THE ANNihilator GRAPH FOR MODULES OVER COMmutative RINGS

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Abstract. Let \( R \) be a commutative ring and \( M \) be an \( R \)-module. The annihilator graph of \( M \), denoted by \( AG(M) \) is a simple undirected graph associated to \( M \) whose the set of vertices is \( Z_R(M) \setminus \operatorname{Ann}_R(M) \) and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( \operatorname{Ann}_M(xy) \neq \operatorname{Ann}_M(x) \cup \operatorname{Ann}_M(y) \). In this paper, we study the diameter and the girth of \( AG(M) \) and we characterize all modules whose annihilator graph is complete. Furthermore, we look for the relationship between the annihilator graph of \( M \) and its zero-divisor graph.

1. Introduction

Let \( R \) be a commutative ring. The zero-divisor graph of \( R \), denoted by \( \Gamma(R) \) is a simple undirected graph whose vertices are the nonzero zero-divisors of \( R \) and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( xy = 0 \), see [1, 2, 6]. The concept of the zero-divisor graph of a ring, has been generalized for modules in many papers, see [7, 9]. Variations of the zero-divisor graph are created by changing the vertex set, the edge condition, or both. The annihilator graph of \( R \) introduced in [5] and studied in some literatures, see [8, 10, 14]. It is a variation of the zero-divisor graph that changes the edge condition. This graph,
denoted by $AG(R)$ is a graph whose vertices are the nonzero zero-divisors of $R$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{Ann}_R(xy) \neq \text{Ann}_R(x) \cup \text{Ann}_R(y)$.

By relying this fact we introduce the annihilator graph for a module. Let $M$ be an $R$-module. The annihilator graph of $M$, denoted by $AG(M)$ is a simple undirected graph associated to $M$ whose vertices are the elements of $Z_R(M) \setminus \text{Ann}_R(M)$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{Ann}_M(xy) \neq \text{Ann}_M(x) \cup \text{Ann}_M(y)$. We investigate the interplay between the graph theoretic properties of $AG(M)$ and some algebraic properties of $M$.

Let $G = (V(G), E(G))$ be a simple undirected graph, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. Let $x, y \in V(G)$. We write $x \sim y$, whenever $x$ and $y$ are adjacent. A universal vertex is a vertex that is adjacent to all other vertices of the graph. We say that $G$ is connected if there is a path between any two distinct vertices. For vertices $x$ and $y$ of $G$, we define $d(x, y)$ to be the length of a shortest path between $x$ and $y$ (if there is no path, then $d(x, y) = \infty$). The open neighborhood of a vertex $x$ is defined to be the set $N(x) = \{y \in V(G) : d(x, y) = 1\}$. The diameter of $G$ is $\text{diam}(G) = \sup\{d(x, y) : x$ and $y$ are vertices of $G\}$. The graph $G$ is complete if any two distinct vertices are adjacent and a complete graph with $n$ vertices is denoted by $K_n$. A complete bipartite graph $G$ is a graph whose vertices can be partitioned into two disjoint nonempty sets $A$ and $B$ such that two distinct vertices are adjacent if and only if they are in distinct sets and it is denoted by $K_{|A|,|B|}$. The girth of $G$, denoted by $\text{gr}(G)$ is the length of a shortest cycle in $G$ ($\text{gr}(G) = \infty$ if $G$ contains no cycle).

Throughout this paper, $R$ denotes a commutative ring with nonzero identity and $M$ is an $R$-module. Recall that $\text{Ann}_R(M) = \{r \in R : rM = 0\}$, $Z_R(M) = \{r \in R : rm = 0$ for some nonzero $m \in M\}$ and $\text{Ass}_R(M) = \{p \in \text{Spec}(R) : p = \text{Ann}_R(m)$ for some nonzero $m \in M\}$. For $x \in R$, $\text{Ann}_M(x) = \{m \in M : xm = 0\}$. The reader is referred to [15], for notations and terminologies not given in this paper.

2. The annihilator graph for modules

In this section we define a simple undirected graph $AG(M)$ and we study the relations between graph theoretic properties of $AG(M)$ and module theoretic properties of $M$.

Definition 2.1. Let $M$ be an $R$-module. The annihilator graph of $M$, denoted by $AG(M)$ is a simple undirected graph associated to $M$ whose the set of vertices is $Z_R(M) \setminus \text{Ann}_R(M)$ and two distinct vertices $x$ and $y$ are adjacent if and only if $\text{Ann}_M(xy) \neq \text{Ann}_M(x) \cup \text{Ann}_M(y)$.
Lemma 2.2. Let $M$ be an $R$-module and $x, y$ be distinct vertices of $AG(M)$. Then the following statements are true:

(i) If $\text{Ann}_M(x) \not\subset \text{Ann}_M(y)$ and $\text{Ann}_M(y) \not\subset \text{Ann}_M(x)$, then $x, y$ are adjacent in $AG(M)$.

(ii) If $x, y$ are not adjacent in $AG(M)$, then either $\text{Ann}_M(x) \subset \text{Ann}_M(y)$ or $\text{Ann}_M(y) \subset \text{Ann}_M(x)$.

(iii) If $x, y$ are not adjacent in $AG(M)$, then either $\text{Ann}_R(xM) \subset \text{Ann}_R(yM)$ or $\text{Ann}_R(yM) \subset \text{Ann}_R(xM)$.

(iv) $x, y$ are not adjacent in $AG(M)$ if and only if either $\text{Ann}_M(xy) = \text{Ann}_M(x)$ or $\text{Ann}_M(xy) = \text{Ann}_M(y)$.

Proof. (i) Suppose that $x, y$ are not adjacent in $AG(M)$. Thus $\text{Ann}_M(x) \cup \text{Ann}_M(y) = \text{Ann}_M(xy)$. So $\text{Ann}_M(xy) = \text{Ann}_M(x)$ or $\text{Ann}_M(xy) = \text{Ann}_M(y)$. Hence, $\text{Ann}_M(x) \subset \text{Ann}_M(y)$ or $\text{Ann}_M(y) \subset \text{Ann}_M(x)$ which is a contradiction.

(ii) It is contrapositive of part (i).

(iii) Suppose that $x, y$ are not adjacent in $AG(M)$. It follows that either $\text{Ann}_M(x) \subset \text{Ann}_M(y)$ or $\text{Ann}_M(y) \subset \text{Ann}_M(x)$, by (ii). Let $\text{Ann}_M(x) \subset \text{Ann}_M(y)$ and $r \in \text{Ann}_R(xM)$. Then $rxM = 0$ and so $rM \subset \text{Ann}_M(x)$. Hence, $rM \subset \text{Ann}_M(y)$ and then $ryM = 0$. Therefore, $r \in \text{Ann}_R(yM)$. So $\text{Ann}_R(xM) \subset \text{Ann}_R(yM)$.

(iv) It is obvious by the proof of part (i). □

Lemma 2.3. Let $M$ be an $R$-module and $x, y$ be distinct vertices of $AG(M)$. Let $x \notin r(\text{Ann}_R(M)) = \{x \in R : x^t \in \text{Ann}_R(M) \text{ for some } t \in \mathbb{N}\}$ and $\text{Ann}_M(x)$ be a prime submodule of $M$. Then $x, y$ are adjacent in $AG(M)$ if and only if $\text{Ann}_M(y) \not\subset \text{Ann}_M(x)$.

Proof. Assume that $\text{Ann}_M(y) \not\subset \text{Ann}_M(x)$ and $m \in \text{Ann}_M(y) \setminus \text{Ann}_M(x)$. Then $ym = 0 \in \text{Ann}_M(x)$. Since $\text{Ann}_M(x)$ is a prime submodule of $M$, $xyM = 0$. So $\text{Ann}_M(x) \cup \text{Ann}_M(y) \neq \text{Ann}_M(xy)$. Conversely, suppose that $\text{Ann}_M(x) \cup \text{Ann}_M(y) \neq \text{Ann}_M(xy)$. Thus there exists $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. If $\text{Ann}_M(y) \subset \text{Ann}_M(x)$, then $xm \in \text{Ann}_M(x)$ and $m \notin \text{Ann}_M(x)$ which implies that $x^2M = 0$ and it is a contradiction. Hence, $\text{Ann}_M(y) \not\subset \text{Ann}_M(x)$. □

Theorem 2.4. Let $M$ be an $R$-module and $x, y$ be distinct vertices of $AG(M)$. Then the following statements are equivalent:

(i) $x, y$ are adjacent in $AG(M)$.

(ii) $xM \cap \text{Ann}_M(y) \neq 0$ and $yM \cap \text{Ann}_M(x) \neq 0$.

(iii) $x \in Z_R(yM)$ and $y \in Z_R(xM)$.

Proof. (i) $\Rightarrow$ (ii) Let $x, y$ be distinct vertices of $AG(M)$. Then there exists $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. So $xM \cap \text{Ann}_M(y) \neq 0$ and $yM \cap \text{Ann}_M(x) \neq 0$. 

Let \( y \) be a minimal element of \( \text{Ass}(M) \), for each \( j \) with \( 1 \leq j \leq n \). Then \( \cap_{i=1}^{n} \text{Ann}(M/Q_{i}) \neq \emptyset \). Suppose that \( a_{j} \in \cap_{i=1, i \neq j}^{n} \text{Ann}(M/Q_{i}) \) \( \setminus p_{j} \). We show that \( \text{Ann}_{M}(a_{j}) = Q_{j} \). We have \( \text{Ann}_{M}(a_{j}) = (0 :_{M} a_{j}) = (\cap_{i=1}^{n} Q_{i} :_{M} a_{j}) = \cap_{i=1}^{n} (Q_{i} :_{M} a_{j}) = (Q_{j} :_{M} a_{j}) \). It is clear that \( Q_{j} \subseteq (Q_{j} :_{M} a_{j}) \). If there exists \( m \in (Q_{j} :_{M} a_{j}) \) with \( m \not\in Q_{j} \), then \( a_{j}^{t} M \subseteq Q_{j} \) for some \( t \in \mathbb{N} \) and so \( a_{j} \in p_{j} \) which is a contradiction. Hence, \( Q_{j} = \text{Ann}_{M}(a_{j}) \).

Let \( M \) be an \( R \)-module. Then the zero submodule is a primary submodule of \( M \) if and only if \( Z_{R}(M) = r(\text{Ann}_{R}(M)) \).

**Theorem 2.6.** Let \( M \) be a Noetherian \( R \)-module. Then \( AG(M) \) is a complete graph if and only if \( Z_{R}(M) = r(\text{Ann}_{R}(M)) \).

**Proof.** Let \( 0 = \cap_{i=1}^{n} Q_{i} \) be a minimal primary decomposition of the zero submodule of \( M \) with \( r(\text{Ann}_{R}(M/Q_{i})) = p_{i} \), for each \( i = 1, \ldots, n \). Let \( p_{j} \) be a minimal element of \( \text{Ass}(M) \), for some \( 1 \leq j \leq n \). Then by Lemma 2.5, there exists \( a_{j} \in \cap_{i=1, i \neq j}^{n} \text{Ann}(M/Q_{i}) \) \( \setminus p_{j} \) such that \( Q_{j} = \text{Ann}_{M}(a_{j}) \). Suppose that \( c \in Z_{R}(M) \) \( \setminus \text{Ann}_{R}(M) \) and \( c \neq a_{j} \). By the hypothesis \( c, a_{j} \) are adjacent in \( AG(M) \). So \( \text{Ann}_{M}(a_{j}) \cup \text{Ann}_{M}(c) \neq \text{Ann}_{M}(a_{j}c) \). Thus there exists \( m \in M \) such that \( a_{j}cm = 0 \) but \( a_{j}m \neq 0 \). Hence, \( c^{t} M \subseteq Q_{j} \) for some \( t \in \mathbb{N} \) so \( c^{t} \in \text{Ann}_{R}(M/Q_{j}) \subseteq p_{j} \). Therefore,
\( Z_R(M) = p_j \cup \{a_j\} \). Let \( p_j \subseteq p_k \), for some \( 1 \leq k \leq n \). Since \( p_k \subseteq Z_R(M) = p_j \cup \{a_j\} \), \( p_k = p_j \cup \{a_j\} \) which is a contradiction. Hence, \( n = 1 \) and so 0 is a primary submodule of \( M \). So \( \text{Ass}_R(M) = \{p_j\} \) and consequently \( Z_R(M) = r(\text{Ann}_R(M)) \).

\[ \leq \] Let \( Z_R(M) = r(\text{Ann}_R(M)) \) and let \( x, y \in Z_R(M) \setminus \text{Ann}_R(M) \) be two distinct vertices of \( AG(M) \). Then \( \text{Ann}_M(x) \) and \( \text{Ann}_M(y) \) are essential submodules of \( M \) by [3, Theorem 5]. So \( xM \cap \text{Ann}_M(y) \neq 0 \) and \( yM \cap \text{Ann}_M(x) \neq 0 \). Hence, \( x, y \) are adjacent in \( AG(M) \) by Theorem 2.4.

The following example has been presented to show that the property of being Noetherian is a necessary condition in Theorem 2.6.

**Example 2.7.** Consider \( M = \mathbb{Z}_{p^\infty} \) as a \( \mathbb{Z} \)-module, where \( p \) is a prime integer. It is easy to see that \( AG(M) \) is a complete graph but \( Z_\mathbb{Z}(M) = p\mathbb{Z} \) and \( r(\text{Ann}_\mathbb{Z}(M)) = 0 \).

**Proposition 2.8.** Let \( M \) be an \( R \)-module and \( x, y \) be distinct vertices of \( AG(M) \). If \( \text{Ann}_M(x) = \text{Ann}_M(y) \), then \( N_{AG(M)}(x) = N_{AG(M)}(y) \).

**Proof.** Let \( z \in Z_R(M) \setminus \text{Ann}_R(M) \) and \( z \in N_{AG(M)}(x) \). Then there exists \( m \in M \) such that \( xzm = 0 \) but \( xm \neq 0 \neq zm \). So \( zm \in \text{Ann}_M(y) \) and \( ym \neq 0 \neq zm \). It means that \( y, z \) are adjacent in \( AG(M) \). Hence, \( z \in N_{AG(M)}(y) \). The reverse inclusion can be proved similarly. \( \square \)

3. Relation between the zero-divisor graph and the annihilator graph

Let \( M \) be an \( R \)-module. The zero-divisor graph of \( M \), denoted by \( \Gamma(M) \), is a simple undirected graph associated to \( M \) whose vertices are the elements of \( Z_R(M) \setminus \text{Ann}_R(M) \) and two distinct vertices \( x \) and \( y \) are adjacent if and only if \( xyM = 0 \), see [11].

**Lemma 3.1.** Let \( M \) be an \( R \)-module and \( x, y \) be distinct vertices of \( AG(M) \). Then the following statements are true:

(i) If \( x, y \) are adjacent in \( \Gamma(M) \), then \( x, y \) are adjacent in \( AG(M) \).
   In particular, if \( P \) is a path in \( \Gamma(M) \), then \( P \) is a path in \( AG(M) \).

(ii) If \( d_{\Gamma(M)}(x, y) = 3 \), then \( x, y \) are adjacent in \( AG(M) \).

**Proof.** (i) Suppose that \( x, y \) are adjacent in \( \Gamma(M) \). Thus \( xyM = 0 \) and so \( \text{Ann}_M(xy) = M \); but \( \text{Ann}_M(x) \neq M \) and \( \text{Ann}_M(y) \neq M \). Hence, \( \text{Ann}_M(xy) \neq \text{Ann}_M(x) \cup \text{Ann}_M(y) \) and \( x, y \) are adjacent in \( AG(M) \).

(ii) Suppose that \( d_{\Gamma(M)}(x, y) = 3 \). Thus \( xyM \neq 0 \) and there exist \( a, b \in Z_R(M) \setminus \text{Ann}_R(M) \cup \{x, y\} \) such that \( axM = 0, abM = 0 \) and
by $M = 0$. If $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$, then in view of $axM = 0$ it follows that $aM \subseteq \text{Ann}_M(x) \subseteq \text{Ann}_M(y)$. Thus $ayM = 0$ which contradicts to the hypothesis. Hence, $\text{Ann}_M(x) \not\subseteq \text{Ann}_M(y)$. By a similar argument one can show that $\text{Ann}_M(y) \not\subseteq \text{Ann}_M(x)$. Therefore, $x, y$ are adjacent in $AG(M)$ by Lemma 2.2(i).

Lemma 3.2. Let $M$ be an $R$-module and $x, y$ be distinct vertices of $AG(M)$. If $\text{Ann}_M(x)$ and $\text{Ann}_M(y)$ are distinct prime submodules of $M$, then $x, y$ are adjacent in $\Gamma(M)$ and so are adjacent in $AG(M)$.

Proof. Assume that $P_1 = \text{Ann}_M(x), P_2 = \text{Ann}_M(y)$ are two distinct prime submodules of $M$ and $m \in P_1 \setminus P_2$. Thus $xm = 0 \in P_2$ which implies that $xM \subseteq P_2 = \text{Ann}_M(y)$. Hence, $xyM = 0$ and so $x, y$ are adjacent in $\Gamma(M)$. The second assertion follows by Lemma 3.1(i). □

Let $M$ be an $R$-module and $\text{Spec}_R(M)$ denote the set of prime submodules of $M$. Then $m - \text{Ass}_R(M) = \{P \in \text{Spec}_R(M) : P = \text{Ann}_M(a) \text{ for some } 0 \neq a \in R\}$.

Corollary 3.3. Let $M$ be an $R$-module such that for every edge of $AG(M)$, $x \sim y$ say, either $\text{Ann}_M(x) \in m - \text{Ass}_R(M)$ or $\text{Ann}_M(y) \in m - \text{Ass}_R(M)$. Then $\Gamma(M) = AG(M)$.

Proof. In view of Lemma 3.1(i), $\Gamma(M)$ is a subgraph of $AG(M)$. Let $x, y$ be adjacent vertices of $AG(M)$ and let either $\text{Ann}_M(x) \in m - \text{Ass}_R(M)$ or $\text{Ann}_M(y) \in m - \text{Ass}_R(M)$. Without loss of generality we may assume that $\text{Ann}_M(x) \in m - \text{Ass}_R(M)$. Thus $\text{Ann}_M(xy) \neq \text{Ann}_M(x) \cup \text{Ann}_M(y)$. Hence, there is $m \in M$ such that $xym = 0$ but $xm \neq 0 \neq ym$. Therefore, $ym \in \text{Ann}_M(x)$ and $m \notin \text{Ann}_M(x)$. So $xyM = 0$ since $\text{Ann}_M(x)$ is a prime submodule of $M$ and $x$ and $y$ are adjacent in $\Gamma(M)$. □

Theorem 3.4. Let $M$ be an $R$-module and $\Gamma(M)$ be a connected graph. Then $AG(M)$ is a connected graph and $\text{diam}(AG(M)) \leq 2$.

Proof. Suppose that $x, y$ are distinct non-adjacent vertices of $AG(M)$. Thus by Lemma 2.2(ii), either $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$ or $\text{Ann}_M(y) \subseteq \text{Ann}_M(x)$. Without loss of generality we may assume that $\text{Ann}_M(x) \subseteq \text{Ann}_M(y)$. Thus $\text{Ann}_R(xM) \subseteq \text{Ann}_R(yM)$, by Lemma 2.2(iii). Since $x$ is not an isolated vertex of $\Gamma(M)$, thus there exists $z \in \text{Ann}_R(xM) \setminus \text{Ann}_R(M)$ such that $xzM = 0$. So $yzM = 0$. Hence, $x \sim z \sim y$ is a path in $\Gamma(M)$ and so is a path in $AG(M)$. □

Theorem 3.5. Let $M$ be a Noetherian $R$-module and $\Gamma(M)$ be a connected graph. Then $\text{gr}(AG(M)) \in \{3, 4, \infty\}$.
Proof. If \( \Gamma(M) = AG(M) \), then in view of [11, Theorem 3.3], \( \text{gr}(AG(M)) \in \{3, 4, \infty\} \). Now, suppose that \( \Gamma(M) \neq AG(M) \) and \( x, y \) are two distinct adjacent vertices of \( AG(M) \) such that they are non-adjacent in \( \Gamma(M) \). Since \( \Gamma(M) \) is a connected graph, there exist \( a, b \in Z_R(M) \setminus \text{Ann}_R(M) \cup \{x, y\} \) such that \( axM = byM = 0 \). If \( a = b \), then \( x \sim a \sim y \) is a path in \( \Gamma(M) \) and so \( x \sim a \sim y \sim x \) is a cycle in \( AG(M) \) of length three. So we may assume that \( a \neq b \). If \( abM = 0 \), then \( x \sim a \sim b \sim y \) is a path in \( \Gamma(M) \). Thus \( x \sim a \sim b \sim y \sim x \) is a cycle in \( AG(M) \) of length four. If \( abM \neq 0 \), then \( x \sim ab \sim y \sim x \) is a path in \( \Gamma(M) \) and so \( x \sim ab \sim y \sim x \) is a cycle in \( AG(M) \) of length three. Therefore, \( \text{gr}(AG(M)) \in \{3, 4, \infty\} \). \( \square \)

Consider \( Z_8 \) as a \( \mathbb{Z}_8 \)-module. It is easy to see that \( \text{gr}(AG(Z_8)) = 3 \) and \( \text{gr}(\Gamma(Z_8)) = \infty \).

**Theorem 3.6.** Let \( M \) be a Noetherian \( R \)-module and \( AG(M) \) be a complete graph. Then \( c \in Z_R(M) \setminus \text{Ann}_R(M) \) is a universal vertex of \( \Gamma(M) \) if and only if \( \text{Ann}_M(c) \) is a prime submodule of \( M \).

**Proof.** Let \( c \in Z_R(M) \setminus \text{Ann}_R(M) \) be a universal vertex of \( \Gamma(M) \). We show that \( \text{Ann}_M(c) \) is a prime submodule of \( M \). Assume that \( x, m \in M \setminus \text{Ann}_M(c) \) and \( xm \in \text{Ann}_M(c) \). By [11, Theorem 2.1], \( Z_R(M) = \text{Ann}_R(cm) \) and \( x \in Z_R(M) \) thus \( xM \subseteq \text{Ann}_M(c) \) as desired. Hence, \( \text{Ann}_M(c) \) is a prime submodule of \( M \).

Suppose that \( c \in Z_R(M) \setminus \text{Ann}_R(M) \) and \( \text{Ann}_M(c) \) is a prime submodule of \( M \). We show that \( c \) is a universal vertex of \( \Gamma(M) \). Let \( x \in Z_R(M) \setminus \text{Ann}_R(M) \) be a vertex of \( \Gamma(M) \) and \( x \neq c \). In view of the assumption \( AG(M) \) is a complete graph so there exists \( m \in \text{Ann}_M(cx) \) such that \( xm \neq 0 \neq cm \). Thus \( xm \in \text{Ann}_M(c) \) and \( cm \neq 0 \). Hence, \( xcM = 0 \) and so \( c, x \) are adjacent in \( \Gamma(M) \). \( \square \)

**Corollary 3.7.** Let \( M \) be a Noetherian \( R \)-module and \( AG(M) \) be a complete graph with \( |Z_R(M) \setminus \text{Ann}_R(M)| \geq 3 \). If \( \Gamma(M) \) is a star graph, then \( |m - \text{Ass}_R(M)| = 1 \).

**Proof.** Let \( Z_R(M) \setminus \text{Ann}_R(M) = \{a, b, c, \cdots\} \) and let \( \Gamma(M) \) be a star graph. If \( P_1 = \text{Ann}_M(a) \) and \( P_2 = \text{Ann}_M(b) \) are prime submodules of \( M \), then by Theorem 3.6, \( a \) and \( b \) are universal vertices of \( \Gamma(M) \) which is a contradiction. Thus \( |m - \text{Ass}_R(M)| \leq 1 \). Since \( M \) is Noetherian, \( |m - \text{Ass}_R(M)| \geq 1 \). \( \square \)

Consider \( Z_8 \) as a \( \mathbb{Z}_8 \)-module. It is easy to check that \( AG(Z_8) \) is a complete graph and \( m - \text{Ass}_{\mathbb{Z}_8}(Z_8) = \{2\mathbb{Z}\} \) but \( \Gamma(Z_8) \) is not a star graph. Note that 4 is a universal vertex of \( \Gamma(Z_8) \). Also, 2 \( \sim 12 \) in \( \Gamma(Z_8) \).
Theorem 3.8. Let \( M \) be an \( R \)-module and \( \Gamma(M) \) be a star graph with the universal vertex \( c \). Then the following statements are true:

(i) If \( c \notin r(\text{Ann}_R(M)) \), then \( \Gamma(M) = K_1 \).

(ii) If \( c \in r(\text{Ann}_R(M)) \), then \( \Gamma(M) = K_{1,1} \) or \( Rc = cZ_R(M) \cup \{c\} \).

Proof. (i) In [11, Theorem 2.1], it has been proved that \( Z_R(M) = \text{Ann}_R(cM) \cup \{c\} \) and \( c = c^2 \). If there exists \( a \in R \setminus Z_R(M) \) such that \( ac \neq c \), then \( ac \) and \( x \) are adjacent for all \( x \in Z_R(M) \setminus \text{Ann}_R(M) \) which is a contradiction. So \( ac = c \) and \( \Gamma(M) = K_1 \). Let \( ac = c \), for all \( a \in R \setminus Z_R(M) \). Then \( Rc = cZ_R(M) \cup c(R \setminus Z_R(M)) = cZ_R(M) \cup \{c\} = c\text{Ann}_R(cM) \cup \{c\} \). In this case we have \( R = \mathbb{Z}_2 \oplus R' \) and \( M = \mathbb{Z}_2 \oplus M' \), where \( R' \) is a subring of \( R \) and \( M' \) is an \( R \)-submodule of \( M \). Moreover \( c = (1,0) \) and \( \text{Ann}_R(cM) = 0 \times R' \), see [11, Theorem 2.2]. Thus \( c\text{Ann}_R(cM) = c(0 \times R') = \{(0,0)\} \). Hence, \( Rc = \{(0,0), c = (1,0)\} \).

(ii) It is easy to see that \( c \neq c^2 \). If \( c^2 \notin \text{Ann}_R(M) \), then \( \Gamma(M) = K_1 \). Let \( c^2M = 0 \). If there exists \( a \in R \setminus Z_R(M) \) such that \( ac \neq c \), then \( \Gamma(M) = K_{1,1} \). Suppose that \( ac = c \) for each \( a \in R \setminus Z_R(M) \). Thus \( Rc = cZ_R(M) \cup c(R \setminus Z_R(M)) = cZ_R(M) \cup \{c\} \).

A proper submodule \( P \) of \( M \) is said to be a weakly prime submodule whenever \( 0 \neq rm \in P \) with \( r \in R \) and \( m \in M \), then either \( m \in P \) or \( r \in \text{Ann}_R(M/P) \).

Lemma 3.9. Let \( M \) be an \( R \)-module and \( x \in Z_R(M) \setminus \text{Ann}_R(M) \). Then \( \text{Ann}_M(x) \) is a weakly prime submodule of \( M \) if and only if \( N_{\Gamma(M)}(x) = N_{AG(M)}(x) \).

Proof. \( \Rightarrow \) It is enough to show that \( N_{AG(M)}(x) \subseteq N_{\Gamma(M)}(x) \). Suppose that \( x, y \) are adjacent in \( AG(M) \). Then there exists \( m \in \text{Ann}_M(xy) \) such that \( m \notin \text{Ann}_M(x) \cup \text{Ann}_M(y) \). So \( 0 \neq ym \in \text{Ann}_M(x) \) and \( m \notin \text{Ann}_M(x) \). Since \( \text{Ann}_M(x) \) is a weakly prime submodule of \( M \), thus \( xyM = 0 \). Hence, \( x, y \) are adjacent in \( \Gamma(M) \) and the proof is completed.

\( \Leftarrow \) Suppose that \( x \in Z_R(M) \setminus \text{Ann}_R(M) \) and \( N_{\Gamma(M)}(x) = N_{AG(M)}(x) \). We have to show that \( \text{Ann}_M(x) \) is a weakly prime submodule of \( M \). Let \( 0 \neq ym \in \text{Ann}_M(x) \), for some \( m \in M \) and \( y \in R \) with \( x \neq y \). If \( xym = 0 \) we are done; otherwise \( y \in Z_R(M) \setminus \text{Ann}_R(M) \) and \( xym = 0 \). Thus \( m \in \text{Ann}_M(xy) \setminus \text{Ann}_M(x) \cup \text{Ann}_M(y) \). It means that \( x, y \) are adjacent in \( AG(M) \) and so they are adjacent in \( \Gamma(M) \). Hence, \( xyM = 0 \) and \( yM \subseteq \text{Ann}_M(x) \), as desired. Now, assume that \( 0 \neq xM \in \text{Ann}_M(x) \), for some \( m \in M \). Thus \( x^2M = 0 \) and so \( x \neq x^2 \). We show that \( x^2M = 0 \). In this case \( (x - x^2)m = xM \neq 0 \), so \( x - x^2 \) is a vertex of \( AG(M) \) and let \( x \neq x - x^2 \). Moreover \( x(x - x^2)M = 0 \) thus \( x, x - x^2 \) are adjacent in \( AG(M) \) so by the hypotheses \( x(x - x^2)M = 0 \). Hence, \( x^2(1 - x)M = 0 \).
If $1 - x \not\in Z_R(M)$, then $x^2M = 0$ and we are done. Otherwise, $1 - x \in Z_R(M)$. Since $(x - x^2)m \neq 0$, $1 - x \in Z_R(M) \setminus \text{Ann}_R(M)$. Hence, $1 - x$ is a vertex of $AG(M)$; moreover $\text{Ann}_M(x) \cap \text{Ann}_M(1 - x) = 0$. Therefore, $\text{Ann}_M(1 - x) \not\subseteq \text{Ann}_M(x)$ and $\text{Ann}_M(x) \not\subseteq \text{Ann}_M(1 - x)$. So $x, 1 - x$ are adjacent in $AG(M)$, by Lemma 2.2(ii). Thus $(1 - x)M = 0$ which implies that $(x - x^2)m = xm = 0$ contrary to the assumption.\[\square\]

**Lemma 3.10.** Let $M$ be an $R$-module and $x \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. Then $\text{Ann}_M(x)$ is a prime submodule of $M$ if and only if $N_{\Gamma(M)}(x) = N_{\text{AG}(M)}(x)$.

**Proof.** $\Rightarrow$ It is clear that a prime submodule of $M$ is a weakly prime submodule so the result follows by Lemma 3.9.

$\Leftarrow$ Let $x \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. We show that $\text{Ann}_M(x)$ is a prime submodule of $M$. Assume that $xm \in \text{Ann}_M(x)$, for some $m \in M$. If $xm = 0$ there is nothing to prove; so suppose that $xm \neq 0$. Thus $x \neq x^2$. We show that $x^2M = 0$. If $x^2M \neq 0$, then $x^2 \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$ and so $x, x^2$ are adjacent in $AG(M)$, see [3, Theorem 5] and Theorem 2.4, so $x, x^2$ are adjacent in $\Gamma(M)$. Hence, $x^3M = 0$. In this case $x - x^2$ is a vertex of $AG(M)$ and $x \neq x - x^2$. Moreover $x, x - x^2$ are adjacent in $AG(M)$ and so $x(x - x^2)M = 0$. Thus $0 = x^2M - x^3M = x^2M$ contrary to the assumption. Therefore, $x^2M = 0$, as desired. Let $0 \neq ym' \in \text{Ann}_M(x)$, for some $m' \in M$ and $y \in R$ with $x \neq y$. If either $xm' = 0$ or $yM = 0$, then there is nothing to prove. Otherwise, $xm' \neq 0$ and $y \in Z_R(M) \setminus \text{Ann}_R(M)$. Thus $m' \in \text{Ann}_M(xy) \setminus \text{Ann}_M(x) \cup \text{Ann}_M(y)$. It means that $x, y$ are adjacent in $AG(M)$ and so $x, y$ are adjacent in $\Gamma(M)$. Hence, $xyM = 0$ and so $yM \subseteq \text{Ann}_M(x)$ as desired. If $ym' = 0$ and $xyM \neq 0$, then $m' \in \text{Ann}_M(y) \setminus \text{Ann}_M(x)$ and there exists $m'' \in M$ such that $xm'' \in \text{Ann}_M(x) \setminus \text{Ann}_M(y)$. By Lemma 2.2(i), $x, y$ are adjacent in $AG(M)$ and so are adjacent in $\Gamma(M)$ which is a contradiction. Hence, $xyM = 0$.\[\square\]

**Corollary 3.11.** Let $M$ be an $R$-module. If $\Gamma(M) = AG(M)$, then $\text{Ann}_M(x) \in m - \text{Ass}_R(M)$, for each $x \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$.

4. **Two absorbing submodules and the annihilator graph**

Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is called $2$-absorbing if whenever $abm \in N$ for $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in \text{Ann}_R(M/N)$. The reader is referred to [12, 13] for more results and examples about 2-absorbing submodules.

**Theorem 4.1.** Let $M$ be an $R$-module. Then $\Gamma(M) = AG(M)$ if and only if $0$ is a 2-absorbing submodule of $M$. 

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Proof. \(\Rightarrow\) Let \(\Gamma(M) = AG(M), \) \(x, y \in R\) and \(m \in M\) be such that \(xym = 0\). First of all assume that \(x = y\). In this case \(x^2m = 0\). If \(xym = 0\) we are done; otherwise \(x \in Z_R(M) \setminus \text{Ann}_R(M)\). By Lemma 3.9, \(\text{Ann}_M(x)\) is a weakly prime submodule of \(M\). \(x^2m = 0\) and \(xym \neq 0\) imply that \(x^2M = 0\). Hence, 0 is a 2-absorbing submodule of \(M\). Now suppose that \(x \neq y\). If either \(xym = 0\) or \(ym = 0\), we are done. Let \(xym \neq 0\) and \(ym \neq 0\). Then \(x, y \in Z_R(M) \setminus \text{Ann}_R(M)\) and \(m \in \text{Ann}_M(xy) \setminus \text{Ann}_M(x) \cup \text{Ann}_M(y)\). It means that \(x, y\) are adjacent in \(AG(M)\) and so they are adjacent in \(\Gamma(M)\). So \(xyM = 0\) which implies that 0 is a 2-absorbing submodule of \(M\).

\(\Leftarrow\) It is enough to show that an arbitrary edge of \(AG(M)\) is an edge of \(\Gamma(M)\). Let \(x, y \in Z_R(M) \setminus \text{Ann}_R(M)\) be distinct adjacent vertices of \(AG(M)\). Then there exists \(m \in M\) such that \(xym = 0\) but \(xym \neq 0 \neq ym\). Hence, \(axyM = 0\) since 0 is a 2-absorbing submodule of \(M\). Therefore, \(x \) and \(y\) are adjacent in \(\Gamma(M)\). \(\square\)

The following corollary is a generalization of [5, Theorem 3.6].

**Corollary 4.2.** Let \(M\) be an \(R\)-module. If \(\Gamma(M) = AG(M)\), then \(|\text{MinAss}(M)| \leq 2\).

**Proof.** It follows easily by Theorem 4.1, [12, Theorem 2.3] and [4, Theorem 2.4]. \(\square\)

**Theorem 4.3.** Let \(N\) be a 2-absorbing submodule of a Noetherian \(R\)-module \(M\) such that \(r(N :_R M) = p \cap q\), where \(p\) and \(q\) are distinct prime ideals of \(R\) that are minimal over \(N :_R M\). Then \(\text{Ass}_R(M/N)\) is union of two totally ordered sets.

**Proof.** Let \(N = \cap_{i=1}^n Q_i\) be a minimal primary decomposition of \(N\) with \(r(\text{Ann}_R(M/Q_i)) = p_i\), for each \(1 \leq i \leq n\). Then \(r(N :_R M) = \cap_{i=1}^n r(Q_i :_R M) = \cap_{i=1}^n p_i\) and so \(p \cap q = \cap_{i=1}^n p_i\). Without loss of generality we may assume that \(p = p_1\) and \(q = p_2\). Suppose that \(3 \leq k, t \leq n\) and \(k \neq t\). By the definition of a minimal primary decomposition there exist \(m_k \in \cap_{i \neq k} Q_i \setminus Q_k\) and \(m_t \in \cap_{i \neq t} Q_i \setminus Q_t\). Thus \(r(N :_R m_k) = r(\cap_{i=1}^n Q_i :_R m_k) = \cap_{i=1}^n r(Q_i :_R m_k) = r(Q_k :_R M) = p_k\) and \(r(N :_R m_t) = r(\cap_{i=1}^n Q_i :_R m_t) = r(Q_t :_R M) = p_t\). Let \(p_k \not\subseteq p_t\); we show that \(p_k \not\subseteq p_t\). By the hypotheses we may assume that \(p_1 \subseteq p_t\) moreover we can assume that \(p_t \not\subseteq p_k \cup p_2\). Suppose that \(a \in p_k\) and \(b \in p_t \setminus p_k \cup p_2\). So there exists \(s \in N\) such that \(a^s m_k \in N, b^s m_t \in N\) and \(b^s m_k \not\in N\). If \(a^s (m_k + m_t) \in N\), then \(a \in p_t\) and the proof is completed. Now, let \(a^s (m_k + m_t) \not\in N\). Then \(a^s b^s \in N :_R M\) since \(b^s (m_k + m_t) \not\in N\) and \(a^s b^s (m_k + m_t) \in N\). From \(ab \in p_1 \cap p_2\) and \(b \not\in p_1 \cup p_2\) it follows that \(a \in p_1 \cap p_2\). So \(a^s M \not\subseteq N\) and
\( a \cdot m_t \in N \) which implies that \( a \in p_t \). Hence, \( \text{Ass}_R(M/N) \) is union of two totally ordered sets such as \( \text{Ass}_R(M/N) = \{ p = p_1 \} \cup \{ p_2, p_3, \ldots, p_n \} \) or \( \text{Ass}_R(M/N) = \{ q = p_2 \} \cup \{ p_1, p_3, \ldots, p_n \} \).

In [10, Theorem 2.5], it is shown that \( \Gamma(R) = AG(R) \) whenever for every edges of \( AG(R) \), \( x \sim y \) say, either \( \text{Ann}_R(x) \in \text{Ass}(R) \) or \( \text{Ann}_R(y) \in \text{Ass}(R) \). Also the following question is posed: Let \( R \) be a non-reduced ring and \( x \sim y \) be an edge of \( AG(R) \). If \( \Gamma(R) = AG(R) \), then is it true either \( \text{Ann}_R(x) \in \text{Ass}(R) \) or \( \text{Ann}_R(y) \in \text{Ass}(R) \)?

The following theorem is an affirmative answer to this question.

**Theorem 4.4.** Let \( M \) be a Noetherian \( R \)-module. Then the following statements are equivalent:

1. For each edge of \( AG(M) \), \( x \sim y \) say, \( \text{Ann}_M(x) \in m - \text{Ass}_R(M) \) or \( \text{Ann}_M(y) \in m - \text{Ass}_R(M) \).
2. \( \Gamma(M) = AG(M) \).
3. For each \( x \in Z_R(M) \setminus \text{Ann}_R(M) \), \( \text{Ann}_M(x) \) is a weakly prime submodule of \( M \).

**Proof.** It is enough to prove \( (ii) \Rightarrow (i) \). Let \( x \sim y \) be an edge of \( AG(M) \). Since \( \Gamma(M) = AG(M) \) by Theorem 4.1 the zero submodule of \( M \) is 2-absorbing. Thus \( r(\text{Ann}_R(M)) = p \) or \( r(\text{Ann}_R(M)) = p_1 \cap p_2 \), where \( p_1, p_2 \) are prime ideals of \( R \) that are minimal over \( \text{Ann}_R(M) \). If \( r(\text{Ann}_R(M)) = p \), then by \( xyM = 0 \) it follows that \( xy \in \text{Ann}_R(M) \subseteq p \). So \( x \in p \) or \( y \in p \). Hence, \( \text{Ann}_M(x) \in m - \text{Ass}_R(M) \) or \( \text{Ann}_M(y) \in m - \text{Ass}_R(M) \). Now, let \( r(\text{Ann}_R(M)) = p_1 \cap p_2 \). If either \( x \) or \( y \) belongs to \( r(\text{Ann}_R(M)) \), there is nothing to prove. So assume that \( x \in p_1 \setminus p_2 \) and \( y \in p_2 \setminus p_1 \). Then by using Theorem 4.3 we get either \( \text{Ann}_M(x) = Q_2 \) or \( \text{Ann}_M(y) = Q_1 \). Without loss of generality suppose that \( \text{Ann}_M(x) = Q_2 \). We show that the primary submodule \( \text{Ann}_M(x) = Q_2 \) is prime. Let \( a \in R, m \in M \setminus \text{Ann}_M(x) \) and \( am \in \text{Ann}_M(x) = Q_2 \). Then \( a \in p_2 \) and so \( ax \in p_1 p_2 \subseteq \text{Ann}_R(M) \) which implies that \( aM \subseteq \text{Ann}_M(x) \). Therefore, \( \text{Ann}_M(x) \) is a prime submodule of \( M \). \( \square \)

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THE ANNIHILATOR GRAPH FOR MODULES OVER COMMUTATIVE RINGS

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Abstract

The annihilator graph $AG(M)$ of a module $M$ over a commutative ring $R$ is defined as the graph with vertices $R$-modules and two modules $M$ and $N$ are adjacent if the annihilator of $M$ is contained in the annihilator of $N$. In this paper, we study the annihilator graph and its properties, including its diameter, girth, and connectivity. We also investigate the relationship between the annihilator graph and other graph invariants, such as the clique number and chromatic number. Our results provide new insights into the structure of modules and their annihilator graphs.

Keywords: annihilator graph, module graph, commutative ring.