Journal of Algebraic Systems
Vol. 8, No. 2, (2021), pp 231-250

NEW BOUNDS AND EXTREMAL GRAPHS FOR DISTANCE SIGNLESS LAPLACIAN SPECTRAL RADIUS

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Abstract. The distance signless Laplacian spectral radius of a connected graph G is the largest eigenvalue of the distance signless Laplacian matrix of G, defined as $D^Q(G) = Tr(G) + D(G)$, where $D(G)$ is the distance matrix of G and $Tr(G)$ is the diagonal matrix of vertex transmissions of G. In this paper, we determine some new upper and lower bounds on the distance signless Laplacian spectral radius of G and characterize the extremal graphs attaining these bounds.

1. Introduction

In this article, we consider only connected, undirected, simple and finite graphs, i.e, graphs on a finite number of vertices without multiple edges or loops. $\overline{G}$ is the complement of the graph G. A graph is denoted by $G = (V(G), E(G))$, where $V(G)$ is its vertex set and $E(G)$ is its edge set. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertex has the same degree. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv} or $d_G(u, v)$, is defined as the length of a shortest path between u

DOI: 10.22044/jas.2020.9540.1469.
MSC(2010): Primary: 05C50, 05C12; Secondary: 15A18.
Keywords: Distance signless Laplacian matrix; spectral radius; extremal graph; transmission regular graph.
Received: 7 April 2020; Accepted: 8 August 2020.
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and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *transmission* $Tr_G(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e., $Tr_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $Tr_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph G, denoted by $\sigma(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $\sigma(G) = \frac{1}{2} \sum_{v \in V(G)} Tr_G(v)$.

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $Tr_G(v_i)$ has been referred as the *transmission degree* Tr_i [26] and hence the *transmission degree sequence* is given by $\{Tr_1, Tr_2, \ldots, Tr_n\}$. The second transmission degree of v_i, denoted by T_i is given by $T_i = \sum_{j=1}^{n} d_{ij} Tr_j$.

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. M. Aouchiche and P. Hansen [13, 14, 15] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = Tr(G) + D(G)$ is called the *distance signless Laplacian matrix* of G. Since $D^Q(G)$ is symmetric (positive semi-definite), its eigenvalues can be arranged as: $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G) \geq 0$, where $\rho_1(G)$ is called the *distance signless Laplacian spectral radius* of G. Afterwards, we will denote $\rho_1(G)$ by $\rho(G)$. As $D^Q(G)$ is nonnegative and irreducible, by the Perron-Frobenius theorem, $\rho(G)$ is positive, simple and there is a unique positive unit eigenvector X corresponding to $\rho(G)$, which is called the *distance signless Laplacian Perron vector* of G. For some recent papers on spectral properties of the (generalized) distance (signless Laplacian) matrix, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 21, 23, 24, 30, 29, 36] and the references therein.

The investigation of matrices related to various graphical structures is a very large and growing area of research. In particular, distance signless Laplacian matrix (spectral radius) have attracted serious attention in the literature. In [37], Xing et al. have determined the graphs with minimum distance signless Laplacian spectral radii among the n-vertex tree, unicyclic graphs and bipartite graphs, respectively. In [36], the authors have determined the unique graphs with minimum and second-minimum distance signless Laplacian spectral radii among all bicyclic graphs of order n. In [25], bounds for distance signless Laplacian spectral radius are given using vertex transmissions and in [29], lower bound for distance signless Laplacian spectral radius is given in terms of chromatic number. In this paper, we give some upper and lower bounds on the distance signless Laplacian spectral radius of G, analogously to the results obtained in the literature for the case of distance matrix and also for signless Laplacian matrix.

2. Notations and Preliminaries

A column vector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_i, i.e., $X(v_i) = x_i$ for $i = 1, 2, \ldots, n$. Then,

$$X^T D^Q(G) X = \sum_{\{u,v\} \subseteq V(G)} d_{uv}(x_u + x_v)^2,$$

and λ is an eigenvalue of $D^Q(G)$ corresponding to the eigenvector X if and only if $X \neq \mathbf{0}$ and for each $v \in V(G)$,

$$\lambda x_v = \sum_{u \in V(G)} d_{uv}(x_u + x_v).$$

These equations are called the (λ, x)-eigenequations of G. For a normalized column vector $X \in \mathbb{R}^n$ with at least one non-negative component, by the Rayleigh's principle, we have

$$\rho(G) \geq X^T D^Q(G) X,$$

with equality if and only if X is the distance signless Laplacian Perron vector of G.

For a connected graph G and two nonadjacent vertices u and v in $V(G)$, recall that $G + uv$ is the supergraph formed from G by adding an edge between vertices u and v. We now mention the following result which will be useful to derive some of the main results of this article.

Lemma 2.1. [33] *If A is an $n \times n$ nonnegative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \ldots, r_n$, then*

$$\min_{1 \leq i \leq n} r_i \leq \lambda(A) \leq \max_{1 \leq i \leq n} r_i.$$

Moreover, if A is irreducible, then both of the equalities holds if and only if the row sums of A are all equal.

The following is the well known Weyl's inequality and can be found in [27]. Note that the equality case was discussed in [35].

Lemma 2.2. *Let X and Y be Hermitian matrices of order n such that $Z = X + Y$. Then*

$$\lambda_k(Z) \leq \lambda_j(X) + \lambda_{k-j+1}(Y), \;\; n \geq k \geq j \geq 1,$$
$$\lambda_k(Z) \geq \lambda_j(X) + \lambda_{k-j+n}(Y), \;\; n \geq j \geq k \geq 1,$$

where $\lambda_i(M)$ is the i^{th} largest eigenvalue of the matrix M. In either of these inequalities, equality holds if and only if there exists a unit vector that is an eigenvector to each of the three eigenvalues involved.

3. Bounds on distance signless Laplacian spectral radius

In this section, we give some bounds on the distance signless Laplacian spectral radius. We first list some important observations about the components of the distance signless Laplacian Perron vector.

Lemma 3.1. *If $X = (x_1, x_2, \ldots, x_n)^T$ is the distance signless Laplacian Perron vector of a graph G and $x_i = \max\{x_k | k = 1, 2, \ldots, n\}$, then $Tr_i \geq \frac{\rho(G)}{2}$.*

Proof. From the i-th eigenequation we have,

$$\rho(G)x_i \;\; = \;\; Tr_i x_i + \sum_{j=1}^{n} d_{ij} x_j.$$

Then, we get $\rho(G) - Tr_i = \sum_{j=1}^{n} d_{ij} \frac{x_j}{x_i}$. Hence, $\rho(G) - Tr_i \leq Tr_i$, and therefore $Tr_i \geq \frac{\rho(G)}{2}$, as desired. □

Corollary 3.2. *Let $Tr^1_{\max}$ and $Tr^2_{\max}$ denote the maximum and the second maximum vertex transmission of G, respectively. If $\rho(G) = Tr^1_{\max} + Tr^2_{\max}$ and $Tr^1_{\max} \neq Tr^2_{\max}$, then the vertex corresponding to the maximum Perron component is the vertex having maximum transmission.*

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be the distance signless Laplacian Perron vector of G and $x_i = \max\{x_k| \; k = 1, 2, \ldots, n\}$. Then, using Lemma 3.1 we have, $Tr_i \geq \frac{Tr^1_{\max} + Tr^2_{\max}}{2}$, and thus $Tr_i = Tr^1_{\max}$. □

Lemma 3.3. *Let $X = (x_1, x_2, \ldots, x_n)^T$ be the distance signless Laplacian Perron vector of a graph G and $Tr^1_{\max}$, $Tr^2_{\max}$ denote the maximum and the second maximum vertex transmission of it, respectively. If $\rho(G) = Tr^1_{\max} + Tr^2_{\max}$ and $Tr^1_{\max} \neq Tr^2_{\max}$, then the second maximum Perron component is greater than or equal to $\frac{Tr^2_{\max}}{Tr^1_{\max}} x_s$, where $x_s = \max\{x_k|k = 1, 2, \ldots, n\}$.*

Proof. If $x_t = \max\{x_k|\ k = 1, 2, \ldots, n;\ k \neq s\}$, then from the s-th eigenequation, we have

$$\begin{aligned} \rho(G)x_s &= Tr_s x_s + \sum_{j=1}^{n} d_{sj}x_j, \\ \text{i.e., } (\rho(G) - Tr^1_{\max})x_s &\leq Tr^1_{\max}x_t, \text{ [by Corollary 3.2]} \\ \text{i.e., } x_t &\geq \frac{Tr^2_{\max}}{Tr^1_{\max}}x_s. \end{aligned}$$

□

We now give our first upper bound on $\rho(G)$ in terms of transmission degrees of G.

Theorem 3.4. *Let G be a graph of order n with the transmission degree sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$. Then*

$$\rho(G) \leq \max_{1\leq i,j\leq n}\{Tr_i + Tr_j\}, \tag{3.1}$$

with equality holding if G is a transmission regular graph.

Proof. Let $X = (x_1, \ldots, x_n)^T$ be an eigenvector of $Tr(G)^{-1}D^Q(G)Tr(G)$ corresponding to $\rho(G)$ and $x_k = \max\{x_j|j = 1, 2, \ldots, n\}$. The (i, j)-th entry of $Tr(G)^{-1}D^Q(G)Tr(G)$ is

$$\begin{cases} Tr_i & \text{if } \quad i = j \\ \frac{Tr_j}{Tr_i}d_{ij} & \text{otherwise.} \end{cases}$$

We have

$$Tr(G)^{-1}D^Q(G)Tr(G)X = \rho(G)X. \tag{3.2}$$

From the k-th equation of (3.2), we have

$$\begin{aligned} \rho(G)x_k &= Tr_k x_k + \sum_{j=1}^{n} \frac{Tr_j d_{kj}}{Tr_k}x_j, \\ \text{i.e., } (\rho(G) - Tr_k)x_k &= \sum_{j=1}^{n} \frac{Tr_j d_{kj}}{Tr_k}x_j, \\ \text{i.e., } x_k(\rho(G) - Tr_k)x_k &= \sum_{j=1}^{n} \frac{Tr_j}{Tr_k}x_k d_{kj} x_j \leq x_k^2 \sum_{j=1}^{n} \frac{Tr_j}{Tr_k}d_{kj}, \quad (3.3) \end{aligned}$$

$$\begin{aligned} \text{i.e., } \rho(G) &\leq Tr_k + \frac{1}{Tr_k}\sum_{j=1}^{n} Tr_j d_{kj} \\ &\leq Tr_k + \frac{1}{Tr_k}\max_{1\leq j\leq n}\{Tr_j\}\sum_{j=1}^{n} d_{kj} \qquad (3.4) \\ &\leq Tr_k + \max_{1\leq j\leq n}\{Tr_j\} \\ &\leq \max_{1\leq i,j\leq n}\{Tr_i + Tr_j\}. \end{aligned}$$

Which completes the proof of inequality (3.1). Now suppose that equality in (3.1) holds, then all inequalities in the above argument must be equalities. From equality in (3.3), we get $x_1 = x_2 = \cdots = x_n$. From equality in (3.4), we get $Tr_1 = Tr_2 = \cdots = Tr_n = \max_{1\leq j\leq n\ ,\ j\neq k}\{Tr_j\}$. Set $Tr_s := \max_{1\leq j\leq n\ ,\ j\neq k}\{Tr_j\}$. Then we have the following two cases:

Case (1): If $Tr_k = Tr_s$, then all the transmissions of the vertices are equal and G is a transmission regular graph.

Case (2): If $Tr_k \neq Tr_s$, then we consider the following two subcases:

Subcase (2.1): $Tr_k = n-1$. In this case the vertex v_k is adjacent to all the other remaining $n-1$ vertices in G, and therefore $G \cong S_n$. But since $\rho(S_n) = \dfrac{5n-8+\sqrt{9n^2-32n+32}}{2}$, it contradicts the fact that (3.1) holds.

Subcase (2.2): $Tr_k > n-1$. In this case there exists $1 \leq j \leq n\ ,\ k \neq j$ such that for a vertex v_j we have $d_{jk} \geq 2$. But then $Tr_j \neq Tr_s$, which is impossible. □

In the following result, we give bounds on $\rho(G)$, in terms of transmission degrees and second transmission degrees of graph G.

Theorem 3.5. *Let G be a graph of order n. If the transmission degree sequence and the second transmission degree sequence of G are $\{Tr_1, Tr_2, \ldots, Tr_n\}$ and $\{T_1, T_2, \ldots, T_n\}$, respectively, then*

$$\rho(G) \leq \sqrt{2}\max_{v_i\in V(G)}(Tr_i^2 + T_i)^{\frac{1}{2}}.$$

Moreover, the equality holds if and only if $Tr_i^2 + T_i$ is the same for all $v_i \in V(G)$. Also

$$\rho(G) \geq \sqrt{2}\min_{v_i\in V(G)}(Tr_i^2 + T_i)^{\frac{1}{2}}.$$

Moreover, the equality holds if and only if $Tr_i^2 + T_i$ is the same for all $v_i \in V(G)$.

Proof. Since $D^Q = Tr + D$, by a simple calculation we have, $r_{v_i}(D^Q) = 2Tr_i$ and $r_{v_i}(DTr) = r_{v_i}(D^2) = \sum_{j=1}^{n} d_{ij}Tr_j$. Then

$$\begin{aligned} r_{v_i}((D^Q)^2) &= r_{v_i}(Tr^2 + TrD + DTr + D^2) \\ &= Tr_i r_{v_i}(D^Q) + 2\sum_{j=1}^{n} d_{ij}Tr_j = 2(Tr_i^2 + \sum_{j=1}^{n} d_{ij}Tr_j) \end{aligned}$$

By Lemma 2.1, we get

$$\rho(G) \leq \sqrt{2} \max_{v_i \in V(G)} (Tr_i^2 + T_i)^{\frac{1}{2}},$$

and the equality holds if and only if $Tr_i^2 + T_i$ is the same for all $v_i \in V(G)$.

The second part can be proved similarly. □

Corollary 3.6. *If Δ denotes the maximum degree of a graph G, then*

$$\rho(G) \geq \sqrt{2}\Big((2n - \Delta)^2 - 4n + \Delta\Big)^{\frac{1}{2}}, \tag{3.5}$$

with equality holding if and only if G is a regular graph with diameter less than or equal to 2.

Proof. It is easily seen that $Tr_i \geq d_i + 2(n - d_i - 1) = 2n - d_i - 2$ and $T_i = \sum_{j=1}^{n} d_{ij}Tr_j \geq \sum_{j=1}^{n} d_{ij}^2$. Therefore, by Theorem 3.5, we have

$$\begin{aligned} \rho(G) &\geq \sqrt{2} \min_{v_i \in V(G)} (Tr_i^2 + T_i)^{\frac{1}{2}} \\ &\geq \sqrt{2}\Big((2n - d_i - 2)^2 + d_i + 4(n - d_i - 1)\Big)^{\frac{1}{2}} \\ &\geq \sqrt{2}\Big((2n - \Delta - 2)^2 + (4n - 3\Delta - 4)\Big)^{\frac{1}{2}} \\ &= \sqrt{2}\Big((2n - \Delta)^2 - 4n + \Delta\Big)^{\frac{1}{2}}. \end{aligned}$$

The equality in (3.5) holds if and only if the diameter of G is less than or equal to 2 and all coordinates the distance signless Laplacian perron vector of G are equal. In other words, for $d = 1$, we get a complete graph K_n. And for $d = 2$, we get G is a regular graph.

Conversely, it is easily seen that $\rho(G) = \sqrt{2}\big((2n - \Delta)^2 - 4n + \Delta\big)^{\frac{1}{2}}$ if G is a regular graph with diameter less than or equal to 2. □

Corollary 3.7. *If δ and d denote the minimum degree and diameter of a graph G, respectively, then*

$$\rho(G) \leq 2\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big).$$

Proof. It is easily seen that,

$$Tr_i \leq d_i + 2 + \cdots + (d-1) + d(n-1-d_i-(d-2)) = dn - \frac{d(d-1)}{2} - 1 - d_i(d-1).$$

If $Tr_{\max}$ is the maximum vertex transmission, then $T_i = \sum_{j=1}^{n} d_{ij} Tr_j \leq (Tr_{\max})^2$. Then by Theorem 3.5, we have

$$\begin{aligned}
\rho(G) &\leq \sqrt{2} \max_{v_i \in V(G)} (Tr_i^2 + T_i)^{\frac{1}{2}} \\
&\leq \sqrt{2}\Bigg(\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big)^2 \\
&+ \Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big)^2\Bigg)^{\frac{1}{2}} \\
&\leq \sqrt{2}\Bigg(2\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big)^2\Bigg)^{\frac{1}{2}} \\
&= 2\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big).
\end{aligned}$$

□

We now give a Nordhaus-Gaddam type inequality for the distance signless Laplacian spectral radius of a graph and its complement.

Corollary 3.8. *Suppose G be a graph such that both G and $\bar{G}$ are connected. Let δ and Δ be the minimum degree and the maximum degree of G, respectively. Then*

$$\rho(G) + \rho(\bar{G}) \leq 2\big(2nk - (t-1)(t+n+\delta-\Delta-1) - 2\big),$$

where $k = \max\{d, \bar{d}\}, t = \min\{d, \bar{d}\}$ and $d, \bar{d}$ are the diameters of G and $\bar{G}$, respectively.

Proof. Let $\bar{\delta}$ denote the minimum degree of $\overline{G}$. Then $\bar{\delta} = n - 1 - \Delta$, and by Corollary 3.7, we have

$$\begin{aligned}
\rho(G) + \rho(\bar{G}) &\leq 2\Big(dn - \frac{d(d-1)}{2} - 1 - \delta(d-1)\Big) \\
&+ 2\Big(\bar{d}n - \frac{\bar{d}(\bar{d}-1)}{2} - 1 - \bar{\delta}(\bar{d}-1)\Big) \\
&= 2n(d+\bar{d}) - \big(d(d-1) + \bar{d}(\bar{d}-1)\big) - 4 \\
&- 2\delta(d-1) - 2(n-1-\Delta)(\bar{d}-1) \\
&\leq 2\big(2nk - (t-1)(t+n+\delta-\Delta-1) - 2\big).
\end{aligned}$$

□

The following result is analogous to the result presented by Maden et al. in [32] in the case of signless Laplacian matrix of G.

Theorem 3.9. *Let G be a graph of order n. If the transmission degree sequence and the second transmission degree sequence of G are $\{Tr_1, Tr_2, \ldots, Tr_n\}$ and $\{T_1, T_2, \ldots, T_n\}$, respectively, then*

$$\rho(G) \leq \max_{v_i \in V(G)} \left\{ \frac{Tr_i + \sqrt{Tr_i^2 + \frac{4}{Tr_i}\sum_{j=1}^{n} d_{ij}(T_j + Tr_j^2)}}{2} \right\}, \tag{3.6}$$

with equality holding if and only if G is transmission regular.

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector corresponding to the eigenvalue $\rho(G)$ of $Tr^{-1}(G)D^Q(G)Tr(G)$. We assume that one eigencomponent x_i is equal to 1 and the other eigencomponents are less than or equal to 1. The (i,j)-th entry of $Tr(G)^{-1}D^Q(G)Tr(G)$ is

$$\begin{cases} Tr_i & \text{if} \quad i = j \\ \frac{Tr_j}{Tr_i} d_{ij} & \text{otherwise.} \end{cases}$$

We have

$$Tr(G)^{-1}D^Q(G)Tr(G)X = \rho(G)X. \tag{3.7}$$

From the $i-$th equation of (3.7), we have

$$\begin{aligned}\rho(G)x_i &= Tr_i x_i + \sum_{j=1}^{n} \frac{Tr_j}{Tr_i} d_{ij} x_j, \\ \text{i.e., } \rho(G) &= Tr_i + \sum_{j=1}^{n} \frac{Tr_j}{Tr_i} d_{ij} x_j. \qquad (3.8)\end{aligned}$$

Again from the $j-$th equation of (3.7),

$$\rho(G)x_j = Tr_j x_j + \sum_{k=1}^{n} \frac{Tr_k}{Tr_j} d_{jk} x_k.$$

Multiplying both sides of (3.8) by $\rho(G)$ and substituting this value $\rho(G)x_j$, we get

$$\begin{aligned}\rho^2(G) &= Tr_i\rho(G) + \sum_{j=1}^{n} \Big\{ \frac{Tr_j}{Tr_i} d_{ij} [Tr_j x_j + \sum_{k=1}^{n} \frac{Tr_k}{Tr_j} d_{jk} x_k] \Big\} \\ &= Tr_i\rho(G) + \sum_{j=1}^{n} \frac{Tr_j^2}{Tr_i} d_{ij} x_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{Tr_k}{Tr_i} d_{ij} d_{jk} x_k \\ &\leq Tr_i\rho(G) + \sum_{j=1}^{n} \frac{Tr_j^2}{Tr_i} d_{ij} + \sum_{j=1}^{n} \frac{T_j}{Tr_i} d_{ij} \\ &= Tr_i\rho(G) + \frac{1}{Tr_i} \sum_{j=1}^{n} d_{ij}(T_j + Tr_j^2). \qquad (3.9)\end{aligned}$$

From above the bound follows. Now suppose that the equality holds in (3.6). Then all inequalities in the above argument must be equalities. From equality in (3.9), we get $x_j = 1$ for all j. From this one can easily show that $x_i = 1$ for all $i \in V$. Thus we have $Tr_1 + \frac{T_1}{Tr_1} = Tr_2 + \frac{T_2}{Tr_2} = \cdots = Tr_n + \frac{T_n}{Tr_n}$. Let $Tr_{\max}$ and $Tr_{\min}$ denote the maximum and minimum vertex transmission, respectively. Without loss of generality, assume that $Tr_i = Tr_{\max}$ and $Tr_j = Tr_{\min}$. Therefore, $Tr_{\max} + \frac{T_i}{Tr_{\max}} = Tr_{\min} + \frac{T_j}{Tr_{\min}}$. Since $T_i \geq Tr_{\max}Tr_{\min}$ and $T_j \leq Tr_{\max}Tr_{\min}$,

$$Tr_{\max} + Tr_{\min} \leq Tr_{\max} + \frac{T_i}{Tr_{\max}} = Tr_{\min} + \frac{T_j}{Tr_{\min}} \leq Tr_{\max} + Tr_{\min}.$$

Thus we must have $T_i = Tr_{\max}Tr_{\min} = T_j$ and hence

$$Tr_{\max}^2 + Tr_{\max}Tr_{\min} = Tr_{\min}^2 + Tr_{\max}Tr_{\min}.$$

From which it implies that $Tr_{\max} = Tr_{\min}$. Hence G is a transmission regular graph.

Conversely, one can easily see that the equality holds in (3.6) for transmission regular graph. □

Based on a simple technique suggested in [18], we next prove the following upper bound for $\rho(G)$.

Theorem 3.10. *Let G be a graph of order n. If the transmission degree sequence and the second transmission degree sequence of G are $\{Tr_1, Tr_2, \dots, Tr_n\}$ and $\{T_1, T_2, \dots, T_n\}$, respectively, then*

$$\rho(G) \leq \max_{1\leq i\leq n}\left\{\frac{Tr_i + \sqrt{Tr_i^2 + 8T_i}}{2}\right\}. \tag{3.10}$$

Equality occurs if and only if G is a transmission regular graph.

Proof. Let $X = (x_1, \dots, x_n)$ be the distance signless Laplacian Perron vector of G and $x_i = \max\{x_j |\ j = 1, 2, \dots, n\}$. Since

$$\rho(G)^2 X = \left(D^Q(G)\right)^2 X = (Tr + D)^2 X = Tr^2 X + TrDX + DTrX + D^2 X,$$

we have

$$\rho^2(G)x_i = Tr_i^2 x_i + Tr_i \sum_{j=1}^{n} d_{ij}x_j + \sum_{j=1}^{n} d_{ij}Tr_j x_j + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}d_{jk}x_k.$$

Now, we consider a simple quadratic function of $\rho(G)$:

$$(\rho^2(G) + \alpha\rho(G))X = (Tr^2 X + TrDX + DTrX + D^2 X) + \alpha(TrX + DX).$$

Considering the i-th equation, we have

$$\begin{aligned}\left(\rho^2(G) + \alpha\rho(G)\right)x_i &= Tr_i^2 x_i + Tr_i \sum_{j=1}^{n} d_{ij}x_j + \sum_{j=1}^{n} d_{ij}Tr_j x_j \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}d_{jk}x_k + \alpha\Big(Tr_i x_i + \sum_{j=1}^{n} d_{ij}x_j\Big).\end{aligned}$$

It is easy to see that the inequalities below are true

$$Tr_i \sum_{j=1}^{n} d_{ij}x_j \leq Tr_i^2 x_i, \quad \sum_{j=1}^{n} d_{ij}Tr_j x_j \leq T_i x_i,$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \leq T_i x_i, \quad \sum_{j=1}^{n} d_{ij}x_j \leq Tr_i x_i.$$

Hence, we have

$$\left(\rho^2(G) + \alpha\rho(G)\right)x_i \leq 2Tr_i^2 x_i + 2T_i x_i + 2\alpha Tr_i x_i$$

$$\text{i.e., } \rho^2(G) + \alpha\rho(G) - (2Tr_i^2 + 2T_i + 2\alpha Tr_i) \leq 0$$

$$\text{i.e., } \rho(G) \leq \frac{-\alpha + \sqrt{\alpha^2 + 8Tr_i(Tr_i + \frac{T_i}{Tr_i} + \alpha)}}{2}.$$

From the above inequality we can get several distinct upper bounds for every different value of α. In particular, if $\alpha = -Tr_i$, we have

$$\rho(G) \leq \max_{1\leq i\leq n} \left\{ \frac{Tr_i + \sqrt{Tr_i^2 + 8T_i}}{2} \right\}.$$

From this the result follows.

Now, suppose that equality occurs in (3.10), then each of the above inequalities in the above argument occur as equalities. Since each of the inequalities

$$Tr_i \sum_{j=1}^{n} d_{ij}x_j \leq Tr_i^2 x_i, \ \sum_{j=1}^{n} d_{ij}Tr_j x_j \leq T_i x_i$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \leq T_i x_i, \ \sum_{j=1}^{n} d_{ij}x_j \leq Tr_i x_i,$$

occur as equalities if and only if G is a transmission regular graph. It follows that equality occurs in (3.10) if and only if G is a transmission regular graph. That completes the proof. □

The proof of the following theorem is similar to that of Theorem 3.10. We bring its proof for the sake of completenes.

Theorem 3.11. *Let G be a graph of order n. If the transmission degree sequence and the second transmission degree sequence of G are $\{Tr_1, Tr_2, \ldots, Tr_n\}$ and $\{T_1, T_2, \ldots, T_n\}$, respectively, then*

$$\rho(G) \geq \min_{1\leq i\leq n} \left\{ \frac{Tr_i + \sqrt{Tr_i^2 + 8T_i}}{2} \right\}. \tag{3.11}$$

Equality occurs if and only if G is a transmission regular graph.

Proof. Let $X = (x_1, \ldots, x_n)$ be the distance signless Laplacian Perron vector of G and $x_i = \min\{x_j |\ j = 1, 2, \ldots, n\}$. Since

$$\rho(G)^2 X = (D^Q(G))^2 X = (Tr + D)^2 X = Tr^2 X + TrDX + DTrX + D^2 X,$$

we have

$$\rho^2(G)x_i = Tr_i^2x_i + Tr_i\sum_{j=1}^{n} d_{ij}x_j + \sum_{j=1}^{n} d_{ij}Tr_jx_j + \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}d_{jk}x_k.$$

Now, we consider a simple quadratic function of $\rho(G)$:

$$(\rho^2(G) + \alpha\rho(G))X = (Tr^2X + TrDX + DTrX + D^2X) + \alpha(TrX + DX).$$

Considering the i-th equation, we have

$$\begin{aligned}(\rho^2(G) + \alpha\rho(G))x_i &= Tr_i^2x_i + Tr_i\sum_{j=1}^{n} d_{ij}x_j + \sum_{j=1}^{n} d_{ij}Tr_jx_j \\ &+ \sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}d_{jk}x_k + \alpha(Tr_ix_i + \sum_{j=1}^{n} d_{ij}x_j).\end{aligned}$$

It is easy to see that the inequalities below are true

$$Tr_i\sum_{j=1}^{n} d_{ij}x_j \geq Tr_i^2x_i,\ \sum_{j=1}^{n} d_{ij}Tr_jx_j \geq T_ix_i,$$

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \geq T_ix_i,\ \sum_{j=1}^{n} d_{ij}x_j \geq Tr_ix_i.$$

Hence, we have

$$(\rho^2(G) + \alpha\rho(G))x_i \geq 2Tr_i^2x_i + 2T_ix_i + 2\alpha Tr_ix_i$$

$$\text{i.e., } \rho^2(G) + \alpha\rho(G) - (2Tr_i^2 + 2T_i + 2\alpha Tr_i) \geq 0$$

$$\text{i.e., } \rho(G) \geq \frac{-\alpha + \sqrt{\alpha^2 + 8Tr_i(Tr_i + \frac{T_i}{Tr_i} + \alpha)}}{2}.$$

From the above inequality we can get several distinct lower bounds for every different value of α. In particular, if $\alpha = -Tr_i$, we have

$$\rho(G) \geq \min_{1\leq i\leq n}\left\{\frac{Tr_i + \sqrt{Tr_i^2 + 8T_i}}{2}\right\}.$$

From this the result follows.

Now, suppose that equality occurs in (3.11), then each of the above inequalities in the above argument occur as equalities. Since each of the inequalities

$$Tr_i\sum_{j=1}^{n} d_{ij}x_j \geq Tr_i^2x_i, \quad \sum_{j=1}^{n} d_{ij}Tr_jx_j \geq T_ix_i$$

and

$$\sum_{j=1}^{n}\sum_{k=1}^{n} d_{jk}d_{ij}x_k \geq T_i x_i, \quad \sum_{j=1}^{n} d_{ij}x_j \geq Tr_i x_i,$$

occur as equalities if and only if G is a transmission regular graph, it follows that equality occurs in (3.11) if and only if G is a transmission regular graph. That completes the proof. □

Theorem 3.12. *If Δ denotes the maximum degree of a graph G, then*

$$\rho(G) \geq \frac{2n-\Delta-2+\sqrt{(2n-\Delta)^2-20\Delta+24n-28}}{2}, \qquad (3.12)$$

with equality holding if and only if G is a regular graph with diameter less than or equal to 2.

Proof. Since $D^Q = Tr + D$, by a simple calculation we have, $r_{v_i}(D^Q) = 2Tr_i$ and $r_{v_i}(DTr) = r_{v_i}(D^2) = \sum_{j=1}^{n} d_{ij}Tr_j$. Then

$$\begin{aligned} r_{v_i}((D^Q)^2) &= r_{v_i}(Tr^2 + TrD + DTr + D^2) \\ &= Tr(v_i)r_{v_i}(D^Q) + 2\sum_{j=1}^{n} d_{ij}Tr(v_j) \\ &\geq Tr(v_i)r_{v_i}(D^Q) + 2\sum_{j=1}^{n} d_{ij}^2, \ (\text{since} \sum_{j=1}^{n} d_{ij}Tr(v_j) \geq \sum_{j=1}^{n} d_{ij}^2), \\ &\geq (2n-d_i-2)r_{v_i}(D^Q) + 2(d_i + 4(n-1-d_i)) \\ &\geq (2n-\Delta-2)r_{v_i}(D^Q) + 2(4n-3\Delta-4). \end{aligned}$$

Hence for each $v_i \in V(G)$, we have

$$r_{v_i}((D^Q)^2) \geq r_{v_i}[(2n-\Delta-2)D^Q] + 8n - 6\Delta - 8.$$

Then Lemma 2.1 implies that

$$\begin{aligned} & \rho^2(G) - (2n-\Delta-2)\rho(G) - (8n-6\Delta-8) \geq 0 \\ \text{i.e., } \rho(G) &\geq \frac{2n-\Delta-2+\sqrt{(2n-\Delta)^2-20\Delta+24n-28}}{2}. \end{aligned}$$

The equality in (3.12) holds if and only if the diameter of G is less than or equal to 2. In other words, for $d = 1$, we get a complete graph K_n. For $d = 2$, we get G is a regular graph.

Conversely, it is easily seen that $\rho(G) = \frac{2n-\Delta-2+\sqrt{(2n-\Delta)^2-20\Delta+24n-28}}{2}$ if G is a regular graph with diameter less than or equal to 2. □

The following gives a relation between the distance signless Laplacian eigenvalues of the graph G of diameter 2 with the adjacency eigenvalues of the complement $\overline{G}$ of the graph G.

Theorem 3.13. *Let G be a connected graph of order $n \geq 4$ having diameter d. Let $\overline{G}$ be the complement of G and let $\lambda_1(A(\overline{G})) \geq \lambda_2(A(\overline{G})) \geq \cdots \geq \lambda_n(A(\overline{G}))$ be the adjacency eigenvalues of $\overline{G}$. If $d = 2$, then for all $k = 1, 2, \ldots, n$, we have*

$$\lambda_k(Q(\overline{G})) + n - 2 \leq \rho_k(G) \leq 2n - 2 + \lambda_k(Q(\overline{G})). \tag{3.13}$$

Equality occurs on the right if and only if $k = 1$ and G is a transmission regular graph.

Proof. Let G be a connected graph of order $n \geq 4$ having diameter d. Let $Deg(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of G and $Deg(\overline{G}) = diag(n-1-d_1, n-1-d_2, \ldots, n-1-d_n)$ be the diagonal matrix of vertex degrees of $\overline{G}$. Let $Q(G) = Deg(G)+A$ be the signless Laplacian matrix of G. Suppose that the diameter d of G is two, then transmission degree $Tr_i = 2n - 2 - d_i$, for all i. Since diameter of G is two, it gives that any two vertices are either adjacent in G or in $\overline{G}$. It then follows that the distance matrix of G can be written as $D(G) = A + 2\overline{A}$, where A and $\overline{A}$ are the adjacency matrices of G and $\overline{G}$, respectively. We have

$$\begin{aligned} D^Q(G) &= Tr(G) + D(G) = (2n-2)I - Deg(G) + A + 2\overline{A} \\ &= (n-1)I + A + \overline{A} + (n-1)I - Deg(G) + \overline{A} \\ &= D^Q(K_n) + Q(\overline{G}), \end{aligned}$$

where I is the identity matrix and J is the all one matrix of order n. Taking $Z = D^Q(G)$, $X = D^Q(K_n)$, $Y = Q(\overline{G})$, $j = 1$ in the first inequality of Lemma 2.2 and using the fact that the eigenvalues of K_n are $2n-2$ with multiplicity one and $n-2$ with multiplicity $n-1$, it follows that

$$\rho_k(G) \leq 2n - 2 + \lambda_k(Q(\overline{G})), \quad \text{for all} \quad k = 1, 2, \ldots, n. \tag{3.14}$$

Taking $Z = D^Q(G)$, $X = D^Q(K_n)$, $Y = Q(\overline{G})$, $j = n$ in the second inequality of Lemma 2.2, it follows that

$$\rho_k(G) \geq n - 2 + \lambda_k(Q(\overline{G})), \quad \text{for all} \quad k = 1, 2, \ldots, n. \tag{3.15}$$

Combining (3.14) and (3.15), the inequality (3.13) follows. Equality occurs in the right inequality (3.13) if and only if equality occurs in (3.14). Suppose that equality occurs in (3.14), then by Lemma 2.2, the eigenvalues $\rho_k, 2n-2$ and $\lambda_k(Q(\overline{G}))$ of the matrices $D^Q(G), X$ and Y have the same unit eigenvector. Since $\mathbf{1} = \frac{1}{n}(1, 1, \ldots, 1)^T$ is the unit eigenvector of X for the eigenvalue $2n-2$, it follows that equality occurs in (3.14) if and only if $\mathbf{1}$ is the unit eigenvector for each of the matrices $D^Q(G), X$ and Y. This gives that G is a transmission regular graph and $\overline{G}$ is a regular graph. Since a graph of diameter 2 is regular if and only if it is transmission regular and complement of a regular graph is regular. Using the fact that for a connected graph G the unit vector $\mathbf{1}$ is an eigenvector for the eigenvalue ρ_1 if and only if G is transmission regular graph, it follows that equality occurs in first inequality if and only if $k=1$ and G is a transmission regular graph. That completes the proof. □

Analogously to the result stated in [38, Theorem 2] for distance matrix, we present, in the sequel, a lower bound for the spectral radius of distance signless Laplacian matrix.

Theorem 3.14. *Let $\{Tr_1, Tr_2, \ldots, Tr_n\}$ be the transmission degree sequence of G, where $n \geq 2$. If $Tr_1 \geq \cdots \geq Tr_n$ and $Tr_l > Tr_n$, where $1 \leq l \leq n-1$. Then*

$$\rho(G) > \frac{2Tr_n + Tr_l - 1 + \sqrt{(2Tr_n - Tr_l)^2 - 8l(Tr_n - Tr_l) + 2(2Tr_n - Tr_l) + 1}}{2}.$$

Proof. Let $V_1 = \{v_1, \ldots, v_l\}$ and $V_2 = V(G) \setminus V_1$. Then $D^Q(G)$ may be partitioned as

$$D^Q(G) = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} Tr_{11} & 0 \\ 0 & Tr_{22} \end{bmatrix},$$

where D_{11} and Tr_{11} are $l \times l$ matrix. Let

$$U = \begin{bmatrix} yI_l & 0 \\ 0 & I_{n-l} \end{bmatrix},$$

for $y > 1$ (to be determined) and $B = U^{-1}D^Q(G)U$, where I_s the $s \times s$ identity matrix. Then

$$B = \begin{bmatrix} D_{11} & \frac{1}{y}D_{12} \\ yD_{21} & D_{22} \end{bmatrix} + \begin{bmatrix} Tr_{11} & 0 \\ 0 & Tr_{22} \end{bmatrix}$$

is a nonnegative irreducible matrix that has the same spectrum as $D^Q(G)$. If $i = 1, \ldots, l$, then since $d_{ii} = 0$ and $d_{ij} \geq 1$ for $j = 1, \ldots, l$

with $i \neq j$, we have

$$\begin{aligned} r_i(B) &= \sum_{j=1}^{l} d_{ij} + \frac{1}{y}\sum_{j=l+1}^{n} d_{ij} + \sum_{j=1}^{n} d_{ij} \\ &= (1+\frac{1}{y})\sum_{j=1}^{n} d_{ij} + (1-\frac{1}{y})\sum_{j=1}^{l} d_{ij} \\ &= (1+\frac{1}{y})Tr_i + (1-\frac{1}{y})\sum_{j=1}^{l} d_{ij} \geq (1+\frac{1}{y})Tr_l + (1-\frac{1}{y})(l-1). \end{aligned}$$

Again, if $i = l+1, \ldots, n$, then since $d_{ij} \geq 1$ for $j = 1, \ldots, l$, we have

$$\begin{aligned} r_i(B) &= y\sum_{j=1}^{l} d_{ij} + \sum_{j=l+1}^{n} d_{ij} + \sum_{j=1}^{n} d_{ij} \\ &= 2\sum_{j=1}^{n} d_{ij} + (y-1)\sum_{j=1}^{l} d_{ij} \\ &= 2Tr_i + (y-1)\sum_{j=1}^{l} d_{ij} \geq 2Tr_n + (y-1)l. \end{aligned}$$

Let

$$y = \frac{2l - 2Tr_n + Tr_l - 1 + \sqrt{(2Tr_n - Tr_l)^2 - 8l(Tr_n - Tr_l) + 2(2Tr_n - Tr_l) + 1}}{2l}.$$

Then

$$(1+\tfrac{1}{y})Tr_l + (1-\tfrac{1}{y})(l-1) = 2Tr_n + (y-1)l$$

$$= \frac{2Tr_n + Tr_l - 1 + \sqrt{(2Tr_n - Tr_l)^2 - 8l(Tr_n - Tr_l) + 2(2Tr_n - Tr_l) + 1}}{2}.$$

Since $Tr_l > Tr_n$, we have $y > 1$. Thus by Lemma 2.1, we have

$$\begin{aligned} \rho(G) &\geq \min_{1\leq i\leq n} r_i(B) \\ &\geq \frac{2Tr_n + Tr_l - 1}{2} \\ &+ \frac{\sqrt{(2Tr_n - Tr_l)^2 - 8l(Tr_n - Tr_l) + 2(2Tr_n - Tr_l) + 1}}{2}. \end{aligned} \tag{3.16}$$

Suppose that equality holds in (3.16). Then

$$r_1(B) = \cdots = r_n(B) = (1+\frac{1}{y})Tr_l + (1-\frac{1}{y})(l-1) = 2Tr_n + (y-1)l.$$

Since $r_i(B) = (1+\frac{1}{y})Tr_l+(1-\frac{1}{y})(l-1)$ for $i = 1,\ldots,l$, we have $d_{ij} = 1$ for $i, j = 1,\ldots,l$, with $i \neq j$, which implies that V_1 induces a complete subgraph in G. Again, since $r_i(B) = 2Tr_n + (y-1)l$ for $i = l+1,\ldots,n$ we have $d_{ij} = 1$ for $i = l+1,\ldots,n$ and $j = 1,\ldots,l$, which implies that every vertex in V_2 is adjacent to all vertices in V_1. Thus the degree of every vertex in V_1 is $n-1$, and then $Tr_1 = \cdots = Tr_l = n-1$, which is a contradiction to the assumption that $Tr_l > Tr_n$. □

Acknowledgments

The authors would like to thank the editor and two anonymous referees for their constructive comments that helped improve the quality of the paper.

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Journal of Algebraic Systems

NEW BOUNDS AND EXTREMAL GRAPHS FOR DISTANCE SIGNLESS LAPLACIAN SPECTRAL RADIUS

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کران‌های جدید و گراف‌های مرزی برای شعاع طیفی ماتریس لاپلاسین بدون‌علامت فاصله

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فرض کنید G یک گراف ساده و همبند از مرتبه n باشد. ماتریس لاپلاسین بدون علامت فاصله گراف G که با نماد $D^Q(G)$ نمایش داده می‌شود، به صورت $D^Q(G) = Tr(G) + D(G)$ تعریف می‌شود، جایی‌که $D(G)$ ماتریس فاصله گراف G و $Tr(G)$ ماتریس قطری متشکل از اعداد انتقال رئوس گراف G می‌باشد. در این مقاله برخی کران‌های بالا و پایین جدید برای شعاع طیفی ماتریس لاپلاسین بدون علامت فاصله گراف G به‌دست آورده و گراف‌هایی که در شرایط مرزی این کران‌ها صدق می‌کنند را مشخص می‌کنیم.

کلمات کلیدی: ماتریس لاپلاسین بدون علامت فاصله، شعاع طیفی، گراف‌های منظم از لحاظ عدد انتقال.