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# ANNIHILATING-IDEAL GRAPH OF $C(X)$

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ABSTRACT. The annihilating-ideal graph of the ring $C(X)$ is studied. It is tried to associate the graph properties of $\mathbb{AG}(X)$, the ring properties of $C(X)$ and the topological properties of $X$. It is shown that $X$ has an isolated point if and only if $\mathbb{R}$ is a direct summand of $C(X)$ and this happens if and only if $\mathbb{AG}(X)$ is not triangulated. Radius, girth, dominating number and clique number of $\mathbb{AG}(X)$ are investigated. It is proved that $c(X) \leqslant \mathrm{dt}(\mathbb{AG}(X)) \leqslant w(X)$ and $\omega\mathbb{AG}(X) = \chi\mathbb{AG}(X) = c(X)$.

## 1. Introduction

Let $G = \langle V(G), E(G) \rangle$ be an undirected graph. A vertex adjacent to just one vertex is called a *pendant vertex*. The degree of a vertex of $G$ is the number of vertices adjacent to the vertex. If $G$ has a vertex adjacent to all other vertices and all other vertices are pendant, then $G$ is called a *star graph*. For each pair of vertices $u$ and $v$ in $V(G)$, the length of the shortest path between $u$ and $v$, denoted by $d(u, v)$, is called the *distance* between $u$ and $v$. The *diameter* of $G$ is defined by $\mathrm{diam}(G) = \sup\{d(u, v) : u, v \in V(G)\}$. The *eccentricity* of a vertex $u$ of $G$, denoted by $\mathrm{ecc}(u)$, is defined to be $\max\{d(u, v) : v \in G\}$. The minimum of $\{\mathrm{ecc}(u) : u \in G\}$, denoted by $\mathrm{Rad}(G)$, is called the *radius* of $G$. For every $u, v \in V(G)$, we denote the length of the shortest cycle containing $u$ and $v$ by $\mathrm{gi}(u, v)$ and the minimum

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length of cycles in $G$, is denoted by $\operatorname{girth}(G)$ and is called the *girth* of graph, so $\operatorname{girth}(G) = \min\{\operatorname{gi}(u,v) : u,v \in V(G)\}$. We say $G$ is *triangulated* (*hypertriangulated*) if each vertex (edge) of $G$ is a vertex (edge) of some triangle. A subset $D$ of $V(G)$ is called a *dominating set* if for each $u \in V(G) \setminus D$, there is some $v \in D$, such that $v$ is adjacent to $u$. The *dominating number* of $G$, denoted by $\operatorname{dt}(G)$, is the smallest cardinal number of dominating sets of $G$. Two vertices $u$ and $v$ are called *orthogonal* and is denote by $u \perp v$, if $u$ and $v$ are adjacent and there is no vertex which adjacent to both vertices $u$ and $v$. If for every $u \in V(G)$, there is some $v \in V(G)$ such that $u \perp v$, then $G$ is called *complemented*. A subset of a graph $G$ is called a *clique* of $G$ if each pair of vertices of this subset are adjacent. The supremum of the cardinality of cliques of $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the minimum cardinal number of colors needed to color vertices of $G$ so that no two vertices have the same color. Clearly, $\omega(G) \leqslant \chi(G)$. A subset of vertices of a graph is called *independent* if no two adjacent vertices of this subset are adjacent. A *bipartite* graph is a graph whose vertices can be divided into two disjoint and independent sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$.

Throughout the paper, $R$ denotes a commutative ring with unity. For each subset $S$ of $R$ and each element $a$ of $R$, we denote $\{x \in R : ax \in S\}$ by $(S : a)$. When $I = \langle 0\rangle$ we write $\operatorname{Ann}(a)$ instead of $\big(\langle 0\rangle : a\big)$ and call this the *annihilator* of $a$. If for each subset $S$ of $R$, there is some $a \in R$ such that $\operatorname{Ann}(S) = \operatorname{Ann}(a)$, then we say $R$ satisfies *infinite annihilating condition* ($R$ is an i.a.c ring). An ideal $I$ is called annihilating ideal if $\operatorname{Ann}(I) \neq \{0\}$. We denote the family of all non-zero annihilating ideals of $R$ by $\mathbb{A}(R)^*$. We denote by $\mathbb{AG}(R)$ the graph with vertices $\mathbb{A}(R)^*$, and two distinct vertices $I$ and $J$ are adjacent, if $IJ = \{0\}$.

In this paper, $X$ denotes a Tychonoff space and $C(X)$ denotes the set of all real-valued continuous functions on $X$. The *weight* of $X$, denoted by $w(X)$, is the infimum of the cardinalities of bases of $X$. The *cellularity* of $X$, denoted by $c(X)$, is defined as

$$\sup\{|\mathcal{U}| : \mathcal{U} \text{ is a family of mutually disjoint nonempty open subsets of } X\}.$$

For any $f \in C(X)$, we denote $f^{-1}\{0\}$ and $X \setminus f^{-1}\{0\}$ by $Z(f)$ and $\operatorname{Coz}(f)$, respectively. Every set of the form $Z(f)$(resp., $\operatorname{Coz}(f)$) is called a *zeroset* (resp., *cozero set*). The family of all zerosets of $X$ is denoted by $Z(X)$. An ideal $I$ of $C(X)$ is called *fixed* (*free*) if $\bigcap_{f\in I} Z(f) \neq \emptyset$ ($\bigcap_{f\in I} Z(f) = \emptyset$). For a subset $A$ of $X$, we denote $\{f \in C(X) : A \subseteq Z(f)\}$ and $\{f \in C(X) : A \subseteq Z(f)^\circ\}$ by $M_A$ and $O_A$, respectively. When $A = \{p\}$, we write $M_p$ and $O_p$

instead of $M_{\{p\}}$ and $O_{\{p\}}$, respectively; it is clear that $M_A = \bigcap_{p \in A} M_p$ and $O_A = \bigcap_{p \in A} O_p$. For each space $X$, $\beta X$ denotes Stone-Cěch compactification of $X$. By [11, Theorem 7.3(Gelfand-Kolmogoroff)], $\{M^p : p \in \beta X\}$ is the family of all maximal ideal of $C(X)$. An ideal $I$ of $C(X)$ is called a $z$-ideal, if the conditions $Z(f) = Z(g)$ and $f \in I$, implies $g \in I$. For each subset $\mathcal{F}$ of $Z(X)$ and $S$ of $C(X)$, we denote $\{f \in C(X) : Z(f) \in \mathcal{F}\}$ and $\{Z(f) : f \in S\}$ by $Z^{-1}(\mathcal{F})$ and $Z(S)$, respectively. Clearly, for every ideal $I$ of $C(X)$, $Z^{-1}(Z(I))$ is the smallest $z$-ideal containing $I$. For more details, we refer the reader to [4, 9, 11, 15].

Graphs on $C(X)$ are studied in a number of interesting investigations, in which attempts are made to associate the ring properties of $C(X)$, the graph properties of graphs on $C(X)$ and the topological properties of $X$. In [3, 5, 6], the zero-divisor graph, the comaximal ideal graph of $C(X)$ and comaximal graph of $C(X)$ were studied. Papers [7, 8] are studies that embarked on investigating the annihilating-ideal graph of commutative rings. Later on, this line of research was pursued in several papers, including [1, 2, 10, 12, 13, 14].

The main purpose of this paper is studying the annihilating-ideal graph of $C(X)$. We abbreviate $\mathbb{A}(C(X))^*$ and $\mathbb{AG}(C(X))$ by $\mathbb{A}(X)^*$ and $\mathbb{AG}(X)$, respectively. If $|X| = 1$, then $\mathbb{A}(X)^* = \emptyset$, so we assume $|X| > 1$, throughout the paper.

In the rest part of this section, we put forward a number of propositions immediately concluded from the native algebraic properties of $C(X)$ and [5, 7, 8]. In Section 2, we define maps $\mathbf{O}$ from the family of all subsets of $C(X)$ onto the family of all open subsets of $X$ and $\mathbf{I}$ from the family of all subsets of $X$ into the family of all ideals of $C(X)$. We study these maps and apply these notions to study the graph. We show that $I$ is adjacent to $J$ if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, the non-zero ideal $I$ is an annihilating ideal if and only if $\overline{\mathbf{O}(I)} \neq X$, $\mathbf{I}(U) \in \mathbb{A}(X)^*$ if and only if $\overline{U}^{\circ} \neq \emptyset$. In Section 3, we investigate the radius of the graph and we show that $\mathbb{AG}(X)$ is a star graph if and only if $|X| = 2$. Section 4, is devoted to the girth of the graph. In this section we show that if $|X| > 2$, then girth$\mathbb{AG}(X) = 3$; also, we show that an ideal $I$ in $\mathbb{A}(X)^*$ is a pendant vertex if and only if $X \setminus \overline{\mathbf{O}(I)}$ is singleton. The study of dominating number of the graph is the subject of Section 5. In this section we show that both $\omega\mathbb{AG}(X)$ and $\chi\mathbb{AG}(X)$ are identical with the cellularity of $X$.

**Proposition 1.1.** *The following statements are equivalent.*

(a) $|X| = 2$.
(b) $\operatorname{diam}(\mathbb{AG}(X)) = 1$.
(c) $\omega\mathbb{AG}(X) = 2$.
(d) $\mathbb{AG}(X)$ *is a bipartite graph by two nonempty parts.*
(e) $\mathbb{AG}(X)$ *is a complete bipartite graph by two nonempty parts.*

*Proof.* It is concluded from [8, Theorem 1.4] and [14, Corollary 2.1]. □

**Proposition 1.2.** *The following statements hold.*

(a) *$X$ has at least 3 points, if and only if* $\operatorname{diam}(\mathbb{AG}(X)) = 3$.
(b) $\chi(\mathbb{AG}(X)) = \omega(\mathbb{AG}(X))$.

*Proof.* (a). It is concluded from [8, Proposition 1.1], [5, Corollary 1.3] and the previous proposition.

(b). It is evident, by [8, Corollary 2.11]. □

The following proposition is an immediate consequence of [7, Theorem 1.4], [8, Corollaries 2.11 and 2.12] and the fact that $X$ is finite if and only if $C(X)$ has just finitely many ideals. We note that for each ring $R$ the zero-divisor graph $\Gamma(R)$ is a graph with vertices of all nonzero zero-divisor elements of $R$, and two vertices $x$ and $y$ are adjacent, if $xy = 0$.

**Proposition 1.3.** *The following statements are equivalent.*

(a) $\mathbb{AG}(X)$ *is a finite graph.*
(b) $C(X)$ *has only finitely many ideals.*
(c) *Every vertex of* $\mathbb{AG}(X)$ *has a finite degree.*
(d) $X$ *is finite.*
(e) $\chi(\mathbb{AG}(X))$ *is finite.*
(f) $\omega(\mathbb{AG}(X))$ *is finite.*
(g) $\mathbb{AG}(X)$ *does not have an infinite clique.*
(h) $\chi(\Gamma(C(X)))$ *is finite.*

## 2. $\mathbf{I}(U)$ AND $\mathbf{O}(I)$

We denote (for simplicity and studying map properties) two concepts in the form of maps. For each subset $S$ of $C(X)$, we denote $\bigcup_{f\in S} \operatorname{Coz}(f)$ by $\mathbf{O}(S)$, and for each subset $U$ of $X$, we denote $\{f \in C(X) : U \subseteq Z(f)\} = M_U = \bigcap_{a\in U} M_a$ by $\mathbf{I}(U)$. It is clear that $\mathbf{O}(S) = X \setminus \left(\bigcap_{f\in S} Z(f)\right)$ and if $G$ is an open set in $X$, then $\mathbf{I}(G) = O_G$. First, in this section we study the properties of these maps, then utilizing the maps, the edges and vertices of $\mathbb{AG}(X)$ are investigated.

**Lemma 2.1.** *Let $S$ and $T$ be two subsets of $C(X)$, $f$ be an element of $C(X)$ and $U$, $V$ be two subsets of $X$. The following hold.*

(a) *If $S \subseteq T$, then $\mathbf{O}(S) \subseteq \mathbf{O}(T)$.*
(b) *If $U \subseteq V$, then $\mathbf{I}(V) \subseteq \mathbf{I}(U)$.*
(c) *$\mathbf{O}(S) = \emptyset$ if and only if $S = \{0\}$.*
(d) *$\mathbf{O}(S) = X$ if and only if $\langle S\rangle$ is a free ideal.*
(e) *$\mathbf{I}(U) = \{0\}$ if and only if $U$ is dense in $X$.*
(f) *$\mathbf{I}(U) = C(X)$ if and only if $U = \emptyset$.*
(g) *$\mathbf{O}(\langle f\rangle) = \operatorname{Coz}(f)$.*
(h) *$\mathbf{I}(U) = \mathbf{I}(\overline{U})$.*

*Proof.* It is straightforward. □

**Proposition 2.2.** *Let $S$ be a subset of $C(X)$. If $I = \langle S \rangle$, then $\mathbf{O}(I) = \mathbf{O}(S)$.*

*Proof.* It is straightforward. □

**Proposition 2.3.** *Let $\{I_\alpha\}_{\alpha\in A}$ be a family of ideals of $C(X)$, $I$ and $J$ be ideals of $C(X)$, $\{U_\alpha\}_{\alpha\in A}$ be a family of subsets of $X$ and $U$ and $V$ be subsets of $X$. Then the following hold.*

(a) $\mathbf{O}\left(\sum_{\alpha\in A} I_\alpha\right) = \bigcup_{\alpha\in A} \mathbf{O}(I_\alpha)$.
(b) $\mathbf{O}\left(\bigcap_{\alpha\in A} I_\alpha\right) \subseteq \bigcap_{\alpha\in A} \mathbf{O}(I_\alpha)$.
(c) $\mathbf{I}\left(\bigcup_{\alpha\in A} U_\alpha\right) = \bigcap_{\alpha\in A} \mathbf{I}(U_\alpha)$.
(d) $\mathbf{O}(I\cap J) = \mathbf{O}(I)\cap \mathbf{O}(J)$.
(e) $\mathbf{I}(U\cap V) \supseteq \mathbf{I}(U) + \mathbf{I}(V)$.

*Proof.* It is straightforward. □

In the following examples we show that the equality in parts (b) and (e) of the above proposition need not be established.

**Example 2.4.** Consider the ring $C(\mathbb{R})$. For each $r \in \mathbb{Q}$, we have $\mathbf{O}(M_r) = \mathbb{R}\setminus\{r\}$, thus $\bigcap_{r\in Q} \mathbf{O}(M_r) = \mathbb{R}\setminus\mathbb{Q}$. Also $\bigcap_{r\in\mathbb{Q}} M_r = M_{\mathbb{Q}} = \{0\}$, so $\mathbf{O}\left(\bigcap_{r\in\mathbb{Q}} M_r\right) = \mathbf{O}(\{0\}) = \emptyset$.

**Example 2.5.** Consider $C(\mathbb{R})$. Easily we can see that, $\mathbf{I}(\mathbb{Q}) = \{0\} = \mathbf{I}(\mathbb{R}\setminus\mathbb{Q})$, and thus $\mathbf{I}\left[\mathbb{Q}\cap(\mathbb{R}\setminus\mathbb{Q})\right] = \mathbf{I}(\emptyset) = C(\mathbb{R}) \neq \{0\} = \mathbf{I}(\mathbb{Q}) + \mathbf{I}(\mathbb{R}\setminus\mathbb{Q})$.

**Corollary 2.6.** *Let $U$ and $V$ be subsets of $X$. The following are equivalent.*

(a) *$U\cup V$ is dense in $X$.*
(b) $\mathbf{I}(U)\cap\mathbf{I}(V) = \{0\}$.
(c) $\mathbf{I}(U)\mathbf{I}(V) = \{0\}$.

*Proof.* It follows from Lemma 2.1 and Proposition 2.3. □

**Proposition 2.7.** *Let $I$ be an ideal of $C(X)$ and $U$ be a subset of $X$. The following hold.*

(a) $\mathbf{O}(\mathbf{I}(U)) = (X\setminus U)^\circ$.
(b) $\mathbf{I}(\mathbf{O}(I)) = \mathrm{Ann}(I)$.
(c) $(\mathbf{IO})^3(I) = (\mathbf{IO})(I)$.
(d) $\mathbf{O}(\mathrm{Ann}(I)) = (X\setminus\mathbf{O}(I))^\circ$.

*Proof.* (a). $\mathbf{O}(\mathbf{I}(U)) = \bigcup_{f\in\mathbf{I}(U)} \mathrm{Coz}(f) = \bigcup_{Z(f)\supseteq U} \mathrm{Coz}(f) = \bigcup_{\mathrm{Coz}(f)\subseteq X\setminus U} \mathrm{Coz}(f) = (X\setminus U)^\circ$.

(b).

$$\begin{aligned} f \in \mathrm{Ann}(I) &\Leftrightarrow \forall g \in I \quad fg = 0 \\ &\Leftrightarrow \forall g \in I \quad \mathrm{Z}(f) \cup \mathrm{Z}(g) = X \\ &\Leftrightarrow \forall g \in I \quad \mathrm{Coz}(g) \subseteq \mathrm{Z}(f) \\ &\Leftrightarrow \bigcup_{g \in I} \mathrm{Coz}(g) \subseteq \mathrm{Z}(f) \\ &\Leftrightarrow \mathbf{O}(I) \subseteq \mathrm{Z}(f) \\ &\Leftrightarrow f \in \mathbf{I}(\mathbf{O}(I)) \end{aligned}$$

Thus $\mathrm{Ann}(I) = \mathbf{I}(\mathbf{O}(I))$.

(c). Since $\mathrm{Ann}^3(I) = \mathrm{Ann}(I)$, it follows from (b), immediately.

(d). By (b) and (a), $\mathbf{O}(\mathrm{Ann}(I)) = \mathbf{O}(\mathbf{I}(\mathbf{O}(I))) = (X \setminus \mathbf{O}(I))^\circ$. □

Now the following corollary can be concluded from parts (a) and (e) of Proposition 2.7 and Lemma 2.1(e).

**Corollary 2.8.** *Suppose that $I$ is a non-zero ideal of $C(X)$ and $U \subseteq X$.*

(a) *$I \in \mathbb{A}(X)^*$ if and only if $\overline{\mathbf{O}(I)} \neq X$.*

(b) *$\mathbf{I}(U) \in \mathbb{A}(X)^*$ if and only if $\overline{U}^\circ \neq \emptyset$.*

**Corollary 2.9.** *If $I$ is an annihilating ideal of $C(X)$, then $I$ is a fixed ideal.*

*Proof.* Since $I$ is annihilating, $\mathrm{Ann}(I) \neq \{0\}$, so Proposition 2.7, deduces $\mathbf{I}(\mathbf{O}(I)) \neq \{0\}$, hence $\mathbf{O}(I)$ is not dense, by Lemma 2.1. Thus $\overline{\mathbf{O}(I)} \neq X$ and therefore $\bigcap_{f \in I} \mathrm{Z}(f) = X \setminus \mathbf{O}(I) \neq \emptyset$, so $I$ is a fixed ideal. □

The converse of the above corollary need not be true, for instance $M_0 \subseteq C(\mathbb{R})$ is a fixed ideal which is not an annihilating ideal.

**Corollary 2.10.** *Let $P$ be a prime ideal of $C(X)$. $P$ is annihilating if and only if there is some isolated point $p$ in $X$ such that $P = M_p = O_p$.*

*Proof.* ($\Rightarrow$). By Corollary 2.9, $P$ is fixed, so there is some $p \in X$ such that $O_p \subseteq P \subseteq M_p$. Thus $\mathbf{O}(P) = X \setminus \left(\bigcap_{f \in P} \mathrm{Z}(f)\right) = X \setminus \{p\}$. Since $P$ is annihilating, $\mathrm{Ann}(P) \neq \{0\}$ and therefore $\mathbf{I}(\mathbf{O}(P)) \neq \{0\}$, by Proposition 2.7. Now Lemma 2.1, deduces $X \setminus \{p\}$ is not dense and thus $p$ is an isolated point. Consequently, $P = M_p = O_p$.

($\Leftarrow$). Since $P = M_p$, $\mathbf{O}(P) = X \setminus \left(\bigcap_{f \in P} \mathrm{Z}(f)\right) = X \setminus \{p\}$. Since $p$ is an isolated point, it follows that $\mathbf{O}(P)$ is not dense in $X$, thus $\mathbf{I}(\mathbf{O}(P)) \neq \{0\}$, by Lemma 2.1. Now Proposition 2.7, entails that $\mathrm{Ann}(P) \neq \{0\}$ and therefore $P$ is annihilating. □

**Lemma 2.11.** *If $G$ is an open subset of $X$, then an ideal $I$ exists such that $\mathbf{O}(I) = G$. In other words, $\mathbf{O}$ maps the family of all ideals of $C(X)$ onto the family of all open subsets of $X$.*

*Proof.* Put $I = \Big\langle \{f \in C(X) : \mathrm{Coz}(f) \subseteq G\} \Big\rangle$. Then by Proposition 2.2,

$$\mathbf{O}(I) = \mathbf{O}\left(\langle \{f : \mathrm{Coz}(f) \subseteq G\} \rangle\right) = \bigcup_{\mathrm{Coz}(f)\subseteq G} \mathrm{Coz}(f) = G \qquad \square$$

Now we note that for each ideal $I$ of $C(X)$, the ideal $I_z$ means the smallest $z$-ideal containing $I$; i.e. $I_z$ is the intersection of all $z$-ideals containing $I$.

**Lemma 2.12.** *For each ideal $I$ of $C(X)$, we have $\mathbf{O}(I_z) = \mathbf{O}(I)$.*

*Proof.* Since $\mathrm{Z}(I) = \mathrm{Z}(I_z)$, so $\{\mathrm{Coz}(f) : f \in I\} = \{\mathrm{Coz}(f) : f \in I_z\}$ and therefore $\mathbf{O}(I_z) = \mathbf{O}(I)$. $\square$

**Theorem 2.13.** *$\mathbf{O}$ is a map from the family of all $z$-ideals of $C(X)$ onto the family of all open sets of $X$.*

*Proof.* It is clear by Lemmas 2.11 and 2.12. $\square$

**Theorem 2.14.** *Let $I$ and $J$ be two ideals of $C(X)$. The following statements hold*

(a) $IJ = \{0\}$ *if and only if* $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$.
(b) $I\mathrm{Ann}(J) = \{0\}$ *if and only if* $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$.
(c) $\mathrm{Ann}(I)\mathrm{Ann}(J) = \{0\}$ *if and only if* $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$.
(d) $\overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)}$ *if and only if* $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.
(e) $\mathbf{I}(U)I = \{0\}$ *if and only if* $\mathbf{O}(I) \subseteq \overline{U}$.

*Proof.* (a $\Rightarrow$). Since $IJ = \{0\}$, $I \subseteq \mathrm{Ann}(J)$, thus $I \subseteq \mathbf{I}(\mathbf{O}(J))$, by Proposition 2.7(b). Now suppose that $f \in I$, then $f \in \mathbf{I}(\mathbf{O}(J))$, hence $\mathrm{Z}(f) \supseteq \mathbf{O}(J)$, so $\mathrm{Coz}(f) \subseteq X \setminus \mathbf{O}(J)$. It follows that $\mathbf{O}(I) = \bigcup_{f\in I} \mathrm{Coz}(f) \subseteq X \setminus \mathbf{O}(J)$ and therefore $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$.

(a $\Leftarrow$). Suppose that $f \in I$ and $g \in J$, then $\mathrm{Coz}(f) \subseteq \mathbf{O}(I)$ and $\mathrm{Coz}(g) \subseteq \mathbf{O}(J)$, thus $\mathrm{Coz}(f) \cap \mathrm{Coz}(g) \subseteq \mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, so $fg = 0$ and therefore $IJ = \{0\}$.

(b). Considering part (a), $I\mathrm{Ann}(J) = \{0\}$ if and only if $\mathbf{O}(I) \cap \mathbf{O}(\mathrm{Ann}(J)) = \emptyset$. By Proposition 2.7, it is equivalent to $\mathbf{O}(I) \cap (X \setminus \mathbf{O}(J))^\circ = \emptyset$. It is equivalent to $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$.

(c). According to part (a), $\mathrm{Ann}(I)\mathrm{Ann}(J) = \{0\}$ if and only if $\mathbf{O}(\mathrm{Ann}(I)) \cap \mathbf{O}(\mathrm{Ann}(J)) = \emptyset$ if and only if $(X \setminus \mathbf{O}(J))^\circ \cap (X \setminus \mathbf{O}(I))^\circ = \emptyset$, By Proposition 2.7. It is equivalent to stating that $\overline{\mathbf{O}(I) \cup \mathbf{O}(I)} = X$.

(d ). Via part (b),

$$\begin{aligned} \overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)} \quad &\Leftrightarrow \quad \mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)} \ \text{ and } \ \mathbf{O}(J) \subseteq \overline{\mathbf{O}(I)} \\ &\Leftrightarrow \quad I\mathrm{Ann}(J) = \{0\} \ \text{ and } \ J\mathrm{Ann}(I) = \{0\} \\ &\Leftrightarrow \quad \mathrm{Ann}(J) \subseteq \mathrm{Ann}(I) \ \text{ and } \ \mathrm{Ann}(I) \subseteq \mathrm{Ann}(J) \\ &\Leftrightarrow \quad \mathrm{Ann}(I) = \mathrm{Ann}(J)\ . \end{aligned}$$

(e). Through part (a) and Proposition 2.7,

$$\begin{aligned} I\mathbf{I}(U) = \{0\} \quad &\Leftrightarrow \quad \mathbf{O}(I) \cap \mathbf{O}(\mathbf{I}(U)) = \emptyset \\ &\Leftrightarrow \quad \mathbf{O}(I) \cap (X \setminus U)^{\circ} = \emptyset \\ &\Leftrightarrow \quad \mathbf{O}(I) \cap X \setminus \overline{U} = \emptyset \\ &\Leftrightarrow \quad \mathbf{O}(I) \subseteq \overline{U}\ . \end{aligned}$$

□

**Proposition 2.15.** *Suppose that $I, J \in \mathbb{A}(R)^*$. Then $I$ and $J$ are adjacent if and only if each maximal ideal of $C(X)$ contains either $I$ or $J$.*

*Proof.* (⇒). It is clear.

(⇐). By the assumption, we have

$$\begin{aligned} & \forall p \in X \qquad I \subseteq M_p \quad \text{or} \quad J \subseteq M_p \\ \Rightarrow \quad & \forall p \in X \qquad p \in \bigcap_{f\in I} Z(f) \quad \text{or} \quad p \in \bigcap_{f\in J} Z(f) \\ \Rightarrow \quad & \left(\bigcap_{f\in I} Z(f)\right) \cup \left(\bigcap_{f\in J} Z(f)\right) = X \\ \Rightarrow \quad & \mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset \end{aligned}$$

Hence $I$ and $J$ are adjacent, by Theorem 2.14(a). □

**Corollary 2.16.** *Suppose that $I, J \in \mathbb{A}(X)^*$. Then $I$ and $J$ are orthogonal if and only if $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ and $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$.*

*Proof.* It is verifiable through Theorem 2.14(a) and Corollary 2.8(a). □

**Proposition 2.17.** *If each closed set of $X$ is a zero set, then $C(X)$ is an i.a.c. ring.*

*Proof.* By the assumption, each open set of $X$ is a cozero set and thus $\mathbf{O}$ is a map from the family all ideals of $C(X)$ onto the family all cozero sets of $X$, by Lemma 2.11 . Suppose that $S$ is a subset of $X$ and set $I = \langle S \rangle$. Now we can conclude that there is some $f \in C(X)$ such that $\mathbf{O}(I) = \mathbf{O}(S) = \mathrm{Coz}(f)$. Thus, by Lemma 2.1(g), Proposition 2.2 and Theorem 2.14(a),

$$\begin{aligned} g \in \mathrm{Ann}(I) \quad &\Leftrightarrow \quad gI = \{0\} \quad \Leftrightarrow \quad \mathbf{O}(\langle g \rangle) \cap \mathbf{O}(I) = \emptyset \\ &\Leftrightarrow \quad \mathrm{Coz}(g) \cap \mathrm{Coz}(f) = \emptyset \quad \Leftrightarrow \quad gf = 0 \\ &\Leftrightarrow \quad g \in \mathrm{Ann}(f) \end{aligned}$$

Hence $\mathrm{Ann}(S) = \mathrm{Ann}(I) = \mathrm{Ann}(f)$, i.e. $C(X)$ is an i.a.c. ring. □

## 3. Radius of the graph

In this section, some topological properties of $X$ are linked to the distance and eccentricity of vertices of $\mathbb{AG}(X)$, then by these facts we study the radius of the graph.

**Lemma 3.1.** *For any ideals $I$ and $J$ in $\mathbb{A}(X)^*$,*

(a) $d(I,J)=1$ *if and only if* $\mathbf{O}(I)\cap\mathbf{O}(J)=\emptyset$.
(b) $d(I,J)=2$ *if and only if* $\mathbf{O}(I)\cap\mathbf{O}(J)\neq\emptyset$ *and* $\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}\neq X$.
(c) $d(I,J)=3$ *if and only if* $\mathbf{O}(I)\cap\mathbf{O}(J)\neq\emptyset$ *and* $\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}=X$.

*Proof.* (a). It is evident, by Theorem 2.14.

(b $\Rightarrow$). Since $I$ is not adjacent to $J$, $\mathbf{O}(I)\cap\mathbf{O}(J)\neq\emptyset$, by Theorem 2.14. By the assumption there is an ideal $K$ in $\mathbb{A}(X)^*$ such that $K$ is adjacent to both ideals $I$ and $J$. Now Lemma 2.1 concludes that $\mathbf{O}(K)\neq\emptyset$ and also Theorem 2.14 implies that $\mathbf{O}(I)\cap\mathbf{O}(K)=\emptyset$ and $\mathbf{O}(J)\cap\mathbf{O}(K)=\emptyset$, hence $\mathbf{O}(K)\cap(\mathbf{O}(I)\cup\mathbf{O}(J))=\emptyset$ and thus $\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}\neq X$.

(b $\Leftarrow$). Theorem 2.14 follows that $I$ is not adjacent to $J$. Set $H=\mathbf{O}(I)\cup\mathbf{O}(J)$ and $K=\mathbf{I}(H)$. Since $\emptyset\neq H\subseteq\overline{H}^{\circ}\neq\emptyset$, by Corollary 2.8, $\mathbf{I}(H)\in\mathbb{A}(X)^*$. Since $\mathbf{O}(I),\mathbf{O}(J)\subseteq H\subseteq\overline{H}$, $IK=JK=\{0\}$, by Theorem 2.14. Hence $K$ is adjacent to both ideals $I$ and $J$, thus $d(I,J)=2$.

(c). It follows from (a), (b) and [7, Theorem 2.1]. ☐

**Lemma 3.2.** *Let $f\in C(X)$, $I$ be an ideal of $C(X)$ and $p\in\mathbf{O}(I)$. If* $\mathrm{Coz}(f)\subseteq\{p\}$ *and $p$ is an isolated point of $X$, then $f\in I$.*

*Proof.* Since $p\in\mathbf{O}(I)$, there is some $g\in I$ such that $p\in\mathrm{Coz}(g)$. Set

$$h(x)=\begin{cases}\frac{f(p)}{g(p)} & x=p\\ 0 & x\neq p\end{cases}$$

Since $p$ is an isolated point, $h\in C(X)$. Now we have $f=gh$ and therefore $f\in I$. ☐

**Proposition 3.3.** *Suppose that $I$ is a non-zero annihilating ideal of $C(X)$. The following statements hold.*

(a) $\mathrm{ecc}(I)=3$ *if and only if* $\mathbf{O}(I)$ *is not a singleton.*
(b) $\mathrm{ecc}(I)=2$ *if and only if* $\mathbf{O}(I)$ *is a singleton and* $|X|>2$.
(c) $\mathrm{ecc}(I)=1$ *if and only if* $\mathbf{O}(I)$ *is a singleton and* $|X|=2$.

*Proof.* (a $\Rightarrow$). There is some $J\in\mathbb{A}(X)^*$ such that $d(I,J)=3$. Lemma 3.1, concludes that $\mathbf{O}(I)\cap\mathbf{O}(J)\neq\emptyset$ and $\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}=X$. If $\mathbf{O}(I)$ is a singleton, then $\mathbf{O}(I)\subseteq\mathbf{O}(J)$ and therefore $\overline{\mathbf{O}(J)}=\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}=X$, so $J\notin\mathbb{A}(X)^*$, by Corollary 2.8, which is a contradiction.

(a $\Leftarrow$). There are distinct points $p$ and $q$ in $\mathbf{O}(I)$, so there are disjoint open sets $H,K\subseteq\mathbf{O}(I)$ such that $p\in H$ and $q\in K$. By Lemma 2.11, there is some ideal $J$ such that $\mathbf{O}(J)=H\cup X\setminus\overline{\mathbf{O}(I)}$. Since $q\notin\overline{\mathbf{O}(J)}$ and $p\in\mathbf{O}(J)$, Lemma 2.1 and Corollary 2.8, conclude that $J\in\mathbb{A}(X)^*$. Then

$$H\subseteq\mathbf{O}(I)\cap\mathbf{O}(J)\quad\Rightarrow\quad\mathbf{O}(I)\cap\mathbf{O}(J)\neq\emptyset$$

$$\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}\supseteq\overline{\mathbf{O}(I)}\cup\left(X\setminus\overline{\mathbf{O}(I)}\right)=X\quad\Rightarrow\quad\overline{\mathbf{O}(I)\cup\mathbf{O}(J)}=X$$

Hence $d(I,J)=3$, by Lemma 3.1. Consequently, $\mathrm{ecc}(I)=3$.

(c $\Rightarrow$). Since $\mathrm{ecc}(I) = 1$, $I$ is adjacent to any element of $\mathbb{A}(X)^*$. By (a), $\mathbf{O}(I)$ is a singleton, thus there is some isolated point $p \in X$ such that $\mathbf{O}(I) = \{p\}$. Since $\emptyset \neq X \setminus \{p\}$ is open, by Lemma 2.11, there is some ideal $J$, such that $\mathbf{O}(J) = X \setminus \{p\}$. Since $\mathbf{O}(J) \neq \emptyset$ and $\overline{\mathbf{O}(J)} = X \setminus \{p\} \neq X$, we obtain that $J \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Since $\mathrm{ecc}(I) = 1$, $\mathrm{ecc}(J) \leqslant 2$, so $\mathbf{O}(J)$ is a singleton, by part (a), and therefore $|X| = 2$.

(c $\Leftarrow$). $C(X) \cong \mathbb{R} \oplus \mathbb{R}$, so $\mathbb{AG}(X)$ is a star graph, by [7, Corollary 2.3]. Since $\mathbb{A}(X)^*$ has just two elements, it follows that $\mathrm{ecc}(I) = 1$.

(b). It concludes from (a) and (c). □

The following corollary is an immediate consequence of the above theorem.

**Corollary 3.4.** *$|X| = 2$ if and only if $\mathbb{AG}(X)$ is a star.*

Now we can determine the radius of the graph.

**Theorem 3.5.** *For any topological space $X$,*

$$\mathrm{Rad}(\mathbb{AG}(X)) = \begin{cases} 1 & \text{if } |X| = 2 \\ 2 & \text{if } |X| > 2 \text{ and } X \text{ has an isolated point.} \\ 3 & \text{if } |X| > 2 \text{ and } X \text{ does not have any isolated point.} \end{cases}$$

*Proof.* It is a straight consequence of Lemma 2.11 and Proposition 3.3. □

## 4. Girth of the graph

In this section, first we provide an equivalent topological property to pendant vertices, then we show that if $\mathbb{AG}(X)$ has a cycle then $\mathrm{girth}\mathbb{AG}(X) = 3$. Finally we attempt to associate the graph properties of $\mathbb{AG}(X)$, the ring properties of $C(X)$ and the topological properties of $X$.

**Lemma 4.1.** *Suppose that $Y$ is a clopen subset of $X$. Then for each ideal $I$ of $C(X)$, there are ideals $I_1, I_2$ of $C(X)$ such that $I = I_1 \oplus I_2$ and $I_1$ and $I_2$ are ideals of $M_Y \cong C(X \setminus Y)$ and $M_{X\setminus Y} \cong C(Y)$, respectively.*

*Proof.* Considering the fact that $Y$ is clopen, $C(X) \cong C(Y) \oplus C(X \setminus Y)$, it is straightforward. □

**Proposition 4.2.** *Let $I \in \mathbb{A}(X)^*$. Then $X \setminus \overline{\mathbf{O}(I)}$ is a singleton if and only if $I$ is a pendant vertex.*

*Proof.* $\Rightarrow$). Suppose that $X \setminus \overline{\mathbf{O}(I)} = \{p\}$. Since $\{p\}$ is open, by Lemma 2.11, there is an ideal $J$ such that $\mathbf{O}(J) = \{p\}$, then $\overline{\mathbf{O}(J)} = \{p\}$, and therefore $J \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Also $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, so $I$ is adjacent to $J$, by Theorem 2.14. Suppose that $K$ is adjacent to $I$ and $Y = \overline{\mathbf{O}(I)}$. Then $\mathbf{O}(K) \cap \mathbf{O}(I) = \emptyset$, by Theorem 2.14, thus $\mathbf{O}(K) \subseteq X \setminus \overline{\mathbf{O}(I)} = \{p\}$. By Lemma 2.1, $\mathbf{O}(K) \neq \emptyset$, so $\mathbf{O}(K) = \{p\}$. Since $\{p\}$ is clopen, by Lemma 4.1, it follows that there are ideals $K_1$ and $K_2$ of $M_p \cong C(Y)$ and $M_Y \cong C(\{p\}) \cong \mathbb{R}$, respectively, such that $K = K_1 \oplus K_2$.

If $K_1 \neq \{0\}$, then $0 \neq f \in K_1 \subseteq K$ exists, so there is a $q \in Y$ such that $f(q) \neq 0$, thus $p \neq q \in \mathrm{Coz}(f) \subseteq \mathbf{O}(K)$, which is a contradiction. Hence $K_1 = \{0\}$, since $K \neq \{0\}$, it follows that $K_2 = M_Y$, thus $K = M_Y$, and this completes the proof.

$\Leftarrow$). Suppose that $X \setminus \overline{\mathbf{O}(I)}$ is not a singleton, so distinct points $p, q$ in $X \setminus \overline{\mathbf{O}(I)}$ exist. Since $X \setminus \overline{\mathbf{O}(I)}$ is open and $X$ is Hausdorff, there are disjointed open sets $H_1$ and $H_2$ containing $p$ and $q$, respectively, in which $H_1 \cap \mathbf{O}(I) = H_2 \cap \mathbf{O}(I) = \emptyset$. Now Lemma 2.11, implies that there are ideals $J_1$ and $J_2$ such that $\mathbf{O}(J_1) = H_1$ and $\mathbf{O}(J_2) = H_2$, clearly $J_1, J_2 \in \mathbb{A}(X)^*$. Then $\mathbf{O}(I) \cap \mathbf{O}(J_1) = \mathbf{O}(I) \cap \mathbf{O}(J_2) = \emptyset$. So, by Theorem 2.14, $I$ is adjacent to both ideals $J_1$ and $J_2$. □

**Lemma 4.3.** *Suppose that $I, J \in \mathbb{A}(X)^*$ are not pendant vertices. The following statements hold.*

(a) $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ *and* $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} \neq X$ *if and only if* $\mathrm{gi}(I, J) = 3$.
(b) *If* $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$ *and* $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X$, *then* $\mathrm{gi}(I, J) = 4$.
(c) *If* $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ *and* $\overline{\mathbf{O}(I)} = \overline{\mathbf{O}(J)}$, *then* $\mathrm{gi}(I, J) = 4$.
(d) *Suppose that* $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$ *and* $\overline{\mathbf{O}(I)} \neq \overline{\mathbf{O}(J)}$. *Then* $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$ *is not a singleton if and only if* $\mathrm{gi}(I, J) = 4$.
(e) $\mathbf{O}(I) \cap \mathbf{O}(J) \neq \emptyset$, $\overline{\mathbf{O}(I)} \neq \overline{\mathbf{O}(J)}$ *and* $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$ *is a singleton if and only if* $\mathrm{gi}(I, J) = 5$.

*Proof.* (a $\Rightarrow$). Set $H = \mathbf{O}(I) \cup \mathbf{O}(J)$ and $K = \mathbf{I}(H)$. Since $\overline{H} \neq X$ and $\overline{H}^{\circ} \neq \emptyset$, $K \in \mathbb{A}(X)^*$, by Corollary 2.8(b). Since $\mathbf{O}(I), \mathbf{O}(J) \subseteq H \subseteq \overline{H}$, by Theorem 2.14, $K$ is adjacent to both ideals $I$ and $J$. By the assumption and Theorem 2.14, $I$ is adjacent to $J$, hence $\mathrm{gi}(I, J) = 3$.

(a $\Leftarrow$). By the assumption, $I$ is adjacent to $J$ and some $K \in \mathbb{A}(X)^*$ exists such that $K$ is adjacent to both ideals $I$ and $J$, so $\mathbf{O}(I) \cap \mathbf{O}(J) = \emptyset$, $\mathbf{O}(I) \cap \mathbf{O}(K) = \emptyset$ and $\mathbf{O}(J) \cap \mathbf{O}(K) = \emptyset$, by Theorem 2.14. Hence $(\mathbf{O}(I) \cup \mathbf{O}(J)) \cap \mathbf{O}(K) = \emptyset$. Since $K \neq \{0\}$, $\mathbf{O}(K) \neq \emptyset$, by Lemma 2.1, and therefore $\overline{\mathbf{O}(I) \cup \mathbf{O}(J)} \neq X$.

(b). The assumption and part (a) imply that $\mathrm{gi}(I, J) \geqslant 4$ and Theorem 2.14, concludes that $IJ = \mathrm{Ann}(I)\mathrm{Ann}(J) = \{0\}$. Since $I$ and $J$ are not pendant vertices, there are $I_1, J_1 \in \mathbb{A}(X)^*$ such that $I$ is adjacent to $I_1 \neq J$ and $J$ is adjacent to $J_1 \neq I$, so $II_1 = JJ_1 = \{0\}$, thus $I_1 \subseteq \mathrm{Ann}(I)$ and $J_1 \subseteq \mathrm{Ann}(J)$, hence $I_1 J_1 \subseteq \mathrm{Ann}(I)\mathrm{Ann}(J) = \{0\}$ and therefore $I_1 J_1 = \{0\}$. Consequently, $I$ is adjacent to $J$, $J$ is adjacent to $J_1$, $J_1$ is adjacent to $I_1$ and $I_1$ is adjacent to $I$, they imply that $\mathrm{gi}(I, J) = 4$.

(c). We can conclude from the assumption and part (a), that $\mathrm{gi}(I, J) \geqslant 4$. Since $\overline{\mathbf{O}(J)} = \overline{\mathbf{O}(J)}$, by Theorem 2.14, it follows that $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. Since $I$ is adjacent to $\mathrm{Ann}(I)$ and $I$ is not a pendant vertex, it follows there is some vertex $I_1 \in \mathbb{A}(X)^*$ distinct from $\mathrm{Ann}(I)$ such that $I$ is adjacent to $I_1$, then $I_1 I = \{0\}$, so $I_1 \subseteq \mathrm{Ann}(I) = \mathrm{Ann}(J)$ and therefore $I_1 J = \{0\}$. Consequently, $I$ is adjacent to $\mathrm{Ann}(I)$, $\mathrm{Ann}(J)$ is adjacent to $J$, $J$ is adjacent to $I_1$ and $I_1$ is adjacent to $I$ and thus $\mathrm{gi}(I, J) = 4$.

(d $\Rightarrow$). Evidently, there are two distinct nonempty open sets $H_1$ and $H_2$ such that $H_1 \cap \mathbf{O}(I) = H_1 \cap \mathbf{O}(J) = H_2 \cap \mathbf{O}(I) = H_2 \cap \mathbf{O}(J) = \emptyset$. Then, by Lemma 2.11, there are two ideals $K_1$ and $K_2$ such that $\mathbf{O}(K_1) = H_1$ and $\mathbf{O}(K_2) = H_2$, it is clear that $K_1, K_2 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Now Theorem 2.14, concludes that both vertices $I$ and $J$ are adjacent to both vertices $K_1$ and $K_2$, thus $\mathrm{gi}(I, J) = 4$, by part (a).

(d $\Leftarrow$). By Theorem 2.14, $I$ is not adjacent to $J$. Since $\mathrm{gi}(I, J) = 4$, it follows that there are distinct vertices $K_1$ and $K_2$ which are adjacent to both vertices $I$ and $J$, so $I + J$ is adjacent to both vertices $K_1$ and $K_2$. Now Propositions 2.3 and 4.2, conclude that $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X \setminus \overline{\mathbf{O}(I + J)}$ is not a singleton.

(e $\Rightarrow$). By parts (a) and (d), $\mathrm{gi}(I, J) \geqslant 5$. If $\mathbf{O}(I) \subseteq \overline{\mathbf{O}(J)}$, then $\overline{\mathbf{O}(I)} \subseteq \overline{\mathbf{O}(J)}$, so $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)} = X \setminus \overline{\mathbf{O}(J)}$ and therefore $X \setminus \overline{\mathbf{O}(J)}$ is a singleton, by the assumption. Now Proposition 4.2, concludes that $J$ is a pendant vertex, which contradicts the assumption, so $\mathbf{O}(I) \not\subseteq \overline{\mathbf{O}(J)}$, similarly, it can been shown that $\mathbf{O}(J) \not\subseteq \overline{\mathbf{O}(I)}$, so $H_1 = \mathbf{O}(I) \setminus \overline{\mathbf{O}(J)}$ and $H_2 = \mathbf{O}(J) \setminus \overline{\mathbf{O}(I)}$ are nonempty open sets; thus, Lemma 2.11, implies that there are ideals $K_1$ and $K_2$ such that $\mathbf{O}(K_1) = H_1$ and $\mathbf{O}(K_2) = H_2$, it is evident that $K_1, K_2 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Since $X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$ is a nonempty open set, there is an ideal $K_3$ such that $\mathbf{O}(K_3) = X \setminus \overline{\mathbf{O}(I) \cup \mathbf{O}(J)}$, it is clear that $K_3 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Then

$$\begin{aligned}\mathbf{O}(I) \cap \mathbf{O}(K_2) &= \mathbf{O}(K_2) \cap \mathbf{O}(K_1) = \mathbf{O}(K_1) \cap \mathbf{O}(J)\\ &= \mathbf{O}(J) \cap \mathbf{O}(K_3) = \mathbf{O}(K_3) \cap \mathbf{O}(I) = \emptyset\end{aligned}$$

so $\mathrm{gi}(I, J) = 5$.

(e $\Leftarrow$). It is clear, by parts (a)-(d). □

It is clear that if $|X| = 2$, then $\mathbb{AG}(X)$ does not have any cycle. In the following theorem we show that if $\mathbb{AG}(X)$ has a cycle then the girth of the graph is 3.

**Theorem 4.4.** *If* $|X| > 2$, *then* $\mathrm{girth}\mathbb{AG}(X) = 3$.

*Proof.* It is clearly observable that there are mutually disjointed nonempty open sets $G_1$, $G_2$ and $G_3$. By Lemma 2.11, there are ideals $I_1$, $I_2$ and $I_3$, such that $\mathbf{O}(I_1) = G_1$, $\mathbf{O}(I_2) = G_2$ and $\mathbf{O}(I_3) = G_3$, evidently, $I_1, I_2, I_3 \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. By Theorem 2.14, $I_1$ is adjacent to $I_2$, $I_2$ is adjacent to $I_3$ and $I_3$ is adjacent to $I_1$, hence $\mathrm{girth}\mathbb{AG}(X) = 3$. □

**Theorem 4.5.** *The following statements are equivalent.*

(a) *$X$ has an isolated point.*
(b) *$\mathbb{R}$ is a direct summand of $C(X)$.*
(c) *$\mathbb{AG}(X)$ has a pendant vertex.*
(d) *$\mathbb{AG}(X)$ is not triangulated.*

*Proof.* (a $\Leftrightarrow$ b) and (c $\Rightarrow$ d) are clear and (a $\Leftrightarrow$ c) follows from Proposition 4.2.

(d $\Rightarrow$ a) Suppose that $X$ does not have any isolated point and $I \in \mathbb{A}(X)^*$. Then $X \setminus \overline{\mathbf{O}(I)}$ is not a singleton, so it has two distinct points $p$ and $q$, so there are disjoint open sets $G_1$ and $G_2$, such that $G_1 \cap \mathbf{O}(I) = G_2 \cap \mathbf{O}(I) = \emptyset$. By Lemma 2.11, there are $J, K \in \mathbb{A}(X)^*$, such that $\mathbf{O}(J) = G_1$ and $\mathbf{O}(K) = G_2$. Thus $I$ is adjacent to $J$, $J$ is adjacent to $K$ and $K$ is adjacent to $I$. Consequently, $\mathbb{AG}(X)$ is triangulated. □

## 5. Dominating number

In the last section, an upper bound and a lower bound for dominating number of the graph by topological notions are offered, then the chromatic number and the clique number of the graph are studied.

**Theorem 5.1.** *$c(X) \leqslant \mathrm{dt}(\mathbb{AG}(X)) \leqslant w(X)$, for each topological space $X$.*

*Proof.* Suppose that $\mathcal{U}$ is a family of mutually disjointed nonempty open sets. If $\overline{\bigcup \mathcal{U}} \neq X$, then $\mathcal{V} = \mathcal{U} \cup \left\{X \setminus \overline{\bigcup \mathcal{U}}\right\}$ is a family of mutually disjoint open sets which $\overline{\bigcup \mathcal{V}} = X$, so without loss of generality we can assume that $\overline{\bigcup \mathcal{U}} = X$. For each $U \in \mathcal{U}$, there are some $I_U \in \mathbb{A}(X)^*$ such that $\mathbf{O}(I_U) = U$, by Lemma 2.11. Since $U \neq \emptyset$ and $\overline{U} \neq X$, it follows that $I_U \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. Now suppose that $D$ is a dominating set, then for each $U \in \mathcal{U}$, there is an ideal $J_U$ in $D$ such that $J_U$ is adjacent to $\sum_{U \neq V \in \mathcal{U}} I_V$. Now Theorem 2.14, implies that $\mathbf{O}(J_U) \cap \mathbf{O}\left(\sum_{U \neq V \in \mathcal{U}} I_V\right) = \emptyset$, thus $\mathbf{O}(J_U) \cap \left(\bigcup_{U \neq V \in \mathcal{U}} U\right) = \emptyset$. Suppose that $J_U = J_{U'}$, for some $U, U' \in \mathcal{U}$. Then $\mathbf{O}(J_U) = \mathbf{O}(J_{U'})$. If $U \neq U'$, then

$$\begin{aligned}\mathbf{O}(J_U) \cap \bigcup \mathcal{U} &= \mathbf{O}(J_U) \cap \left[\Big(\bigcup_{U \neq V \in \mathcal{U}} V\Big) \cup \Big(\bigcup_{U' \neq V \in \mathcal{U}} V\Big)\right] \\ &= \left[\mathbf{O}(J_U) \cap \Big(\bigcup_{U \neq V \in \mathcal{U}} V\Big)\right] \cup \left[\mathbf{O}(J_U) \cap \Big(\bigcup_{U' \neq V \in \mathcal{U}} V\Big)\right] = \emptyset.\end{aligned}$$

Thus $\overline{\bigcup \mathcal{U}} \neq X$, which contradicts our assumption. Hence $U = U'$, so $|\mathcal{U}| \leqslant |D|$, and consequently $c(X) \leqslant \mathrm{dt}(\mathbb{AG}(X))$.

Now suppose that $\mathcal{B}$ is a base for $X$, without loss of generality we can assume that every element of $\mathcal{B}$ is not empty. For each $B \in \mathcal{B}$, there is some $0 \neq f_B \in C(X)$ such that $\emptyset \neq \mathrm{Coz}(f_B) \subseteq B$. Clearly, we can choose $f_B$ such that $\overline{\mathrm{Coz}(f_B)} \neq X$. Lemma 2.1, concludes that $\mathbf{O}(\langle f_B \rangle) = \mathrm{Coz}(f_B)$, so $\mathbf{O}(\langle f_B \rangle) \neq \emptyset$ and $\overline{\mathbf{O}(\langle f_B \rangle)} \neq X$, for each $B \in \mathcal{B}$, thus $\langle f_B \rangle \in \mathbb{A}(X)^*$, by Lemma 2.1 and Corollary 2.8. For each $J \in \mathbb{A}(X)^*$, $\overline{\mathbf{O}(I)} \neq X$, by Corollary 2.8, so $(X \setminus \mathbf{O}(I))^\circ \neq \emptyset$, thus $B \in \mathcal{B}$ exists such that $B \subseteq (X \setminus \mathbf{O}(I))^\circ$, hence $\mathbf{O}(\langle f_B \rangle) \subseteq X \setminus \mathbf{O}(I)$, consequently, $\mathbf{O}(\langle f_B \rangle) \cap \mathbf{O}(I) = \emptyset$, therefore Theorem 2.14, implies that $\langle f_B \rangle$ is adjacent to $I$. Hence $\{\langle f_B \rangle : B \in \mathcal{B}\}$ is a dominating set. Since $\left|\{\langle f_B\rangle : B \in \mathcal{B}\}\right| \leqslant |\mathcal{B}|$, it follows that $\mathrm{dt}(\mathbb{AG}(X)) \leqslant w(X)$. □

Now we can conclude the following corollary from the above theorem.

**Corollary 5.2.** *If $X$ is discrete, then* $\mathrm{dt}(\mathbb{AG}(X)) = |X|$.

**Theorem 5.3.** $\mathrm{dt}(\mathbb{AG}(X))$ *is finite if and only if $|X|$ is finite. In this case,* $\mathrm{dt}(\mathbb{AG}(X)) = |X|$.

*Proof.* $\Rightarrow$). Suppose that $|X|$ is infinite. Clearly $c(X)$ is infinite, so $\mathrm{dt}(\mathbb{AG}(X))$ is infinite, by Theorem 5.1.

$\Leftarrow$). If $|X|$ is finite, then $X$ is discrete, so $\mathrm{dt}(\mathbb{AG}(X)) = |X|$ is finite, by Corollary 5.2. □

**Theorem 5.4.** $\chi\mathbb{AG}(X) = \omega\mathbb{AG}(X) = c(X)$*, for each topological space $X$.*

*Proof.* It is an immediate consequence of Proposition 1.2, Lemma 2.11 and Theorem 2.14. □


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# ANNIHILATING-IDEAL GRAPH OF $C(X)$

M. BADIE

# گراف ایده‌آل-پوچساز $C(X)$

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با مطالعه‌ی ایده‌آل-پوچساز حلقه‌ی $C(X)$ سعی کرده‌ایم که رابطه‌هایی بین خواص گراف $\mathbb{AG}(X)$، حلقه‌ی $C(X)$ و توپولوژی $X$ بیابیم. نشان داده‌ایم که $X$ یک نقطه‌ی منفرد دارد اگر و تنها اگر $\mathbb{R}$ یک جمعوند مستقیم $C(X)$ باشد و این دو نیز معادل با این هستند که $\mathbb{AG}(X)$ یک گراف مثلثی‌شدنی نباشد. همچنین شعاع، کمر و اعداد احاطه‌گر و خوشه‌ای $\mathbb{AG}(X)$ مطالعه شده‌اند و ثابت کرده‌ایم $c(X) \leqslant \mathrm{dt}\left(\mathbb{AG}(X)\right) \leqslant w(X)$ و $w\mathbb{AG}(X) = \chi\mathbb{AG}(X) = c(X)$.

کلمات کلیدی: حلقه‌ی توابع پیوسته، گراف ایده‌آل-پوچساز، عدد رنگی، عدد خوشه‌ای، سلولیت.