ON REGULAR PRIME INJECTIVITY OF S-POSETS

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Abstract. In this paper, we define the notion of regular prime monomorphism for S-posets over a pomonoid S and investigate some categorical properties including products, coproducts and pullbacks. We study M-injectivity in the category of S-posets where M is the class of regular prime monomorphisms and show that the Skornjakov criterion fails for the regular prime injectivity. Considering a weaker form of such kind of injectivity, we obtain some classifications for pomonoids.

1. Introduction and preliminaries

Recall that a monoid (group) S is said to be a pomonoid (pogroup) if it is also a poset whose partial order ≤ is compatible with its binary operation, it means that s ≤ t, s' ≤ t' for each s, t, s', t' ∈ S imply ss' ≤ tt'. A non-empty subset I of a pomonoid S is said to be a right ideal if IS ⊆ I. A right ideal I of a pomonoid S is called a right poideal whenever s ≤ s' and s' ∈ I, s ∈ S imply s ∈ I. For a subset X of a pomonoid S, the right poideal of S generated by X, denoted as ↓(XS), is the set \( \{ t ∈ S \mid t ≤ xs \text{ for some } x ∈ X, s ∈ S \} \). If X is finite, then it is called a finitely generated right poideal, and if X = \{x\}, then it is called a principal right poideal of S which is denoted by ↓(xS). For a pomonoid S, a (right) S-poset is a poset A together with a mapping A × S → A, (a, s) ↦ as for a ∈ A, s ∈ S, called an action, satisfying the following conditions:

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(i) \((as)t = a(st)\) for each \(a \in A, s, t \in S\).
(ii) \(a1 = a\) for each \(a \in A\).
(iii) \(a \leq b, s \leq t\) imply \(as \leq bt\) for each \(a, b \in A, s, t \in S\).

A non-empty subset \(B\) of an \(S\)-poset \(A\) is called a sub \(S\)-poset of \(A\), whenever \(B\) is closed under the action with the same order as \(A\). An element \(\theta\) in an \(S\)-poset \(A\) with \(\theta s = \theta\) for all \(s \in S\) is called a zero element. An \(S\)-poset map (or homomorphism) is an action-preserving as well as order-preserving map between \(S\)-posets. Also a regular monomorphism (a morphism which is an equalizer) is exactly an order-embedding, that is, a homomorphism \(f : A \rightarrow B\) for which \(f(a) \leq f(a')\) if and only if \(a \leq a'\), for all \(a, a' \in A\). We denote the category of all (right) \(S\)-posets and homomorphisms between them by \(\text{Pos-}\mathcal{S}\). Recall that the product of a family of \(S\)-posets is their cartesian product, with componentwise action and order. Also the coproduct is their disjoint union, with natural action and componentwise order. As usual, we use the symbols \(\prod\) and \(\bigsqcup\) for product and coproduct, respectively. For more information on acts and \(S\)-posets, one may consult [8, 7]. Throughout, \(S\) stands for a pomonoid unless otherwise stated.

Recall that a right ideal \(I\) of a monoid \(S\) is said to be prime if for \(s, s' \in S\), the inclusion \(ss' \subseteq I\) implies that either \(s \in I\) or \(s' \in I\) (see [5]). Prime ideals are useful tools in the theory of semigroups. This notion was extended to an arbitrary \(S\)-act by Ahsan [1], analogous to the notion of prime module introduced by Dauns [6]. We say that a sub \(S\)-poset \(B\) of an \(S\)-poset \(A\) is a prime \(S\)-poset of \(A\), or \(A\) is a regular prime extension of \(B\), if \(B\) is a prime subact of \(A\) whenever \(A\) is considered as an act over \(S\) as a monoid, that is, for each \(a \in A\) and \(s \in S\), the inclusion \(aS \subseteq B\) implies either \(a \in B\) or \(As \subseteq B\). So, a right ideal \(I\) of a pomonoid \(S\) is a prime ideal if and only if it is prime as a sub \(S\)-poset of \(S\). An \(S\)-poset homomorphism \(f : A \rightarrow B\) is prime if \(f(A)\) is a prime sub \(S\)-poset of \(B\). Clearly, any surjective \(S\)-poset homomorphism is prime. By a regular prime \(S\)-poset monomorphism we mean a regular \(S\)-poset monomorphism which is prime. We investigate the products, coproducts, direct sums and the pullback stability property of regular prime \(S\)-poset monomorphisms.

Banaschewski [3] indicated the notion of \(\mathcal{M}\)-injectivity in a category \(\mathcal{A}\), when \(\mathcal{M}\) is a subclass of monomorphisms the members of which may be called \(\mathcal{M}\)-morphisms as the following definition. An object \(A\) is said to be \(\mathcal{M}\)-injective if for each \(\mathcal{M}\)-morphism \(g : B \rightarrow C\), any morphism \(f : B \rightarrow A\) can be lifted to a morphism \(\tilde{f} : C \rightarrow A\), that is,
\[ \bar{f}g = f : \]

\[ \begin{array}{ccc}
B & \xrightarrow{g_{\in M}} & C \\
\downarrow f & & \downarrow f \\
A & & \\
\end{array} \]

Here we study \( M \)-injectivity where \( M \) is the class of all regular prime \( S \)-poset monomorphisms in the category \( \text{Pos}-S \) which will be called the regular prime injectivity. Analogously to the case of ordinary regular injectivity of \( S \)-posets, we show that every regular prime injective \( S \)-poset is complete, and the Skornjakov criterion also fails for the regular prime injectivity. Finally, by means of a weaker form of regular prime injectivity, we give some classifications for pomonoids. In particular, for a pomonoid \( S \) in which its identity is the bottom element, all principal right poideals of \( S \) are principally poideal regular prime injective if and only if \( S \) is poregular and principally poideal regular prime injective.

2. Categorical properties of regular prime monomorphisms

In this section, we study some categorical properties of regular prime monomorphisms of \( S \)-posets including the products, coproducts, direct sums and the pullback stability.

**Proposition 2.1.** Let \( f : A \to B \) be an \( S \)-poset homomorphism. Then the following assertions are equivalent:

(i) \( f \) is a regular prime \( S \)-poset monomorphism.

(ii) The product induced homomorphism \( \prod_{i \in I} f : \prod_{i \in I} A \to \prod_{i \in I} B \) is a regular prime \( S \)-poset monomorphism.

(iii) The coproduct induced homomorphism \( \coprod_{i \in I} f : \coprod_{i \in I} A \to \coprod_{i \in I} B \) is a regular prime \( S \)-poset monomorphism.

**Proof.** (i) \( \Rightarrow \) (ii) Assume that \( f : A \to B \) is a regular prime \( S \)-poset monomorphism. Using [9, Proposition 2], it remains to show that \( \prod_{i \in I} f : \prod_{i \in I} A \to \prod_{i \in I} B \) is prime. Note that \( (\prod_{i \in I} f)(\prod_{i \in I} A) = \prod_{i \in I} f(A) \). We must prove that \( \prod_{i \in I} f(A) \) is a prime sub \( S \)-poset of \( \prod_{i \in I} B \). Let \( \langle b_i \rangle_s \subseteq \prod_{i \in I} f(A) \) for each \( s \in S \) and \( \langle b_i \rangle_i \in (\prod_{i \in I} B) \setminus (\prod_{i \in I} f(A)) \). Then \( b_i s \subseteq f(A) \) for each \( i \in I \) and \( b_j \notin f(A) \) for some \( j \in I \). Since \( f(A) \) is a prime sub \( S \)-poset of \( B \), \( b_i \in f(A) \) or \( B_s \subseteq f(A) \) for each \( i \in I \). Particularly, for \( j \in I \), \( b_j s \subseteq f(A) \). As \( b_j \notin f(A) \), then \( B_s \subseteq f(A) \). Hence, \( (\prod_{i \in I} B)s = \prod_{i \in I} (B_s) \subseteq \prod_{i \in I} f(A) \).

(ii) \( \Rightarrow \) (i) Let \( \prod_{i \in I} f \) be a regular prime \( S \)-poset monomorphism. This clearly implies that \( f \) is a regular monomorphism. Now we show that \( f : A \to B \) is prime. Let \( bSs \subseteq f(A) \) for each \( b \in B \setminus f(A) \).
and \( s \in S \). We have \( \langle b \rangle \in \prod_{i \in I} (B \setminus f(A)) \subseteq (\prod_{i \in I} B) \setminus (\prod_{i \in I} f(A)) \) and \( (b)_{S_i} \subseteq \prod_{i \in I} f(A) \). Since \( \prod_{i \in I} f(A) \) is a prime sub \( S \)-poset of \( \prod_{i \in I} B \), \((\prod_{i \in I} B)_{S_i} \subseteq \prod_{i \in I} f(A) \). Consider an arbitrary element \( b' \in B_{S_i} \). Thus \( \langle b' \rangle \in \prod_{i \in I} (B_{S_i}) \subseteq \prod_{i \in I} f(A) \) and then \( b' \in f(A) \). Hence, \( B_{S_i} \subseteq f(A) \), as claimed.

(i) \( \Rightarrow \) (iii) Suppose that \( f : A \to B \) is a regular prime monomorphism. In view of [9, Proposition 5], it suffices to show that \( \prod_{i \in I} f : \prod_{i \in I} A \to \prod_{i \in I} B \) is prime. Note that \( (\prod_{i \in I} f)(\prod_{i \in I} A) = \prod_{i \in I} f(A) \).

It must be proved that \( \prod_{i \in I} f(A) \) is a prime sub \( S \)-poset of \( \prod_{i \in I} B \).

Let \( (i, b)_{S_i} \subseteq \prod_{i \in I} f(A) \) for each \( s \in S \) and \( (i, b) \in (\prod_{i \in I} B) \setminus (\prod_{i \in I} f(A)) \). Since \( (i, b)_{S_i} \subseteq \prod_{i \in I} f(A) \), we have \( (i, b)_{S_i} \subseteq (i, f(A)) \). As \( f(A) \) is a prime sub \( S \)-poset of \( B \), \( b_{S_i} \subseteq f(A) \) and \( b \in B \setminus f(A) \), we get \( B_{S_i} \subseteq f(A) \) and then \( (\prod_{i \in I} B)_{S_i} = (\prod_{i \in I} B_{S_i}) \subseteq \prod_{i \in I} f(A) \).

(iii) \( \Rightarrow \) (i) Let \( \prod_{i \in I} f \) be a regular prime \( S \)-poset monomorphism. This clearly gives that \( f \) is a regular monomorphism. Now we prove that \( f : A \to B \) is prime. Let \( b_{S_i} \subseteq f(A) \) for each \( b \in B \setminus f(A) \) and \( s \in S \). We have \( (i, b)_{S_i} = (i, b)_{S_i} \subseteq (i, f(A)) \subseteq \prod_{i \in I} f(A) \) for each \( i \in I \). Since \( (i, b) \in \prod_{i \in I} (B \setminus f(A)) = (\prod_{i \in I} B) \setminus (\prod_{i \in I} f(A)) \) and \( \prod_{i \in I} f \) is prime, \( (\prod_{i \in I} B)_{S_i} \subseteq \prod_{i \in I} f(A) \). Now consider an arbitrary element \( b' \in B_{S_i} \). Then \( (i, b')_{S_i} \in \prod_{i \in I} (B_{S_i}) = (\prod_{i \in I} B_{S_i}) \subseteq \prod_{i \in I} f(A) \) for each \( i \in I \) and so \( b' \in f(A) \). Therefore, \( B_{S_i} \subseteq f(A) \), as required. \( \square \)

Recall that for a family \( \{ A_i \mid i \in I \} \) of \( S \)-posets with a unique zero element \( \theta \), the direct sum \( \bigoplus_{i \in I} A_i \) is defined to be the sub \( S \)-poset of the product \( \prod_{i \in I} A_i \) consisting of all \( \langle a_i \rangle \) such that \( a_i = \theta \) for all \( i \in I \) except a finite number.

**Remark 2.2.** Let \( f : A \to B \) be an \( S \)-poset homomorphism where \( A \) and \( B \) have a unique zero element and \( \bigoplus_{i \in I} f : \bigoplus_{i \in I} A \to \bigoplus_{i \in I} B \) be the homomorphism induced by the product of \( f \). In fact, \( \bigoplus_{i \in I} f = (\prod_{i \in I} f)|_{\bigoplus_{i \in I} A} \). It follows directly from Proposition 2.1 that the map \( f \) is a regular prime monomorphism if and only if so is \( \bigoplus_{i \in I} f \).

The following example shows that the product and coproduct induced homomorphisms of two non-equal (as maps) regular prime homomorphisms on an \( S \)-poset are not necessarily regular prime.

**Example 2.3.** Consider the pomonoid \( (\mathbb{N}, \cdot, \leq) \) and the regular monomorphisms \( f_1, f_2 : \mathbb{N} \to \mathbb{N} \) given by \( f_1(n) = 2n \) and \( f_2(n) = 3n \) for each \( n \in \mathbb{N} \). Clearly, \( f_1(\mathbb{N}) = 2\mathbb{N} \) and \( f_2(\mathbb{N}) = 3\mathbb{N} \) are prime sub \( \mathbb{N} \)-posets of \( \mathbb{N} \). But the product and coproduct induced homomorphisms of \( f_1 \) and \( f_2 \) are not prime. Indeed, \( (f_1 \times f_2)(\mathbb{N} \times \mathbb{N}) = 2\mathbb{N} \times 3\mathbb{N} \) is not a prime sub \( \mathbb{N} \)-poset of \( \mathbb{N} \times \mathbb{N} \) because \( (2, 2)\mathbb{N} = 6\mathbb{N} \subseteq 2\mathbb{N} \times 3\mathbb{N} \) for \((2, 2) \in \mathbb{N} \times \mathbb{N} \).
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\[ P \xleftarrow{p_1} A \xrightarrow{g} C \]

\[ P \xleftarrow{p_B} B \xrightarrow{f} C \]

We say that a subclass \( \mathcal{M} \) of monomorphisms of a category is stable under pullbacks or pullbacks transfer \( \mathcal{M} \) if the morphism \( p_1 \in \mathcal{M} \) whenever \( f \in \mathcal{M} \) in the following pullback diagram:

\[ \begin{array}{ccc}
P & \xrightarrow{p_1} & A \\
\downarrow{p_2} & & \downarrow{g} \\
B & \xrightarrow{f} & C
\end{array} \]

We have to show that \( p_A \) is also a regular prime monomorphism. Using [9, Proposition 4], \( p_A \) is a regular monomorphism. We prove that \( p_A(P) \) is a prime sub \( S \)-poset of \( A \). Let \( aSs \subseteq p_A(P) \) for each \( a \in A \) and \( s \in S \). We have to show that \( a \in p_A(P) \) or \( As \subseteq p_A(P) \). Suppose that \( As \nsubseteq p_A(P) \). For any \( x \in aSs \subseteq p_A(P) \), there exists \( b \in B \) such that \( (x,b) \in P \). Set

\[ D := \{ b \in B \mid f(b) = g(x) \text{ for some } x \in aSs \} \subseteq B. \]

We claim that \( f(D) = g(aSs) \). Clearly, \( f(D) \subseteq g(aSs) \). For the reverse inclusion, let \( g(x) \in g(aSs) \) for some \( x \in aSs \subseteq p_A(P) \). Then \( x = p_A(a,b) = a \) for some \( (a,b) \in P \) and so \( g(x) = g(a) = f(b) \) which means \( b \in D \). Hence, \( g(x) \in f(D) \) and then \( g(aSs) \subseteq f(D) \). Therefore, \( g(a)Ss \subseteq f(D) \subseteq f(B) \). Since \( f(B) \) is a prime sub \( S \)-poset of \( C \), \( g(a) \in f(B) \) or \( Cs \subseteq f(B) \). Now we show that \( Cs \nsubseteq f(B) \)
and just we have \( g(a) \in f(B) \). It follows from \( A \not\subseteq p_A(P) \) that there exists \( a' \in A \) such that \( a's \notin p_A(P) \). So \( g(a')s = g(a's) \neq f(b) \) for all \( b \in B \). This means that \( C's \notin f(B) \). Hence, \( g(a) \in f(B) \) and then \( g(a) = f(b') \) for some \( b' \in B \). This means that \((a, b') \in P \) and so \( a \in p_A(P) \).

\[ \square \]

3. Regular prime injectivity in \( \text{Pos-}S \)

In this section, we consider regular prime injectivity of \( S \)-posets and a weaker form of such kind of injectivity called principal poideal regular prime injectivity and present some homological classifications for pomonoids.

**Definition 3.1.** Let \( A \) be an \( S \)-poset. Then \( A \) is called **regular prime injective** if for each regular prime monomorphism \( g : B \to C \), any homomorphism \( f : B \to A \) can be lifted to a homomorphism \( f : C \to A \), that is, \( fg = f \).

We call a homomorphism \( f : A \to B \) a **prime embedding** of \( A \) into \( B \) if \( f \) is a regular prime monomorphism. In this case, \( A \) is regularly prime embedded into \( B \) or \( B \) contains an isomorphic copy of \( A \). Obviously, every \( S \)-poset may be replaced by an isomorphic \( S \)-poset in the definition of regular prime injectivity. Hence, we may assume that a regular prime monomorphism \( f : A \to B \) is a prime embedding so that \( A \) can be considered as a prime sub \( S \)-poset of \( B \), and use the definition of regular prime injectivity in a slightly different form. More precisely, an \( S \)-poset \( A \) is regular prime injective if and only if for every \( S \)-poset \( C \), any prime sub \( S \)-poset \( B \) of \( C \) and a homomorphism \( f : B \to A \), there exists a homomorphism \( \bar{f} : C \to A \) such that \( \bar{f}|_B = f \).

**Remark 3.2.** (i) Let \( S \) be a pogroup. Then each sub \( S \)-poset \( B \) of an \( S \)-poset \( A \) is prime. Indeed, \( aSs \subseteq B \) for \( a \in A, s \in S \) implies that \( a = as^{-1}s \in aSs \) and so \( a \in B \). This implies that regular prime injectivity of \( S \)-posets over a pogroup \( S \) coincides to regular injectivity.

(ii) It is easily seen that if an \( S \)-poset \( A \) is regular prime injective, then it is a retract of each of its prime extensions, that is, for each regular prime extension \( B \) of \( A \), there exists a homomorphism \( f : B \to A \) (a retraction) which maps \( A \) identically.

(iii) Every regular prime injective prime right poideal \( I \) of a pomonoid \( S \) is a principal right ideal which is generated by an idempotent element. Indeed, consider the prime embedding \( j : I \to S \) and extend the identity map \( id : I \to I \) to \( \overline{id} : S \to I \) by regular prime injectivity. We show that \( I \) is generated by \( \overline{id}(1) \) which is an idempotent element of \( I \). This is
because, $\overline{id}(1) \in I$ and so $\overline{id}(1)\overline{id}(1) = \overline{id}(1\overline{id}(1)) = \overline{id}(\overline{id}(1)) = \overline{id}(1)$. Also $s = \overline{id}(s) = \overline{id}(1s) = \overline{id}(1)$ for each $s \in I$.

Recall from [7] that a regular injective $S$-poset is bounded by two zero elements. For regular prime injectivity of $S$-posets, we have the same result as follows.

**Lemma 3.3.** Any non-singleton regular prime injective $S$-poset is bounded by two zero elements.

**Proof.** Let $A$ be a non-singleton regular prime injective $S$-poset. Consider the $S$-poset $B = A \sqcup \{\theta_1\} \sqcup \{\theta_2\}$ obtained by adjoining two zero elements to $A$ such that $\theta_1 \leq a \leq \theta_2$ for every $a \in A$. We show that $B$ is a prime extension of $A$. Let $bSs \subseteq A$ for $b \in B$ and $s \in S$. Clearly, $b \neq \theta_1, \theta_2$. Thus $b \in A$ which gives that $A$ is a prime sub $S$-poset of $B$. Since $A$ is regular prime injective, there exists a retraction $f : B \rightarrow A$. Note that $f(\theta_1) \leq f(a) = a \leq f(\theta_2)$ for every $a \in A$ and so $f(\theta_1) \neq f(\theta_2)$ because otherwise $|A| = 1$ which is a contradiction. Hence, the zero elements $f(\theta_1)$ and $f(\theta_2)$ are the bottom and top elements of $A$, respectively. □

Clearly, if $1 \in S$ is a zero element, then $S = \{1\}$. Regarding this fact and Lemma 3.3 we get:

**Corollary 3.4.** Let the identity of a pomonoid $S$ be its top or bottom element. If $S$ as an $S$-poset is regular prime injective, then $S = \{1\}$.

**Remark 3.5.** (i) Let $\{A_i : i \in I\}$ be a non-empty family of $S$-posets. As in the case of regular injectivity, using Lemma 3.3, one can show that $\prod_{i \in I} A_i$ is regular prime injective if and only if so is each $A_i$ (cf. [10, Remark 2.12]). As for the coproduct, if $|I| > 1$, then $\bigsqcup_{i \in I} A_i$ is not regular prime injective; because otherwise, using Theorem 3.3, it would be bounded which is a contradiction. As a conclusion, there exists no pomonoid $S$ over which all $S$-posets are regular prime injective. This is because, for a pomonoid $S$, if $A$ is an $S$-poset, then $A \sqcup A$ is not regular prime injective.

(ii) Skornjakov criterion states that the injectivity of acts with a zero is equivalent to being injective relative to all inclusions into cyclic acts (see [8, Theorem III.1.8]). Example 2.14 in [10] shows that this criterion fails for regular injectivity of $S$-posets. Using this example and Lemma 3.3, the Skornjakov criterion also fails for regular prime injectivity of $S$-posets. Indeed, the pomonoid $S = \{0, 1\}$ with equality relation as an $S$-poset is not regular prime injective by Lemma 3.3 since it is not bounded. But, as it was shown in the example, $S$ is regular injective with respect to embeddings into cyclic $S$-posets which clearly
gives that it is regular prime injective relative to prime embeddings into cyclic $S$-posets.

In what follows, we investigate the relation between regular prime injectivity in $\text{Pos}-S$ and regular injectivity in the category $\text{Pos}$ of all posets and order-preserving maps between them.

Recall from [4] that the free functor $F : \text{Pos} \to \text{Pos}-S$ is given by $F(P) = P \times S$ with componentwise order and the action $(x, s)t = (x, st)$ for $x \in P, s, t \in S$. It is a left adjoint to the forgetful functor $U : \text{Pos}-S \to \text{Pos}$.

The following will be used in the sequel.

Lemma 3.6 ([4]). Let $F : C \to D$ and $G : D \to C$ be two functors such that $F$ be a left adjoint to $G$. Also let $\mathcal{M}, \mathcal{M}'$ be certain subclasses of $C, D$, respectively. If for all $f \in \mathcal{M}, Ff \in \mathcal{M}'$, then for each $\mathcal{M}'$-injective object $D \in D$, $GD$ is an $\mathcal{M}$-injective object of $C$.

Now consider the following lemma:

Lemma 3.7. The free functor $F : \text{Pos} \to \text{Pos}-S$ sends any regular monomorphism in $\text{Pos}$ to a regular prime monomorphism.

Proof. Consider a regular monomorphism $f : A \to B$ in $\text{Pos}$. Using [7, Lemma 2.3], $F(f) : A \times S \to B \times S$ is a regular monomorphism in $\text{Pos}-S$. It suffices to prove that $F(f)$ is prime. Let $(b, t)Ss \subseteq F(f)(A \times S)$ for $(b, t) \in B \times S$ and $s \in S$. Then $(b, tSs) \subseteq F(f)(A \times S) = f(A) \times S$. Therefore, $b \in f(A)$ and so $(b, t) \in F(f)(A \times S)$. This means that $F(f)(A \times S)$ is a prime sub $S$-poset of $B \times S$. \hfill \Box

An $S$-poset $A$ is called complete if it is complete as a poset. It is known that regular injective posets are exactly complete posets (see [2]). So, in light of Lemmas 3.6 and 3.7, the following is immediate:

Corollary 3.8. Any regular prime injective $S$-poset is complete.

Definition 3.9. An $S$-poset $A$ is said to be principally poideal regular prime injective if every $S$-poset homomorphism $f : I \to A$ from a principal prime right poideal $I$ of a pomonoid $S$ can be extended to an $S$-poset homomorphism $\bar{f} : S \to A$.

Note that any $S$-poset homomorphism $f : S \to A$ is of the form $\lambda_a$, where $\lambda_a : S \to A$ is the right translation mapping, for $a = f(1)$ since $f(s) = f(1s) = f(1)s = as$ for each $s \in S$. Thus, the fact that an $S$-poset map $f : I \to A$ from a right poideal $I$ of $S$ to an $S$-poset $A$ can be extended to an $S$-poset map $\bar{f} : S \to A$ is equivalent to $f$ being of the form $\lambda_a$ for some $a \in A$. This means that:
An $S$-poset $A$ is principally poideal regular prime injective if and only if for each $S$-poset map $f : I \to A$, where $I \subseteq S$ is a principal right poideal, there exists an element $a \in A$ such that $f = \lambda_a$.

Recall from [11] that a pomonoid $S$ is called poregular if every $s \in S$ for which $sS$ is a poideal of $S$ is a regular element, that is, there exists $t \in S$ such that $s = sts$. Also we say that $S$ is weakly regular if every $s \in S$ is a weakly regular element, that is, there exists $t \in S$ such that $s \leq sts$.

**Lemma 3.10.** Let the identity of a pomonoid $S$ be its bottom element and $s \in S$. Then the principal right poideal $\downarrow (sS)$ is a prime right ideal of $S$.

**Proof.** Assume that $s_1 s_2 \subseteq sS = \downarrow (sS)$ for each $s_1, s_2 \in S$. We show that $s_1 \in \downarrow (sS)$ or $s_2 \in \downarrow (sS)$. Then for each $t \in S$, there exist $l, r \in S$ such that $s_1 t s_2 \leq sr$. As $1$ is the bottom element, we have $1 \leq ts_2$, and hence $s_1 = s_1 1 \leq s_1 t s_2$. This implies that $s_1 \leq s_1 t s_2 \leq sr$ and then $s_1 \in \downarrow (sS)$. Therefore, $S$ is a regular prime extension of $\downarrow (sS)$. □

**Theorem 3.11.** Let $S$ be a pomonoid whose identity is the bottom element. Then the following statements are equivalent:

(i) All principal right poideals of $S$ which are principal as ideals are principally poideal regular prime injective.

(ii) $S$ is poregular and principally poideal regular prime injective.

**Proof.** (i) $\Rightarrow$ (ii) Since $S$ is itself a principal right poideal, it is principally poideal regular prime injective by (i). We show that $S$ is poregular. Let $s \in S$ and $sS$ is a poideal. It is easily seen that $sS = \downarrow (sS)$. Consider the natural embedding $\iota : \downarrow (sS) \hookrightarrow S$. Using Lemma 3.10, $\iota$ is a prime embedding and so $\downarrow (sS)$ is a retract of $S$ because $\downarrow (sS)$ is principally poideal regular prime injective by the assumption. Let $f : S \to \downarrow (sS)$ denote such a retraction. Now, taking $f(1) = u$, we have $u = st$ for some $t \in S$, since $sS = \downarrow (sS)$, and then $s = f(s) = f(1s) = f(1)s = us = sts$, which means that $s$ is a regular element. Therefore, $S$ is poregular.

(ii) $\Rightarrow$ (i) Consider a principal right poideal of the form $\downarrow (sS) = sS$ of $S$. Let $\iota : I \hookrightarrow S$ be a prime embedding from a principal right poideal $I$ and $f : I \to sS$ be a homomorphism. Since $S$ is poregular, there exists $t \in S$ such that $sts = s$. This gives that $sS$ is a retract of $S$ with the retraction $\phi : S \to sS$ given by $\phi(r) = str$ for each $r \in S$. 


Consider the following diagram:

\[
\begin{array}{c}
I \xrightarrow{\iota} S \\
\downarrow f \\
S_{SS} \\
\phi \downarrow i \\
S
\end{array}
\]

Since \( S \) is principally poideal regular prime injective, there exists an \( S \)-poset map \( h : B \to S \) such that \( hg = if \). Considering the map \( f := \phi h \), we get \( fg = \phi hg = \phi if = id_{sS}f = f \) and then \( sS \) is principally poideal regular prime injective.

Recall from [11] that a pomonoid \( S \) is called right po-PP if for every \( s \in S \) there exists an idempotent \( e \in S \) such that \( s = se \), and \( su \leq sv \) implies \( eu \leq ev \) for all \( u, v \in S \).

**Proposition 3.12.** Let \( S \) be a pomonoid whose identity is the bottom element. Then we have the following assertions:

(i) If any principal right poideal of \( S \) is principally poideal regular prime injective, then \( S \) is weakly regular.

(ii) If \( S \) is a principally poideal regular prime injective as well as a right po-PP pomonoid, then \( S \) is poregular.

**Proof.** (i) Let \( s \in S \). Using the assumption, the principal right poideal \( \downarrow (sS) \) is principally poideal regular prime injective. Consider the natural embedding \( \iota : \downarrow (sS) \hookrightarrow S \). By Lemma 3.10, \( \iota \) is a prime embedding. Then there is a retraction \( f : S \to \downarrow (sS) \), because \( \downarrow (sS) \) is principally poideal regular prime injective. Now, taking \( f(t) = u \) we have \( u \leq st \) for some \( t \in S \), and \( s = f(s) = f(1s) = f(1)s = us \leq sts \). Therefore, \( s \) is a weakly regular element and then \( S \) is weakly regular.

(ii) Consider any pomonoid \( S \) satisfying the assumption. Let \( s \in S \) such that \( sS \) is a poideal. By Lemma 3.10, the natural embedding \( \iota : \downarrow (sS) \hookrightarrow S \) is prime. Using the hypothesis, there exists an idempotent element \( e \in S \) such that \( s = se \) and \( su \leq sv \) implies \( eu \leq ev \) for all \( u, v \in S \). Define \( f : \downarrow (sS) = sS \to S \) by \( f(st) = et \) for each \( t \in S \). Let \( st \leq st' \) for some \( t, t' \in S \). Then \( et \leq et' \) and so \( f(st) \leq f(st') \). This means that \( f \) is an order-preserving and then well-defined. Hence, \( f \) is an \( S \)-poset map. Since \( S \) is principally poideal regular prime injective, \( f = \lambda_x \) for some \( x \in S \). Thus \( e = f(s) = xs \). Then \( s = se = sx \), so \( s \) is regular and hence \( S \) is poregular. \( \square \)
Theorem 3.13. Let \( \{A_i \mid i \in I\} \) be a non-empty family of \( S \)-posets with a zero \( \theta_i \in A_i \) for each \( i \in I \). Then the product \( \prod_{i \in I} A_i \) is poideal regular prime injective if and only if so is each \( A_i \).

Proof. The fact that the product of poideal regular prime injective \( S \)-posets is poideal regular prime injective is proved analogously to the case of general injectivity in a category using the universal property of products. To prove the converse, let \( \prod_{i \in I} A_i \) be poideal regular prime injective and take any \( j \in I \). Let \( f : I \to A_j \) be an \( S \)-poset map from a prime right poideal \( I \) of \( S \) to \( A_j \). Consider the \( S \)-poset map \( \tilde{f} : I \to \prod_{i \in I} A_i \) defined by \( \tilde{f}(s) = (\ldots, \theta_j, f(s), \theta_{j+1}, \ldots) \) for each \( s \in S \). It follows from the assumption that there exists an element \( (a_i)\in \prod_{i \in I} A_i \) such that \( \tilde{f} = \lambda_{(a_i)} \). Now it is easily seen that \( f = \lambda_{a_j} \) which shows that \( A_j \) is poideal regular prime injective. □

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References

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ON REGULAR PRIME INJECTIVITY OF S-POSETS

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انژکتیوی اول منظم S-مجموّعه های مرتب

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در این مقاله، مفهوم تکریختی اول منظم را برای S-M-مجموّعه های مرتب روی یک تکاره مرتب S

تعیین و برخی خواص رسته ای آن جمله ضرب ها، هم‌ضربه ها و عقب‌برهای را بررسی می‌کنیم. مأموریت ای

انژکتیوی را در رسته S-M-مجموّعه های مرتب مورد مطالعه قرار می‌دهیم. قاعده کلاس تکریختی‌های اول

منظم است و نشان می‌دهیم مجموعه اسکوئن‌خواف برای انجکتیوی اول منظم برقرار نیست. با

نظر گرفتن تعریف ضعیفاتی از این نوع انژکتیوی، برخی رده‌بدی‌ها را برای تکاره‌های مرتب به‌دست می‌آوریم.

کلمات کلیدی: زیر S-M-مجموّعه مرتب اول، تکریختی اول منظم، انژکتیو اول منظم.