.1, if each element of $\tau$ has a complement in $F$, then $\tau^* = \tau^c = \{t^c : t \in \tau\}$, and we have a structure named topospace which is introduced and studied in [3].
Note that an LG-space need not be an L-space, see [1].

**Remark 1.2.** Let $L$ be a pseudocomplemented lattice. Then the following statements hold concerning the mapping $*: L \to L$.

(i) The mapping “$*$” is decreasing and $a \leq a^{**}$ for every $a \in L$.
(ii) The mapping “$**$” is the identity mapping on $L^*$, i.e., $a^{***} = a^*$ for all $a \in L$ (so the mapping “$**$” on $L$ is a closure operator).
(iii) For every $a, b \in L$, we have
\[a \land b = 0 \iff a \leq b^* \iff b \leq a^* \iff a^{**} \leq b^* \iff a^{**} \land b = 0.\]
(iv) If $L$ is a frame and $S \subseteq L$, then $(\bigvee_{s \in S} s)^* = \bigwedge_{s \in S} s^*$.

**Definition 1.3.** Suppose that $F$ is a frame and $S \subseteq F$. We denote the set of finite meets of elements of $S$ by $Fm(S)$. Set $< S > = \{ \bigvee D : D \subseteq Fm(S) \}$. Clearly, $< S >$ is the smallest subframe of $F$ containing $S$. If $(F, \tau)$ is an LG-space and $\tau = < S >$ for some $S \subseteq F$, then $S$ is said to be a subbase for the topology $\tau$. A set $B \subseteq \tau$ is called a base for an LG-topology $\tau$ if for every $t \in \tau$ there exists $D \subseteq B$ such that $t = \bigvee D$. Moreover, assuming that $F$ is a frame and $B \subseteq F$, we say $B$ is a base for a topology on $F$ if $1 = \bigvee B$, and for every $b_1, b_2 \in B$ there exists $D \subseteq B$ such that $b_1 \land b_2 = \bigvee D$.

**Proposition 1.4.** Let $(F, \tau)$ be an LG-space. The following statements hold:

(a) $0^o = 0$ and $1^o = 1$.
(b) $a^o \leq a$ for every $a \in F$.
(c) If $a, b \in F$ and $a \leq b$, then $a^o \leq b^o$.
(d) For each $a \in F$, $a^o = \bigvee\{ t \in B : t \leq a \}$ where $B$ is a base for $\tau$.
(e) For each $a \in F$, $a^o \in \tau$.
(f) $a = a^o$ if and only if $a \in \tau$.
(g) For each $a \in F$, $(a^o)^o = a^o$.
(h) $a^o$ is the greatest element of $\tau$ that is less than or equal to $a$.
(i) If $a_1, \ldots, a_n \in F$, then $(\bigwedge_{i=1}^n a_i)^o = \bigwedge_{i=1}^n a_i^o$.

Let $\varphi : F \to F$ be a mapping that satisfies (a), (b), (c), (g) and (i). If we define $\tau = \{ a \in F : \varphi(a) = a \}$, then $\tau$ is an LG-topology on $F$, and the interior operator induced by $\tau$ coincides with $\varphi$.

**Proposition 1.5.** Let $(F, \tau)$ be an LG-space. The following statements hold:

(a) $\overline{0} = 0$ and $\overline{1} = 1$.
(b) For each $a \in F$, $a \leq \overline{a}$.
(c) If $a, b \in F$ and $a \leq b$, then $\overline{a} \leq \overline{b}$.
(d) $\overline{a} \in \tau^*$ for every $a \in F$.
(e) $a = \overline{a}$ if and only if $a \in \tau^*$. 
\( (f) \overline{a} = a \) for all \( a \in F \).

\( \overline{a} \) is the smallest closed element that is greater than or equal to \( a \).

\( (g) \overline{a} \) is the smallest closed element that is greater than or equal to \( a \).

\( (h) \) If \((F, \tau)\) is an \( L \)-space and \( a_1, ..., a_n \in F \), then we have \( \overline{\bigvee_{i=1}^{n} a_i} = \bigvee_{i=1}^{n} \overline{a_i} \).

**Definition 1.6.** Suppose that \((F, \tau)\) is an \( LG \)-space and \( a \in F \). If we take \( F_a = \downarrow a \) and \( \tau_a = \{ t \land \alpha : t \in \tau \} \), then clearly \((F_a, \tau_a)\) is an \( LG \)-space. We call \((F_a, \tau_a)\) as a subspace of \((F, \tau)\) (briefly, we say that \( F_a \) is a subspace of \( F \)).

In the following, the basic properties of subspaces in \( LG \)-spaces are given.

**Proposition 1.7.** Suppose that \((F, \tau)\) is an \( LG \)-space and \( a \in F \). The following statements hold.

\( (a) \) If \( S \) is a subbase for \( \tau \), then \( S_a = \{ s \land a : s \in S \} \) is a subbase for \( \tau_a \).

\( (b) \) If \( B \) is a base for \( \tau \), then \( B_a = \{ t \land a : t \in B \} \) is a base for \( \tau_a \).

**Proposition 1.8.** Suppose that \((F, \tau)\) is an \( LG \)-space and \( a \in F \). Then, the following statements hold.

\( (a) \) \( \{ (t \land a)^* \land a : t \in \tau \} \) is the set of closed elements of \( F_a \). In particular, if \( a \in F^* = \{ x^* : x \in F \} \), then \( \{ t^* \land a : t \in \tau \} \) is the set of closed elements of \( F_a \).

\( (b) \) If \( x \leq a \), then \( \text{cl}_{\tau_a} x = (\text{int}_{\tau} x^* \land a)^* \land a \).

\( (c) \) If \( a \in F^* \) and \( x \leq a \), then \( \text{cl}_{\tau_a} x = \text{cl}_{\tau} x \land a \). In particular, if \( a \) is a closed element in \( F \), then \( \text{cl}_{\tau_a} x = \text{cl}_{\tau} x \).

\( (d) \) If \( x \leq a \), then \( \text{int}_{\tau} x \leq \text{int}_{\tau_a} x \). The converse of this fact is not necessarily true.

\( (e) \) If \( a \) is an open element in \( F \) and \( x \leq a \), then \( \text{int}_{\tau} x = \text{int}_{\tau_a} x \).

**Definition 1.9.** Suppose that \((F, \tau)\) is an \( LG \)-space. We say \( a \in F \) is \( \tau \)-compact (briefly, compact) if whenever \( S \subseteq \tau \) and \( a \leq \bigvee S \), then there exists a finite subset \( D \) of \( S \) such that \( a \leq \bigvee D \). We can similarly define Lindelöf, countably compact, etc. Whenever 1 (i.e., the top element of \( F \)) is a compact element in \((F, \tau)\), we say \((F, \tau)\) (briefly, \( F \)) is a compact space.

**Definition 1.10.** Suppose that \((F_i, \tau_i)\) is an \( LG \)-space, for every \( i \in I \). Clearly, \( F = \prod_{i \in I} F_i \) with ordinary order is a frame. Two topologies can be defined on \( F \) as follows:

\( (i) \) \( \tau = \{ (t_i)_{i \in I} : t_i \in \tau_i, \text{ and } t_i = 1 \text{ for all except finitely many } i \in I \} \cup \{ 0 \} \). This topology is called product topology on \( F \). When we deal with \( \prod_{i \in I} F_i \) as an \( LG \)-space, we have in view this topology.
(ii) \( \tau_b = \{ t = (t_i)_{i \in I} : \forall i \in I, \ t_i \in \tau \} \). This topology is called the box topology on \( F \).

Clearly, if \( F = \prod_{i \in I} F_i \) and \( \pi_j \) is the projection mapping from \( F \) to \( F_j \), then for every \( S \subseteq F \), we have \( \bigvee S = \bigvee_{s \in S} \pi_i(s)_{i \in I} \).

**Proposition 1.11.** Suppose that \((F_i, \tau_i)\) is an LG-space, for every \( i \in I \), \( F = \prod_{i \in I} F_i \), and \( \tau \) and \( \tau_b \) are the product topology and box topology on \( F \), respectively. Then the following statements hold.

(a) \( \tau^* = \{ x = (x_i)_{i \in I} \in F : \ x_i \in \tau_i^* \}, \) and \( x_0 = 0 \), for all except finitely many \( i \in I \).

(b) \( \tau_b^* = \{ x = (x_i)_{i \in I} \in F : \ x_i \in \tau_i^* \}, \forall i \in I \} \).

(c) For every \( x \in F \), if \( I \) is infinite, then we have \( \text{int}_x \neq 0 \) if and only if \( x_i = 1 \) for all except finitely many \( i \in I \). Also, if \( I \) is finite, then we have \( \text{int}_x \neq 0 \) if and only if there exists \( i \in I \) such that \( x_i^0 \neq 0 \).

(d) For every \( x \in F \), if \( I \) is finite or \( \text{int}_x \neq 0 \), then \( \text{int}_x = (x_i^0)_{i \in I} \).

(e) For every \( x \in F \), if \( I \) is infinite, then we have \( \text{cl}_x \neq 1 \) if and only if \( x_i = 0 \) for all except finitely many \( i \in I \), and if \( I \) is finite, then we have \( \text{cl}_x \neq 1 \) if and only if there exists \( i \in I \) such that \( \text{cl}_{x_i} \neq 1 \).

(f) For every \( x \in F \), if \( I \) is finite or \( \text{cl}_x \neq 1 \), then \( \text{cl}_x = (\text{cl}_{x_i})_{i \in I} \).

(g) If \( x \in F \), then \( \text{int}_{\tau_b} x = (x_i^0)_{i \in I} \).

(h) If \( x \in F \), then \( \text{cl}_{\tau_b} x = (\text{cl}_{x_i})_{i \in I} \).

2. Continuous functions on LG-spaces

We introduce the notion of continuity for LG-spaces by means of adjoint mappings as follows.

**Definition 2.1.** Suppose that \( X \) and \( Y \) are posets and \( f : X \to Y \) and \( g : Y \to X \) are order-preserving mappings. We say that \( f \) is the left adjoint of \( g \) (or \( g \) is the right adjoint of \( f \)) whenever

\[ \forall x \in X, \forall y \in Y \ x \leq g(y) \Leftrightarrow f(x) \leq y. \]

We denote by \( f_\ast \) the right adjoint of \( f \), if it exists.

Some basic properties of adjoint mappings are given in the next remark.

**Remark 2.2.** Suppose that \( X \) and \( Y \) are posets. It is easy to see that if \( f : X \to Y \) and \( g : Y \to X \) are two order-preserving mappings, then the following statements hold.

(a) The right adjoint of \( f \) (left adjoint of \( g \)), if it exists, is unique.

(b) If \( f \) is left adjoint and \( X \) and \( Y \) are bounded, then \( f(x) = 0 \) if and only if \( x \leq f_\ast(0) \). Consequently, \( f(0) = 0 \).

(c) If \( f \) is left adjoint and \( X \) and \( Y \) are bounded, then \( f_\ast(y) = 1 \) if and only if \( f(1) \leq y \). Consequently, \( f_\ast(1) = 1 \).
2.11. (a) that a set function $f$ mapping and $f$ Proposion could extend the concept of adjoint mapping. W e will find out later in $B$

suprema, then $f$ arbitrary infima).

$ff$ is onto if and only if $f(x) = x$ for every $x \in X$.

(h) If $f$ is left adjoint, then $ff_y(y) = y$ if and only if $y \in f(X)$. So, $f$ is onto if and only if $ff_y(y) = y$ for every $y \in Y$.

(i) If $X$ and $Y$ are complete lattices, then $f$ is left adjoint $(g$ is right adjoint) if and only if $f$ preserves arbitrary suprema $(g$ preserves arbitrary infima).

(j) If $F_1$ and $F_2$ are frames and $f : F_1 \to F_2$ preserves arbitrary suprema, then $f$ is left adjoint, $f_*(y) = \bigvee\{x \in F_1 : f(x) \leq y\}$, the mapping $f_*$ preserves arbitrary infima and $f(x) = \bigwedge\{y \in F_2 : x \leq f_*(y)\}$ for every $x \in F_1$.

Suppose that $h : X \to Y$ is a function. Then the set function $f : P(X) \to P(Y)$ defined by $f(A) = \{h(a) : a \in A\}$ is a left adjoint mapping and $f_*(B) = h^{-1}(B) = \{x \in X : h(x) \in B\}$ for every $B \in P(Y)$. It seems that this example is the basic model on which one could extend the concept of adjoint mapping. We will find out later in Proposition 2.11 that a set function $f : P(X) \to P(Y)$ is induced by a function $h : X \to Y$ if and only if $f$ has a right adjoint mapping which is also left adjoint.

As we mentioned in part (b) of the above remark, for every left adjoint mapping $f$ we have $f(0) = 0$. However, this is not true for the right adjoint mappings. For example, let $f : F_1 \to F_2$ be such that $f(x) = 0$ for every $x \in F_1$. Then $f$ is a left adjoint mapping and $f_*(y) = 1$ for every $y \in F_2$. On the other hand, if $f$ is a set function $f : P(X) \to P(Y)$ induced by $h : X \to Y$, then $f_*(B^*) = h^{-1}(B^*) = (h^{-1}(B))^c = (f_*(B))^c$ for every $B \in P(Y)$. The following proposition shows that even the inequality $f_*(y^*) \leq (f_*(y))^*$ does not hold, in general.

**Proposition 2.3.** Let $F_1$ and $F_2$ be two frames. Suppose that $f : F_1 \to F_2$ is a left adjoint mapping. Then the following statements are equivalent.

(a) For every $x \in F_1$, if $f(x) = 0$, then $x = 0$.
(b) For every $y \in F_2$, if $y \wedge f(1) = 0$, then $f_*(y) = 0$.
(c) $f_*(0) = 0$.
(d) For all $y \in F_2$, we have $f_*(y^*) \leq (f_*(y))^*$.
(e) There exists $y \in F_2$ such that $f_*(y^*) \leq (f_*(y))^*$.

**Proof.** (a) $\Rightarrow$ (b). Let $y \in F_2$ and $y \wedge f(1) = 0$. Thus, $ff_*(y) \leq y \wedge f(1) = 0$ and so $f_*(y) = 0$.
(b) $\Rightarrow$ (c). If we take $y = 0$, then we are done.
(c) ⇒ (d). Let \( y \in F_2 \), then \( y \wedge y^* = 0 \) and so \( 0 = f_*(y \wedge y^*) = f_*(y^*) \land f_*(y) \). Therefore, \( f_*(y^*) \leq (f_*(y))^* \).

(d) ⇒ (e). It is clear.

(e) ⇒ (a). Suppose that \( f(x) = 0 \) and \( y \in F_2 \) such that \( f_*(y^*) \leq (f_*(y))^* \). Hence, we can write

\[
0 = f_*(y^*) \land f_*(y) = f_*(y^* \land y) = f_*(0)
\]

\( \Rightarrow x \leq f_*(f(x) = f_*(0) = 0 \quad \therefore x = 0. \)

\( \square \)

**Definition 2.4.** Let \((F_1, \tau_1)\) and \((F_2, \tau_2)\) be LG-spaces and \( f : F_1 \to F_2 \) be a left adjoint mapping. Then

(i) \( f \) is said to be weakly continuous at \( a \in F_1 \), if whenever \( f(a) \leq t \), then \( a \leq \text{int}(f_*(t)) \) where \( t \in \tau_2 \).

(ii) \( f \) is said to be weakly continuous if it is weakly continuous at each element \( a \in F_1 \).

One can easily see that if \( f : F_1 \to F_2 \) is left adjoint, then \( f \) is weakly continuous at \( a \in F_1 \) if and only if for every open element \( t \geq f(a) \), there is \( s \in \tau_1 \) such that \( a \leq s \leq f_*(t) \) or equivalently, \( a \leq s \) and \( f(s) \leq t \).

**Proposition 2.5.** A left adjoint mapping \( f : F_1 \to F_2 \) is weakly continuous if and only if for any \( t \in \tau_2 \) we have \( f_*(t) \in \tau_1 \).

**Proof.** (⇒). Let \( t \in \tau_2 \). We show that \( f_*(t) \in \tau_1 \). Assuming that

\( A = \{ x \in F_1 : f(x) \leq t \} \), we have \( f_*(t) = \bigvee A \). Let \( x \in A \). Then

\( f(x) \leq t \) and by the hypothesis, there exists \( v_x \in \tau_1 \) such that \( x \leq v_x \leq f_*(t) \). Therefore, \( f_*(t) = \bigvee_{x \in A} v_x \leq f_*(t) \) and so \( f_*(t) = \bigvee_{x \in A} v_x \in \tau_1 \).

(⇐). It is clear. \( \square \)

As an immediate consequence of Proposition 2.5 is the fact that, if \( X \) and \( Y \) are topological spaces, then a function \( f : X \to Y \) is continuous if and only if, for every \( A \subseteq X \) and every open subset \( W \) of \( Y \) containing \( f(A) \), we have \( A \subseteq \text{int}(f^{-1}(W)) \).

Assume that \( f : X \to Y \) is a function, \( A \subseteq X \) and \( f(A) \subseteq B \subseteq Y \). We denote the restriction of \( f \) from \( A \) to \( B \) by \( f_B^A \). If \( A = X \) (\( B = Y \)), then, for convenience, we use \( f^B \) (\( f_A \)) instead of \( f_B^A \) (\( f_B^X \)). Also, if \( X \) and \( Y \) are posets, then we use “\( a \)” instead of “\( \downarrow a \)”.

**Proposition 2.6.** Suppose that a left adjoint mapping \( f : (F_1, \tau_1) \to (F_2, \tau_2) \) is weakly continuous and \( \downarrow a \) is an LG-subspace of \( F_1 \) and \( \downarrow b \) is an LG-subspace of \( F_2 \) that contains the range of \( f \). Then \( g = f_b^a \) is weakly continuous.
Clearly, $g$ is a left adjoint mapping. Assume that $t \land b \in \tau_b$ where $t \in \tau_2$ and $\tau_b$ is the $LG$-topology of $\downarrow b$ as a subspace of $(F_2, \tau_2)$. We show that $g_*(t \land b) \in \tau_a$. First, note that $\{x \land a: x \in F_1, f(x \land a) \leq t\} = \{x \land a: x \in F_1, f(x) \leq t\}$ and $g_*(b) = \bigvee \{x \land a: g(x \land a) \leq b\} = a$. Therefore, we can write

$$g_*(t \land b) = g_*(t) \land g_*(b) = (\bigvee \{x \land a: x \in F_1, f(x \land a) \leq t\}) \land a$$

$$= (\bigvee \{x \land a: x \in F_1, f(x) \leq t\}) \land a$$

$$= (\bigvee \{x \in F_1: f(x) \leq t\}) \land a = f_*(t) \land a \in \tau_a.$$ 

\[ \square \]

Let $(F_1, \tau_1)$ and $(F_2, \tau_2)$ be $LG$-spaces, $f: F_1 \to F_2$ be a function and $\downarrow a$ be an arbitrary $LG$-subspace of $F_1$. Then, as mentioned before, we denote by $f_*$ the restriction of $f$ on $\downarrow a$.

**Proposition 2.7.** Let $(F_1, \tau_1)$ and $(F_2, \tau_2)$ be $LG$-spaces, $f: F_1 \to F_2$ be a left adjoint mapping and $\bigvee_{i \in I} x_i = 1$ where $x_i \in \tau_1$ for every $i \in I$. Then $f$ is weakly continuous if and only if $f_{x_i}$ is weakly continuous for every $i \in I$.

**Proof.** ($\Rightarrow$). It is clear by Proposition 2.6.

($\Leftarrow$). Assume that $t \in \tau_2$. As we see in the proof of Proposition 2.6, $(f_{x_i})_*(t) = f_*(t) \land x_i$ is open in $F_1$ for every $i \in I$. So, we can write

$$f_*(t) = f_*(t) \land (\bigvee_{i \in I} x_i) = \bigvee_{i \in I} (f_*(t) \land x_i) = \bigvee_{i \in I} (f_{x_i})_*(t) \in \tau_1.$$ 

\[ \square \]

It is well-known that two continuous functions $f, g: X \to Y$, where $X$ and $Y$ are topological spaces with $Y$ to be Hausdorff, are identical if and only if $f_D = g_D$ for some dense subset $D$ of $X$. Now, we generalize this fact for $LG$-spaces.

**Proposition 2.8.** Let $(F_1, \tau_1)$ and $(F_2, \tau_2)$ be two $LG$-spaces, $f, g: (F_1, \tau_1) \to (F_2, \tau_2)$ be two weakly continuous and $f_*(0) = 0 = g_*(0)$ such that $f_d = g_d$ for some $\tau_1$-dense element $d$ of $F_1$ (i.e., we have $t \land d \neq 0$ for each $t \in \tau_1 \setminus \{0\}$). Also, suppose that for every $x \in F_1$ we have $f(x) = g(x)$ whenever $f^{-1}(\downarrow (f(x) \land t_1)) \cap g^{-1}(\downarrow (g(x) \land t_2)) \subseteq \{0\}$ for every disjoint elements $t_1, t_2 \in \tau_2$. Then, we have $f = g$. 


Proof. Assume on the contrary that \( f(x) \neq g(x) \) for some \( x \in F_1 \). By our hypothesis, there exist \( 0 \neq a \in F_1 \) and \( t_1, t_2 \in \tau_2 \) such that \( f(a) \leq f(x) \wedge t_1, g(a) \leq g(x) \wedge t_2 \) and \( t_1 \wedge t_2 = 0 \). Clearly, \( a \leq f_\ast(t_1) \wedge g_\ast(t_2) \) which implies that \( x_0 = f_\ast(t_1) \wedge g_\ast(t_2) \wedge d \neq 0 \). Therefore, \( f(x_0) \leq t_1, g(x_0) \leq t_2 \) and since \( t_1 \wedge t_2 = 0 \), it follows that \( f(x_0) \wedge g(x_0) = 0 \).

Since \( x_0 \leq d \), it follows that \( f(x_0) = g(x_0) \) and consequently, \( f(x_0) = g(x_0) = 0 \). On the other, since \( f_\ast(0) = g_\ast(0) \) and \( x_0 \neq 0 \), we deduce, by Proposition 2.3, that \( f(x_0) \neq 0 \neq g(x_0) \), and this is a contradiction. \( \square \)

**Proposition 2.9.** Suppose that \( f : F_1 \to F_2 \) and \( g : F_2 \to F_3 \) are left adjoint mappings. Then the following statements hold.

(a) \( gf \) is a left adjoint mapping and \( (gf)_\ast = f_\ast g_\ast \).

(b) If \( f \) is weakly continuous at \( a \in F_1 \) and \( g \) is weakly continuous at \( f(a) \), then \( gf \) is weakly continuous at \( a \).

(c) If \( f \) and \( g \) are weakly continuous, then \( gf \) is also weakly continuous.

Proof. (a). It is clear.

(b). Assume that \( t \) is an open element in \( F_3 \) such that \( (gf)(a) \leq t \). Then, by the hypothesis, there exists an open element \( s \) in \( F_2 \) such that \( f(a) \leq s \leq g_\ast(t) \). Also, since \( f \) is weakly continuous at \( a \), there exists an open element \( r \) in \( F_1 \) such that \( a \leq r \leq f_\ast(s) \). Therefore, \( a \leq r \leq f_\ast(s) \leq f_\ast g_\ast(t) = (gf)_\ast(t) \). Thus, \( gf \) is weakly continuous at \( a \).

(c). By (b), it is clear. \( \square \)

**Definition 2.10.** Let \( F_1 \) and \( F_2 \) be two frames and \( f : F_1 \to F_2 \) be a left adjoint mapping. We say that \( f \) is perfect provided that for every \( y \in F_2 \), we have \( f_\ast(y) = 0 \) if and only if \( y \wedge f(1) = 0 \). Also, \( f \) is said to be semi-perfect if \( f_\ast(0) = 0 \). In addition, a left adjoint mapping \( f \) is called an \( RL \)-adjoint mapping, if \( f_\ast \) preserves arbitrary suprema; i.e., the right adjoint of \( f \) is a left adjoint mapping. Note that the notion of \( RL \)-adjoint mappings has been first introduced in [9] as \( GOH \).

**Example.** (a) Let \( f : P(X) \to P(Y) \) be a set function induced by \( h : X \to Y \). It is easy to see that \( f \) is a perfect \( RL \)-adjoint mapping. In addition, \( f \) is weakly continuous if and only if \( h \) is continuous.

(b) Suppose that \( 2 = \{0, 1\} \), \( F \) is a frame and \( 1 \neq a \in F \). Define \( f : F \to 2 \) with \( f(x) = 0 \) whenever \( x \leq a \) and \( f(x) = 1 \) whenever \( x \nleq a \). It is easy to see that

(i) \( f \) is left adjoint;

(ii) \( f \) is an \( RL \)-adjoint mapping if and only if \( a = 0 \);
A set function 

are also true for continuous

Assume that

Let

and

Assume that

Let

functions.

Lemma 2.13. A set function \( f : P(X) \to P(Y) \) is induced by a function \( h : X \to Y \) if and only if \( f \) is an RL-adjoint mapping.

Proof. Assume that \( f : P(X) \to P(Y) \) is an RL-adjoint mapping. It suffices to show that, for every \( x \in X \) there exists a point \( y \in Y \) such that \( f(\{x\}) = \{y\} \). To see this, suppose that \( x \in X \) and \( f(\{x\}) = B \) for some \( B \in P(Y) \). By the hypothesis, we can write

\[
\{x\} \subseteq f(\{x\}) = f_*(B) = f_*(\cup_{y \in B} \{y\}) = \cup_{y \in B} f_*(\{y\})
\]

\[\Rightarrow \exists y \in B \text{ } x \in f_*(\{y\}) \Rightarrow \{x\} \subseteq f_*(\{y\}) \Rightarrow f(\{x\}) \subseteq \{y\}.
\]

On the other hand, since \( f \) is an RL-adjoint mapping, it is semi-perfect. Therefore, by Proposition 2.3, \( f(\{x\}) \neq \emptyset \) and so \( f(\{x\}) = \{y\} \).

Definition 2.12. Let \((F_1, \tau_1)\) and \((F_2, \tau_2)\) be LG-spaces and \( f : F_1 \to F_2 \) be a left adjoint mapping. Then \( f \) is said to be continuous if it is weakly continuous and RL-adjoint.

Clearly, propositions 2.5, 2.6, 2.7 and 2.8 are also true for continuous functions.

Let \( f : F_1 \to F_2 \) be a continuous function. It is a natural question whether \( f_*(y) \) is closed in \( F_1 \) for every closed element \( y \in F_2 \). We need the following lemma, which is probably well-known, to answer this question and some others.

Lemma 2.13. Assume that \( F_1 \) and \( F_2 \) are frames, \( h : F_1 \to F_2 \) and \( f : F_1 \times F_1 \to F_2 \times F_2 \) with \( f(a, b) = (h(a), h(b)) \). Then the following statements hold.

(i) \( f \) is a frame homomorphism if and only if \( h \) is so.

(ii) \( f \) is a left adjoint function if and only if \( h \) is so, and in this case \( f_*(c, d) = (h_*(c), h_*(d)) \) for every \((c, d) \in F_2 \times F_2\).

(iii) If \( F_1 \) is a finite chain, then \( f \) is a frame homomorphism if and only if \( h \) is a \( \{0, 1\} \)-order homomorphism (i.e., \( h \) is an order-preserving mapping such that \( h(0) = 0 \) and \( h(1) = 1 \)).

The next example shows that an RL-adjoint mappings need not be perfect, in general. Also, it shows that the continuous preimage of a closed element is not necessarily closed.

Example 2.14. (a) Let \( F_1 = \{0, a, 1\} \) be a chain, \( h : F_1 \to F_1 \) with \( h(0) = 0 \) and \( h(a) = h(1) = 1 \). Define \( F = F_1 \times F_1 \) and \( f : F \to F \) with
f(a, b) = (h(a), h(b)). By Lemma 2.13, f is an RL-adjoint mapping. Again from Lemma 2.13, it follows that \( f_*(a, a) = (h_*(a), h_*(a)) \) = (0, 0) = \( 0_F \). Therefore, f is not perfect.

(b) Using the example of part (a) and Lemma 2.13, if we put \( \tau = F \), then it follows that \( g = f_* \) is a continuous function (since \( f_*(a, b) = (h_*(a), h_*(b)) \)). Also, clearly, \( g_*(0_F) = (h_*(0), h_*(0)) = (a, a) \) and \( (a, a) \) is not a closed element in F.

Let \( F_i \) be a frame for every \( i \in I \) and \( F = \prod_{i \in I} F_i \). Suppose that \( j \in I \) and \( x \in F \) are such that \( x_i = 1 \) (resp., \( x_i = 0 \)) for every \( i \neq j \), then for convenience, some times, we denote \( x \) by \( \tilde{x}_j \) (resp., \( \hat{x}_j \)). Clearly, if \( \pi_j : F \to F_j \) is the projection mapping, then we have \( (\pi_j)_*(x_j) = \tilde{x}_j \) for every \( x_j \in F_j \).

**Proposition 2.15.** Suppose that \((F_i, \tau_i)\) is an LG-space for every \( i \in I \) and \( \prod_{i \in I} F_i \), equipped with the product topology (resp., the box topology). For every \( j \in I \), consider the projection mapping \( \pi_j : \prod_{i \in I} F_i \to F_j \). Then the following statements hold.

(a) \( \pi_j \) is an open mapping (i.e., \( \pi_j(t) \in \tau_j \) for every open element \( t \) in \( \prod_{i \in I} F_i \)).

(b) \( \pi_j \) is a closed mapping (i.e., \( \pi_j(t^*) \in \tau_j^* \) for every closed element \( t^* \) in \( \prod_{i \in I} F_i \)).

(c) \( \pi_j \) is weakly continuous.

**Proof.** The proof is routine. \( \square \)

Note that, in Proposition 2.15, \( \pi_j \) distributes over any arbitrary join of nonempty family. To see this, suppose that \( x_\alpha \in F_j \) for every \( \alpha \in A \). Then we can write
\[
(\pi_j)_*(\bigvee_{\alpha \in A} x_\alpha) = \bigvee_{\alpha \in A} \tilde{x}_\alpha = \bigvee_{\alpha \in A} \hat{x}_\alpha = \bigvee_{\alpha \in A} (\pi_j)_*(x_\alpha).
\]

However, if \( |I| \geq 2 \), then \( (\pi_j)_*(0) = \hat{0} \neq 0 \) and so it does not preserve arbitrary suprema. Therefore, if \( |I| \geq 2 \), then \( \pi_j \) is not an RL-adjoint mapping.

**Proposition 2.16.** Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are LG-spaces and \( f : F_1 \to F_2 \) is a left adjoint mapping. Also, suppose that \( B \) and \( S \) are base and subbase for LG-space \( F_2 \), respectively. Then the following statements hold.

(a) If \( f_*(S) = \tau_2 \), then \( f \) is weakly continuous if and only if \( f_*(s) \in \tau_1 \) for every \( s \in S \).

(b) If \( f \) is an RL-adjoint mapping, then the following statements are equivalent.

(i) \( f \) is continuous.

(ii) \( f_*(b) \) is open in \( F_1 \) for every \( b \in B \).
(iii) $f_*(s)$ is open in $F_1$ for every $s \in S$.

Proof. (a) Since $f_*$ preserves finite meets, it is clear.

(b) Since $f_*$ preserves arbitrary suprema and infima, it is clear. □

**Proposition 2.17.** Suppose that $(F_i, \tau_i)$ for every $i \in I$, and $(F, \tau)$ are LG-spaces, $\prod_{i \in I} F_i$ equipped by product topology and $f : (F, \tau) \to \prod_{i \in I} F_i$. Then $f$ is weakly continuous if and only if $\pi_j f$ is so for every $j \in I$.

Proof. $(\Rightarrow)$ By Propositions 2.9 and 2.15, it is clear.

$(\Leftarrow)$ First, we show that $f$ is a left adjoint mapping. To see this, we prove that $f$ preserves arbitrary suprema. Since $\pi_j$ and $\pi_j f$ are left adjoint mappings, we can write

$$f(\bigvee_{\lambda \in A} x_\lambda) = (\pi_j f(\bigvee_{\lambda \in A} x_\lambda))_{j \in I} = (\bigvee_{\lambda \in A} \pi_j f(x_\lambda))_{j \in I}$$

$$= (\pi_j (\bigvee_{\lambda \in A} f(x_\lambda))_{j \in I} = \bigvee_{\lambda \in A} f(x_\lambda).$$

Now, by Proposition 2.16, it is enough to show that $f_*((\pi_j)_*(t_j))$ is open in $F$, where $j \in I$ and $t_j \in \tau_j$. This is easy, since $\pi_j f$ is weakly continuous and $f_*((\pi_j)_*(t_j)) = (\pi_j f)_*(t_j)$. □

Note that, by the proof of Proposition 2.17, $f$ is a left adjoint mapping if and only if $\pi_j f$ is so for every $j \in I$.

**Corollary 2.18.** Let $L$ and $F_i$ ($i \in I$) be LG-spaces, $F = \prod_{i \in I} F_i$ and $f_i$ be a mapping from $L$ to $F_i$ for each $i \in I$. Define $f : L \to F$ with $f(x) = (f_i(x))_{i \in I}$. Then the following statements hold.

(a) $f$ is a left adjoint mapping if and only if $f_i$ is such for every $i \in I$. Also, $f$ is weakly continuous if and only if $f_i$ is such for every $i \in I$.

(b) If $f_*$ exists, then for every $y = (y_i)_{i \in I} \in F$ we have $f_*(y) = \bigwedge_{i \in I} f_i_*(y_i)$.

(c) If there exists $j \in I$ such that $f_j$ is semi-perfect, then $f$ is also semi-perfect.

Proof. (a). Since $\pi_i f = f_i$ for every $i \in I$, it is clear, by Proposition 2.17.

(b). Define $g : F \to L$ with $g(y) = \bigwedge_{i \in I} f_i_*(y_i)$. For every $x \in L$ and every $y = (y_i)_{i \in I} \in F$, we can write

$$gf(x) = \bigwedge_{i \in I} f_i_*(f_i(x)) \geq \bigwedge_{i \in I} x = x,$$

$$fg(y) = f(\bigwedge_{i \in I} f_i_*(y_i)) = (f_j(\bigwedge_{i \in I} f_i_*(y_i)))_{j \in I}$$

$$\leq \left( f_j f_j_*(y_j) \right)_{j \in I} \leq (y_j)_{j \in I} = y.$$
Therefore, \( g = f_* \).

(c). By part (b), it is clear. \( \square \)

Applying Corollary 2.18, we can find some useful examples as follows.
(i) Suppose that \( L = M_2 = \{0, \alpha, \beta, 1\} \) and \( F_1 = F_2 = \{0, 1\} \). Define \( f_1 : L \rightarrow F_1 \) with \( f_1(0) = f_1(\alpha) = 0, f_1(\beta) = f_1(1) = 1 \) and \( f_2 : L \rightarrow F_2 \) with \( f_2(0) = f_2(\beta) = 0, f_2(\alpha) = f_2(1) = 1 \). If we put \( f = (f_1, f_2) \), then, clearly, \( f_1 \) and \( f_2 \) are not even semi-perfect whereas \( f \) is perfect. To see this, note that \( f_1(0) = \alpha, f_2(0) = \beta, f_1(1) = f_2(1) = 1 \). Thus, by Corollary 2.18, \( 0 = f_*(c, d) = f_1(c) \wedge f_2(d) \) if and only if \( f_*(c) = \alpha \) and \( f_*(d) = \beta \), if and only if \( (c, d) = (0, 0) = 0_{F \times F} \).

(ii) Define \( g : [0, 1] \rightarrow [0, 1] \) with \( g(x) = x^{1/n} \). Clearly, \( g \) is an order-isomorphism and so \( g \) is a perfect \( RL \)-adjoint mapping. Now, we show that \( f = (g, g) \) is not \( RL \)-adjoint. Take \( t \in [0, 1] \), then by Corollary 2.18, we have \( f_1((1, t)) \vee f_2((t, 1)) = (g_1(1) \wedge g_1(t)) \vee (g_2(t) \wedge g_2(1)) = t^n \vee t^m \neq 1 = f_*(((1, 1)) = f_*(((1, t) \vee (t, 1))) \). Therefore, \( f \) is not an \( LR \)-adjoint mapping.

**Definition 2.19.** Let \( (F_1, \tau_1) \) and \( (F_2, \tau_2) \) be \( LG \)-spaces and \( f : F_1 \rightarrow F_2 \) be a left adjoint mapping. Then \( f \) is said to be a homeomorphism if it is one-to-one, onto, continuous and, in addition, \( f_* \) is continuous.

The next proposition gives some equivalent conditions for a mapping between \( LG \)-spaces to be a homeomorphism which has a straightforward proof.

**Proposition 2.20.** Suppose that \( (F_1, \tau_1) \) and \( (F_2, \tau_2) \) are \( LG \)-spaces, \( f : F_1 \rightarrow F_2 \) is a one-to-one and onto left adjoint mapping. Then the following statements are equivalent.

(a) \( f \) is homeomorphism.

(b) \( t \in \tau_2 \) if and only if \( f_*(t) \in \tau_1 \).

(c) \( f \) is continuous and open (i.e., \( f(t) \in \tau_2 \) for every \( t \in \tau_1 \)).

(d) \( f \) and \( f_* \) are weakly continuous.

**Proposition 2.21.** Let \( f : (F_1, \tau_1) \rightarrow (F_2, \tau_2) \) be a left adjoint mapping. Then \( f_*(k) \) is closed in \( F_1 \) for each closed element \( k \) in \( F_2 \) if and only if \( f(cl_{F_1}a) \leq cl_{F_2}f(a) \) for each \( a \in F_1 \).

**Proof.** \( \Rightarrow \) For \( a \in F_1 \), we can write

\[
\begin{align*}
 f(a) \leq cl_{F_2}f(a) \Rightarrow a \leq f_*(f(a)) \leq f_*(cl_{F_2}f(a)) \in \tau_1^* \\
 \Rightarrow cl_{F_1}a \leq f_*(cl_{F_2}f(a)) \Rightarrow f(cl_{F_1}a) \leq cl_{F_2}f(a).
\end{align*}
\]

\( \Leftarrow \) Suppose that \( k \in \tau_2^* \) and \( a = f_*(k) \), then we can write

\[
 f(cl_{F_1}a) \leq cl_{F_2}f(a) \leq k \Rightarrow cl_{F_1}a \leq f_*(k) = a
\]
We need the following lemma to show that a continuous image of a compact element is compact.

**Lemma 2.22.** Let $F_1$ and $F_2$ be frames and $f : F_1 \to F_2$ be a left adjoint mapping. Then $f$ is an RL-adjoint mapping if and only if for every $a \in F_1$ and every $R \subseteq F_2$, we have

$$f(a) \leq \bigvee_{r \in R} f_* r \iff a \leq \bigvee_{r \in R} f_* r.$$  

**Proof.** $\Rightarrow$) Assume that $f(a) \leq \bigvee_{r \in R} f_* r$ where $R \subseteq F_2$. By the hypothesis, $f_*$ preserves arbitrary suprema and so we can write

$$a \leq f_* f(a) \leq f_* \bigvee_{r \in R} f_* r = \bigvee_{r \in R} f_* r.$$  

For the reverse inequality, suppose that $a \leq \bigvee_{r \in R} f_* r$. Since $f$ is left adjoint, by previous facts, $f$ preserves arbitrary suprema, and so we can write

$$a \leq \bigvee_{r \in R} f_* r \Rightarrow f(a) \leq f_* \bigvee_{r \in R} f_* r = \bigvee_{r \in R} f_* f_* r.$$  

$\Leftarrow$) Assume that $y_i \in F_2$ for every $i \in I$. Clearly, $\bigvee_{i \in I} f_* y_i \leq f_* \bigvee_{i \in I} y_i$. On the other hand, $f f_* \bigvee_{i \in I} y_i \leq \bigvee_{s \in F} f_* s$, and thus, it follows from (b) that $f_* \bigvee_{i \in I} y_i \leq \bigvee_{i \in I} f_* y_i$.  

**Proposition 2.23.** Let $(F_1, \tau_1)$ and $(F_2, \tau_2)$ be LG-spaces, $f : F_1 \to F_2$ be continuous and $a$ be a compact element in $F_1$. Then $f(a)$ is a compact element in $F_2$.

**Proof.** Assume that $f(a) \leq \bigvee S$ where $S \subseteq \tau_2$. By Lemma 2.22, we have $a \leq \bigvee_{s \in F} f_* s$. Since $a$ is compact, it follows that there exists a finite subset $F \subseteq S$ such that $a \leq \bigvee_{s \in F} f_* s$, and again it follows from Lemma 2.22 that $f(a) \leq \bigvee_{s \in F} f_* s$.  

Clearly, Proposition 2.23 also holds whenever $a$ is countably compact, Lindelöf or other kinds of compactness.

### 3. Continuity and Connectedness

In this section, we are going to find the relations between continuity and connectedness. To this aim, we need some definitions and facts.

**Definition 3.1.** Let $(F, \tau)$ be an LG-space. We denote by $\tau^c$ the set 

$$\{ x \in F : \exists t \in \tau, x = t^c \},$$  

where $t^c$ is the complement of $t$ in $F$ (if it exists). An element $a \in F$ is called nonconnected if there exist $0 \neq r_a, s_a \in \tau_a$ such that $r_a \land s_a = 0$ and $r_a \lor s_a = a$, otherwise, we say...
that \(a\) is connected. If \(1\) is a nonconnected (resp., connected) element, then, some times, we say that \(F\) is nonconnected (resp., connected).

It follows from Definition 3.1 that if \(F\) is a frame and \(a \leq b \in F\), then \(a\) is connected as a point of \(F\) if and only if it is connected as a point of \(b\).

**Remark 3.2.** Let \((F, \tau)\) be an LG-space. Inspired by Definition 3.1, we can give the following two definitions.

(a) \(a \in F\) is said to be relatively connected in \(F\) if whenever \(t_1, t_2 \in \tau\), \(t_1 \land t_2 = 0\) and \(a \leq t_1 \lor t_2\), then \(a \leq t_1\) or \(a \leq t_2\).

(b) \(a \in F\) is said to be weakly relatively connected in \(F\) if whenever \(t\) is complemented in \(
\), then \(a \leq t\) or \(a \leq t^c\).

It is easy to see that if \(a \in F\) is connected, then \(a\) is relatively connected in \(F\), and also if \(a \in F\) is relatively connected in \(F\), then \(a\) is weakly relatively connected in \(F\). The converses of these two facts are not necessarily true. For example:

(i) Suppose that \(X\) is an infinite set with cofinite topology and \(a, b \in X\) are two distinct points. Then \(A = \{a, b\}\) is not connected whereas it is relatively connected in \(P(X)\).

(ii) Suppose that \((X, \tau)\) is a connected space and \(U, V \in \tau\) are two nonempty disjoint open sets. Then \(A = U \cup V\) is not relatively connected in \(P(X)\) whereas \(A\) is weakly relatively connected in \(P(X)\).

In the sequel, consider the lattice \(M_2 = \{0, \alpha, \beta, 1\}\). As we will see in the following proposition, the lattice \(M_2\) has an important role in the connectedness of LG-spaces, as well as the role of discrete space \(\{0, 1\}\) in the realm of the connectedness of topological spaces.

**Proposition 3.3.** Let \((F, \tau)\) be an LG-space. The following statements are equivalent.

(a) \(F\) is connected.

(b) For every continuous \(f : (F, \tau) \to (M_2, M_2)\) we have \(\{\alpha, \beta\} \not\subseteq f(F)\).

(c) \(\tau \cap \tau^c = \{0, 1\}\).

**Proof.** (a) \(\Rightarrow\) (b). Suppose that \(f : (F, \tau) \to (M_2, M_2)\) is continuous and \(\alpha \in f(F)\). We show that \(\beta \not\in f(F)\). Taking \(r = f_*(\alpha)\) and \(s = f_*(\beta)\), it follows that

\[
\begin{align*}
    r, s \in \tau, & \quad r \land s = f_*(\alpha) \land f_*(\beta) = f_*(\alpha \land \beta) = f_*(0) = 0, \\
    r \lor s = f_*(\alpha) \lor f_*(\beta) = f_*(\alpha \lor \beta) = f_*(1) = 1.
\end{align*}
\]

Since \(\alpha \in f(F)\), it follows that \(r \neq 0\). By connectedness of \(F\), we conclude that \(0 = s = f_*(\beta) = \bigvee \{x \in F : f(x) \leq \beta\}\) and consequently, \(\beta \not\in f(F)\).
(b) ⇒ (c). On the contrary, suppose that there exists \( r \in \tau \cap \tau^c \setminus \{0,1\} \). Now, define \( f : F \to M_2 \) with \( f(0) = 0, f(x) = \alpha \) whenever \( 0 \neq x \leq r \), \( f(x) = \beta \) whenever \( 0 \neq x \leq \tau^c \) and \( f(x) = 1 \) whenever \( x \land \tau^c \neq 0 \) and \( x \land \tau^c \neq 0 \). To see that \( f \) is a left adjoint mapping, it is enough to show that \( f \) preserves arbitrary suprema. Suppose \( x_i \in F \) for every \( i \in I \). We will have four cases:

1. \( \bigvee_{i \in I} x_i = 0 \);
2. \( 0 \neq \bigvee_{i \in I} x_i \leq r \);
3. \( 0 \neq \bigvee_{i \in I} x_i \leq \tau^c \);
4. \( \bigvee_{i \in I} x_i \land r = 0 \) and \( \bigvee_{i \in I} x_i \land \tau^c \neq 0 \).

It is easy to see that in any case, we have \( f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i) \). It remains to prove that \( f \) is continuous. This is easy, by the fact that \( f_* (\alpha) = r \) and \( f_* (\beta) = \tau^c \).
(c) ⇒ (a). It is clear. \( \square \)

We know that if \( X \) and \( Y \) are topological spaces, \( X \) is connected and \( h : X \to Y \) is a weakly continuous function (in point set topology weakly continuous and continuous are equivalent), then \( h(X) \) is also connected. This fact is not necessarily true in \( LG \)-spaces. For example, suppose that \( (F_1, \tau_1) \) is an \( LG \)-space in which 1 is a connected element and \( (F_2, \tau_2) \) is an \( LG \)-space in which 1 is a disconnected element. Define \( f : (F_1, \tau_1) \to (F_2, \tau_2) \) with \( f(0) = 0 \) and \( f(x) = 1 \) for every \( x \neq 0 \). Clearly, \( f \) is a weakly continuous function and semi-perfect while 1 is connected and \( f(1) \) is disconnected.

**Definition 3.4.** Let \( X \) be a partially ordered set and \( D, E \subseteq X \). We say \( D \) cuts \( E \) if for every \( 0 \neq e \in E \) there exists \( 0 \neq d \in D \) such that \( d \leq e \).

**Proposition 3.5.** Suppose that \( (F_1, \tau_1) \) and \( (F_2, \tau_2) \) are \( LG \)-spaces and \( f : (F_1, \tau_1) \to (F_2, \tau_2) \) is a continuous function. If 1 is connected in \( F_1 \) and \( f(F_1) \) cuts \( \downarrow f(1) \), then \( f(1) \) is connected.

**Proof.** Let \( g : \downarrow f(1) \to M_2 \) be continuous. Clearly, \( gf : F_1 \to M_2 \) is continuous. Thus, \( \{\alpha, \beta\} \not\subseteq (gf)(F_1) \). Without loss of generality, suppose that \( \beta \notin (gf)(F_1) \). It is enough to show that \( \beta \notin g(\downarrow f(1)) \). On the contrary, assume that \( \beta = g(y) \) for some \( y \in \downarrow f(1) \). By the hypothesis, there exists \( x \in F_1 \) such that \( 0 \neq f(x) \leq y \). Thus, \( 0 \neq g(f(x)) \leq g(y) = \beta \). Therefore, \( \beta = gf(x) \), consequently, \( \beta \in (gf)(F_1) \) and this is a contradiction. \( \square \)

**Lemma 3.6.** Let \( F_1 \) and \( F_2 \) be two frames and \( f : F_1 \to F_2 \) be a left adjoint function. Then \( f \) is semi-perfect and \( f(F_1) \) cuts \( \downarrow f(1) \) if and only if \( f \) is perfect.
Lemma 3.5. Let \((a)\) be a connected element and \(f\) be a perfect continuous function. Then the following statements are equivalent.

\[ f^* \subseteq f(F) \]

\(\Rightarrow\) It suffices to show that \(f(1) = f^*\). Assume that \(0 \neq y \leq f(1)\). By the hypothesis, \(0 \neq f(y)\) and so \(0 \neq f_*(y)\). Hence, \(f(1) = f^*\), we are done.

\(\Leftarrow\) It suffices to show that \(f(F)\) cuts \(f(1)\). Assume that \(0 \neq y \leq f(1)\). By the hypothesis, \(0 \neq f(y)\) and so \(0 \neq f_*(y)\). Now, taking \(x = f(y)\), we are done.

By Proposition 3.5 and Lemma 3.6, we have the following result.

**Proposition 3.7.** Let \((F_1, \tau_1)\) and \((F_2, \tau_2)\) be two LG-spaces, \(a \in F_1\) be a connected element and \(f : F_1 \to F_2\) be a function such that \(f_a\) is a perfect continuous function. Then \(f(a)\) is also a connected element.

Here is an open question. Assuming that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two LG-spaces, \(a \in F_1\) is a connected element and \(f : F_1 \to F_2\) is a perfect continuous function, can we conclude that \(f(a)\) is a connected element?

**Proposition 3.8.** Let \(a\) be a connected element of \(F_1\) and \(a \leq b \leq \bar{a}\). Then \(b\) is also a connected element.

**Proof.** Suppose that \(r_b = r \wedge b\) and \(s_b = s \wedge b\) are two disjoint elements of \(\tau_b\) such that \(b = r_b \vee s_b\). Clearly, if we take \(r_a = r \wedge a\) and \(s_a = s \wedge a\), then \(r_a\) and \(s_a\) are disjoint elements of \(\tau_a\) and \(r_a \vee s_a = a\). Thus, \(r_a = 0\) or \(s_a = 0\), say \(r_a = 0\). Hence, \(a \leq r^*\) and so \(b \leq \bar{a} \leq r^*\). Therefore, \(r_b = r \wedge b = 0\).

**Lemma 3.9.** Let \(F\) be an LG-space and \(f : F \to M_2\) be a continuous function. Then the following statements are equivalent.

(a) \(\{\alpha, \beta\} \subseteq f(F)\).

(b) \(f\) is onto.

(c) \(1 \in f(F)\).

**Proof.** (a) \(\Rightarrow\) (b). It suffices to show that \(1 \in f(F)\). Assume that \(f(a) = \alpha\) and \(f(b) = \beta\). Then, clearly, \(f(a \vee b) = f(a) \vee f(b) = \alpha \vee \beta = 1\).

(b) \(\Rightarrow\) (c). It is clear.

(c) \(\Rightarrow\) (a). Clearly, \(ff_*(\alpha) \leq \alpha\) and \(ff_*(\beta) \leq \beta\). Therefore, since \(f\) is continuous, it follows that

\[ 1 = f(1_F) = ff_*(1) = ff_*(\alpha \vee \beta) = f f_*(\alpha) \vee f f_*(\beta) \]

Hence, we conclude that \(ff_*(\alpha) = \alpha\) and \(ff_*(\beta) = \beta\) and so \(\{\alpha, \beta\} \subseteq f(F)\).
Proposition 3.10. Let $F$ be a frame and $x_i \in F$ be connected for every $i \in I$ and $0 \neq a = \bigwedge_{i \in I} x_i$. Then $x = \bigvee_{i \in I} x_i$ is also a connected element of $F$.

Proof. Suppose that $f : \downarrow x \rightarrow M_2$ be a continuous mapping. Clearly, $f_{x_i}$ is a continuous function from $\downarrow x_i$ to $M_2$ for every $i \in I$. Therefore, by Proposition 3.3 and Lemma 3.9, for every $i \in I$ we have $f(\downarrow x_i) = \{0, \alpha\}$ or $f(\downarrow x_i) = \{0, \beta\}$. On the other hand, $f$ is semi-perfect and so by Proposition 2.3, $f(a) \neq 0$. Since $a \in \bigcap_{i \in I} \downarrow x_i$, it follows that $f(a) = \alpha$ or $f(a) = \beta$. say $f(a) = \alpha$. Then, obviously, $f(\downarrow x_i) = \{0, \alpha\}$ for every $i \in I$. Now, suppose that $c \in \downarrow x$, then we can write

$$c \leq x = \bigvee_{i \in I} x_i \Rightarrow c = \bigvee_{i \in I} (c \land x_i)$$

$$\Rightarrow f(c) = f(\bigvee_{i \in I} (c \land x_i)) = \bigvee_{i \in I} f(c \land x_i) \in \{0, \alpha\}.$$ 

Therefore, $f(\downarrow x) \subseteq \{0, \alpha\}$. Hence, $x$ is connected. \qed

By Propositions 3.8 and 3.10, the following corollary is immediate.

Corollary 3.11. Let $F$ be a frame and $x \in F$. If we take $C(x) = \{c \in F : c$ is connected and $x \leq c\}$ and $c_x = \bigvee C(x)$, then $c_x$ is closed.

Proposition 3.12. Let $F$ be an LG-space and $R$ be a relation on $F \setminus \{0\}$ such as follows. $a R b$ whenever $a = b$ or $c_a = c_b \neq 0$. Then we have the following statements.

(a) $R$ is an equivalence relation on $F \setminus \{0\}$.

(b) $c_x \neq 0$ if and only if $C(x)$ contains at least one non zero connected element.

(c) If we denote by $[x]$ the equivalence class of $x$ with respect to the relation $R$, then $C(x) \subseteq [x]$ for every $x \in F \setminus \{0\}$.

(d) $c_x = 0$ or $\bigvee [x] = c_x \in C(x)$ for every $x \in F \setminus \{0\}$.

(e) $c_a \land c_b \neq 0$ if and only if $[a] = [b]$ and each of classes $[a]$ and $[b]$ contains at least one non zero connected element.

(f) Let $X$ be the set of all connected elements of $F \setminus \{0\}$. Then the mapping $x \rightarrow c_x$ is a closure operator on $X$.

Proof. (a). It is clear.

(b). It is obvious, by the definition of $c_a$.

(c). Assume that $x \in F \setminus \{0\}$ and $a \in C(x)$. Without loss of generality, suppose that $C(x) \neq \emptyset$. We show $c_x = c_a$. Clearly, $C(a) \subseteq C(x)$, so it is enough to prove that $C(a)$ is cofinal with respect to $C(x)$. To see this, let $y \in C(x)$, then by Proposition 3.10, $x \leq y \leq a \lor y \in C(a)$ and we are done.
(d). Assume that $x \in F \setminus \{0\}$ and $c_x \neq 0$. Clearly, by part (c), $C(x) \subseteq [x]$, so it is enough to prove that $C(x)$ is cofinal with respect to $[x]$. To see this, let $y \in [x]$, then $y \leq c_y = c_x \in C(x)$ and we are done.

(e $\Rightarrow$). By the hypothesis, $c_a \neq 0$ and $c_b \neq 0$. Therefore, $\emptyset \neq C(a) \subseteq [a]$ and $\emptyset \neq C(b) \subseteq [b]$. Now, it suffices to prove that $c_a = c_b$. Since $c_a$ and $c_b$ are connected, by Proposition 3.10, $c_a \lor c_b$ is a connected element greater than or equal to $a$ and $b$ and so $c_a = c_a \lor c_b = c_b$. Thus, $[a] = [b]$.

(e $\Leftarrow$). Since $[a]$ and $[b]$ contain at least one non zero connected element, clearly, $c_a \neq 0 \neq c_b$ and so by part (d), $c_a = \bigvee[a] = \bigvee[b] = c_b$. Hence, $c_a \lor c_b = c_a \neq 0$.

(f). Since $x \in X$ is connected, clearly the mapping $x \to c_x$ is well defined on $X$. By the previous parts, the remainder of proof is clear (in fact, if $a, b \in X$ and $a \leq b$, then $[a] = [b]$).

We conclude the paper by generalizing Proposition 3.10.

**Proposition 3.13.** Let $F$ be an LG-space, $\Gamma$ be an ordinal and $\{a_{\lambda}\}_{\lambda < \Gamma}$ be a family of connected elements of $F$ such that for every $0 < \lambda_1 < \Gamma$ there exists $\lambda_0 < \lambda_1$ with $a_{\lambda_0} \land a_{\lambda_1} \neq 0$. Then $x = \bigvee_{\lambda < \Gamma} a_\lambda$ is connected.

**Proof.** Suppose that $f : \downarrow x \to M_2$ is a continuous function. Clearly, for every $\lambda < \Gamma$, $f_{a_\lambda}$ is a continuous function and so, by Proposition 3.3, is not onto. Without loss of generality, assume that $f_0(\downarrow a_0) \subseteq \{0, \alpha\}$. We show, by transfinite induction, that $f_{a_\lambda}(\downarrow a_\lambda) \subseteq \{0, \alpha\}$ for every $\lambda < \Gamma$. Clearly, this is true for $\lambda = 0$. Now, suppose that $\gamma < \Gamma$ and this claim is true for every $\lambda < \gamma$, then we show that this is true for $\lambda = \gamma$. By the hypothesis, there exists $\lambda_0 < \gamma$ such that $x' = a_{\lambda_0} \land a_\gamma \neq 0$. Since $f_{\lambda_0}(\downarrow a_{\lambda_0}) \subseteq \{0, \alpha\}$ and $f$ is semi-perfect, it follows that $f(x') \neq 0$ and so $f(x') = \alpha$. Now, because of $x' \in \downarrow x$, and the fact that $f_\gamma$ is not onto, it turns out that $f_\gamma(\downarrow a_\gamma) \subseteq \{0, \alpha\}$. Now, suppose that $c \in \downarrow x$, then $f(c) \leq f(x) = f(\bigvee_{\lambda < \Gamma} a_\lambda) = \bigvee_{\lambda < \Gamma} f(a_\lambda) \subseteq \{0, \alpha\}$. Therefore, $f(\downarrow x) \subseteq \{0, \alpha\}$. \hfill $\square$

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**References**


**Ali Rezaei Aliabad**  
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.  
Email: aliabady_r@scu.ac.ir

**Hossein Zarepour**  
Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran.  
Email: ho.zarepour@gmail.com
CONTINUOUS FUNCTIONS ON $LG$-SPACES

A. R. ALIABAD AND H. ZAREPOUR

توابع پیوسته روی $LG$-فضا

علی رضایی، دانشگاه شهید حسینیه زرآبادی، ایران

و حسین زارعی

2-فضا در نظر گرفته می‌شود. در این صورت ($F, \tau$) یک فضای تورپولوژی $LG$-فضا می‌باشد. در این مورد مطالعه فضای $LG$-فضا مورد بررسی قرار می‌گیرد. شرایطی دست‌یافتنی مطرح خواهند شد که تعریف یک عضو فشرده در یک $LG$-فضا تحت یک تابع پیوسته، فشرده است. به علاوه، همبندی در $LG$-فضا تعیین شده و مجدداً شرایطی دست‌یافتنی معرفی می‌شود که توصیف یک عضو همبند در یک $LG$-فضا تحت یک تابع پیوسته، همبند است. در واقع با نتایج بدست آمده، به خوبی نشان داده می‌شود که همبندگی یک $LG$-فضا تعیین مناسب برای تورپولوژی این $LG$-فضاها نیز تعیین مناسب برای فضاهای تورپولوژیاند.

کلمات کلیدی: $\tau$، $LG$-فضا، عنصر فشرده، عنصر همبند، تغییر پیوسته.