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SOME RESULTS ON STRONGLY PRIME SUBMODULES

A. R. NAGHIPOUR

Abstract. Let R be a commutative ring with identity and let M be an R-module. A proper submodule P of M is called strongly prime submodule if $(P+Rx:M)y \subseteq P$ for $x, y \in M$, implies that $x \in P$ or $y \in P$. It is shown that a finitely generated R-module M is Artinian if and only if M is Noetherian and every strongly prime submodule of M is maximal. We also study the strongly dimension of a module which is defined to be the length of a longest chain of strongly prime submodules.

1. Introduction

This paper focuses on all rings, which are commutative with identity and all modules which are unitary. Also we consider R to be a ring and M an R-module.

For a submodule N of M, let $(N:M)$ denote the set of all elements r in R such that $rM \subseteq N$. The annihilator of M, denoted by $\mathrm{Ann}(M)$, is $(0:M)$. A proper submodule N of M is called *prime* if $rx \in N$, for $r \in R$ and $x \in M$, implies that either $x \in N$ or $r \in (N:M)$. This notion of prime submodule was first introduced and systematically studied in [6] and recently has received a good deal of attention from several authors, see for example [10], [11] and [17]. The collection of all prime submodules of M is denoted by $\mathrm{Spec}_R(M)$, and the collection of all maximal submodules of M is denoted by $\mathrm{Max}_R(M)$.

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Unfortunately, unlike the rings, not every R-module contains a prime submodule, for example $\operatorname{Spec}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) = \emptyset$ (see [12]). More generally, we know that if R is a domain, then any torsion divisible R-module has no prime submodule (see [15, Lemma 1.3(i)]). If $\operatorname{Spec}_R(M) = \emptyset$, we call such modules M *primeless*.

Notation. Let N be a submodule of M and let $x \in M$. We denote the ideal $(N + Rx : M)$ by $I_x^{M,N}$ or simply by I_x^N when no ambiguity is possible.

Let M be an R-module. A proper submodule P of M is called a *strongly prime submodule* if $I_x^P y \subseteq P$, for $x, y \in M$, implies that either $x \in P$ or $y \in P$. This notion inherits most of the essential properties of the usual notion of prime ideal. In particular, the Generalized Principal Ideal Theorem is extended to modules (see [18] and [20]). We need to mention that this notion is different from the one proposed in [9].

The following remark is used widely in the sequel.

Remark 1.1. ([18, Propositions 1.1 and 1.3]) Let M be an R-module. Then the following should be considered.

(1) Any strongly prime submodule of M is prime.
(2) Any maximal submodule of M is strongly prime. The converse is true if R is a field.

The collection of all strongly prime submodules of M is called the *strongly spectrum* of M and is denoted by $\text{S-Spec}_R(M)$. If $\text{S-Spec}_R(M) = \emptyset$, we call such modules M *strongly primeless*. For example, if R is an integral domain which is not a field and F the field of quotients of R, then $\operatorname{Spec}_R(F) = 0$ (see [12, Theorem 1]). It is easy to see that (0) is not a strongly prime submodule of F and hence $\text{S-Spec}_R(F) = \emptyset$.

The *classical Krull dimension* of a ring R, $\operatorname{cl.K.dim}(R)$, is the supremum of lengths of chains of prime ideals of R. The classical Krull dimension of an R-module M is defined as the classical Krull dimension of the ring $R/\operatorname{Ann}(M)$ and denoted by $\operatorname{cl.K.dim}_R(M)$ (see [19]). The notion of classical Krull dimension has a substantial role in commutative algebra and algebraic geometry, see for example [7, Part II]. Since the classical Krull dimension of a ring is defined in terms of length of ascending chains of prime ideals, one would naturally wonder whether it is possible to define the classical Krull dimension of a module in terms of lengths of ascending chain of prime submodules. Note that an R-module M could have ascending chains of prime submodules of arbitrary length, while $\operatorname{cl.K.dim}_R(M) = 0$ (for example, when R is a field). Abu-Saymeh in [1] defined the dimension of M as the supremum of lengths of chains of distinguished prime submodules. (We recall that if $\mathfrak{p}$ is an ideal of R, then $M(\mathfrak{p}) = \{x \in M : rx \in \mathfrak{p}M$, for some $r \in R\backslash\mathfrak{p}\}$ is called a *distinguished submodule* of M). Also Behboodi in [5], introduced a generalization of the classical Krull dimension for a module M. This is defined to be the length of the longest strong chain of prime submodules of M.

By motivation of [18, Definition 2.2], we introduce a new generalization of the classical Krull dimension of rings to modules via strongly prime submodules such that all Artinian modules with a maximal (strongly prime) submodule as well as, all semisimple modules lie in the class of modules with dimension zero. We define the *strongly dimension* of M (s-$\dim_R(M)$) in terms of ascending chains of strongly prime submodules as follow:
s-$\dim_R(M) = \sup\{n | \exists\ P_0, P_1, \ldots, P_n \in$ S-$\mathrm{Spec}_R(M)$ such that $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n\}$, if S-$\mathrm{Spec}_R(M) \neq \emptyset$, otherwise it is defined to be -1. This dimension seems to be an adequate dimension for modules. Note that if we consider R as an R-module, then strongly prime submodules are exactly prime ideals of R and hence the notion of strongly dimension of R and the classical Krull dimension of R coincides.

It is a well known fact that a ring R is Artinian if and only if R is Noetherian and $\mathrm{Spec}(R) = \mathrm{Max}(R)$ (see, for example [7, Corollary 9.1]). One of our main results of this note is to generalize this fact for modules: If M is a finitely generated R-module, then M is Artinian if and only if M is Noetherian and S-$\mathrm{Spec}_R(M) = \mathrm{Max}_R(M)$. For other generalizations of this fact, the following interesting articles [5] and [23] are suggested.

This article consists of three sections. In Section 2, we prove some preliminary facts about strongly spectrum of modules. In Section 3, by considering the results in Section 2, we prove some important facts about the dimension of modules.

2. Strongly Spectrum of Modules

In this section, we give some facts about strongly spectrum of modules. We start with the following proposition which is similar to its counterpart in prime submodules (see [16, Lemma 4.1]).

Proposition 2.1. *Let M be an R-module and let $N \subseteq P$ be submodules of M. Then P is a strongly prime submodule of M if and only if P/N is a strongly prime submodule of M/N.*

Proof. Obvious. □

Lemma 2.2. *Let P be a proper submodule of M. Then the following are equivalent.*

1. *P is a strongly prime submodule of M.*
2. *$I_x^P I_y^P M \subseteq P$, for $x, y \in M$, implies that either $x \in P$ or $y \in P$.*

Proof. (1)⇒(2) Let $x, y \in M$ and $I_x^P I_y^P M \subseteq P$. If $I_y^P M \subseteq P$, then $y \in P$, since P is a strongly prime submodule. If $I_y^P M \nsubseteq P$, then there exists $z \in I_y^P M$ such that $z \notin P$. Since P is a strongly prime submodule and $I_x^P z \subseteq P$, we must have $x \in P$.
(2)⇒(1) Let $x, y \in M$ and $I_x^P y \subseteq P$. Then

$$I_x^P I_y^P M \subseteq I_x^P (Ry + P) \subseteq P.$$

Therefore $x \in P$ or $y \in P$ and hence P is a strongly prime submodule. □

If P is a strongly prime submodule of M and $\mathfrak{p} = (P : M)$, we say that P is a *strongly $\mathfrak{p}$-prime* submodule of M. The set of all strongly $\mathfrak{p}$-prime submodule of M is denoted by $\text{S-Spec}_{\mathfrak{p}}(M)$.

Theorem 2.3. *Let M be an R-module and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the following hold:*

1. $\text{S-Spec}_{\mathfrak{p}}(M) = \max\{N \leq M | \operatorname{Ann}(M/N) = \mathfrak{p}\}$.
2. *If $M/\mathfrak{p}M$ is a finitely generated R-module and* $\operatorname{Ann}(M) \subseteq \mathfrak{p}$, *then* $\text{S-Spec}_{\mathfrak{p}}(M) \neq \emptyset$.

Proof. (1) $\subseteq$: Let $P \in \text{S-Spec}_{\mathfrak{p}}(M)$. Suppose to the contrary that $P \notin \max\{N \leq M | \operatorname{Ann}(M/N) = \mathfrak{p}\}$. Then there exists a submodule K of M such that $P \subsetneq K$ and $\operatorname{Ann}(M/K) = \mathfrak{p}$. Let $y \in M$ and $x \in K \setminus P$. Then

$$I_x^P y = (P + Rx : M)y \subseteq \operatorname{Ann}(M/K)y = \mathfrak{p}y \subseteq P.$$

Since P is a strongly prime submodule, we must have $y \in P$. It follows that $P = M$, which is a contradiction.
$\supseteq$: Let $P \in \max\{N \leq M | \operatorname{Ann}(M/N) = \mathfrak{p}\}$. We claim that P is a strongly prime submodule. Suppose to the contrary that P is not a strongly prime submodule. Then by the above lemma, we have $I_x^P I_y^P M \subseteq P$, for some $x, y \in M \setminus P$. By maximality of P, we have $\mathfrak{p} \subsetneq I_x^P$ and $\mathfrak{p} \subsetneq I_y^P$. Let $r \in I_x^P \setminus \mathfrak{p}$ and $s \in I_y^P \setminus \mathfrak{p}$. Then $rsM \subseteq I_x^P I_y^P M \subseteq P$. It follows that $rs \in \operatorname{Ann}(M/P) = \mathfrak{p}$. Hence $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$, which is a contradiction.
(2): Set

$$\Sigma = \{N \leq M | \mathfrak{p}M \subseteq N \text{ and } \operatorname{Ann}(M/N) = \mathfrak{p}\}.$$

By [11, Proposition 8], $\mathfrak{p}M \in \Sigma$ and hence $\Sigma \neq \emptyset$. Let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots ,$$

be an ascending chain of Σ and let $K = \bigcup_{i=1}^{\infty} K_i$. We claim that $K \in \Sigma$. Clearly $\mathfrak{p} \subseteq \operatorname{Ann}(M/K)$. Now let $r \in \operatorname{Ann}(M/K)$. Since $M/\mathfrak{p}M$ is finitely generated, there exists $i \in \mathbb{N}$ such that $rM \subseteq K_i$ and hence $r \in \mathfrak{p}$. Therefore $\mathfrak{p} = \operatorname{Ann}(M/K)$ and so $K \in \Sigma$. By Zorn's Lemma Σ has a maximal element P. Now Part (1) implies that $P \in \text{S-Spec}_{\mathfrak{p}}(M)$ and the proof is complete. □

An R-module M is called a *multiplication module* if for each submodule N of M, $N = IM$ for some ideal I of R (see [4]). It is clear that every cyclic R-module is a multiplication module. It is also easy to check that an R-module M is a multiplication module if and only if $N = (N : M)M$ for all submodules N of M (see [22]). In the following we give some conditions under which the prime and strongly prime submodules coincide.

Proposition 2.4. *Let M be an R module. If M is multiplication then* $\text{S-Spec}_R(M) = \text{Spec}_R(M)$. *The converse is true if M is finitely generated.*

Proof. By considering Remark 1.1(1), it is enough to show that every prime submodule of M is strongly prime. Let $P \in \text{Spec}_R(M)$ and let $I_x^P y \subseteq P$ for some $x \in M$ and $y \in M \setminus P$. Because P is a prime submodule, we must have $I_x^P M \subseteq P$. Since M is multiplication, $P + Rx = (P + Rx : M)M = I_x^P M \subseteq P$ and hence $x \in P$.
Now let M be a finitely generated R-module such that $\text{S-Spec}_R(M) = \text{Spec}_R(M)$. Then it is easy to see that
$\text{S-Spec}_{R/\mathfrak{m}}(M/\mathfrak{m}M) = \text{Spec}_{R/\mathfrak{m}}(M/\mathfrak{m}M)$ for each $\mathfrak{m} \in \operatorname{Max}(R)$. Therefore Remark 1.1(2) implies that $M/\mathfrak{m}M$ is cyclic. Thus the assertion follows from [8, Corollary 1.5]. □

We conclude this section by obtaining some conditions under which $\text{S-Spec}_R(M) = \operatorname{Max}_R(M)$. First, we observe that strongly prime submodules behave naturally under localization. The following proposition shows that the second part of [18, Theorem 1.5] is still true if we drop the assumption "M is finitely generated".

Proposition 2.5. *Let M be an arbitrary R-module, and let S be a multiplicatively closed subset of R. Then*

$$\text{S-Spec}_{S^{-1}R}(S^{-1}M) = \{S^{-1}P | P \in \text{S-Spec}_R(M) \ \textit{and}\ (P : M) \subseteq R \setminus S\}.$$

Proof. $\subseteq$: Follows easily from [18, Theorem 1.5].

$\supseteq$: Let $P \in \text{S-Spec}_R(M)$ and $(P : M) \subseteq R \setminus S$. We prove that $S^{-1}P \in \text{S-Spec}_{S^{-1}R}(S^{-1}M)$. First, we show that $S^{-1}P \neq S^{-1}M$. On the contrary, suppose that $S^{-1}P = S^{-1}M$. Let $x \in M \setminus P$. Then $x/1 \in S^{-1}P$. Hence there exist $s \in S$ and $y \in P$ such that $x/1 = y/s$.

Therefore $stx \in P$ for some $t \in S$. Since $st \notin (P : M)$, we must have $x \in P$, which is a contradiction. Now let $x_1/s_1, x_2/s_2 \in S^{-1}M$ and $I_{x_1/s_1}^{S^{-1}M,S^{-1}P} I_{x_2/s_2}^{S^{-1}M,S^{-1}P} S^{-1}M \subseteq S^{-1}P$. Then $I_{x_1}^{M,P} I_{x_2}^{M,P} M \subseteq P$. Therefore $x_1 \in P$ or $x_2 \in P$ and hence $x_1/s_1 \in P$ or $x_2/s_2 \in P$, which completes the proof. □

Theorem 2.6. *Let M be an R-module. Then* $\text{S-Spec}_R(M) = \text{Max}_R(M)$ *in each of the following cases:*

(1) *M is an Artinian module.*
(2) *M is a semisimple R-module.*

Proof. (1): By considering Remark 1.1(2), it is enough to show that $\text{S-Spec}_R(M) \subseteq \text{Max}_R(M)$. First take R to be a local ring with maximal ideal $\mathfrak{m}$. Suppose to the contrary that $P \in \text{S-Spec}_R(M) \setminus \text{Max}_R(M)$. Then there exists $x \in M \setminus P$ such that $P + Rx \neq M$. By [21, Exercise 8.48], there exists a natural number n such that $\mathfrak{m}^n x = 0$. Hence $(I_x^P)^n x \subseteq \mathfrak{m}^n x = 0$. Since P is a strongly prime submodule, it follows that $x \in P$, which is a contradiction. Therefore $\text{S-Spec}_R(M) = \text{Max}_R(M)$. Now we go to the general case. Let R be any ring and let $P \in \text{S-Spec}_R(M)$ and $\mathfrak{p} = \text{Ann}(M/P)$. If $S = R \setminus \mathfrak{p}$, then the above proposition implies that $S^{-1}P \in \text{S-Spec}_{R_\mathfrak{p}}(M_\mathfrak{p})$. By the local case, $S^{-1}P \in \text{Max}_{R_\mathfrak{p}}(M_\mathfrak{p})$. It follows easily that $P \in \text{Max}_R(M)$.

(2): By considering Remark 1.1(2), it is enough to show that $\text{S-Spec}_R(M) \subseteq \text{Max}_R(M)$. Suppose to the contrary that $P \in \text{S-Spec}_R(M) \setminus \text{Max}_R(M)$. Let $\{S_i\}_{i\in I}$ be an indexed set of simple submodules of M such that $M = \bigoplus_{i\in I} S_i$. By the proof of [2, Lemma 9.2], there is a subset $J \subseteq I$ such that $M = P \bigoplus (\bigoplus_{j\in J} S_j)$. Since P is not maximal, $|J| \geq 2$, where $|J|$ denotes the cardinality of J. Let $0 \neq x_{j_1} \in S_{j_1}$ and $0 \neq y_{j_2} \in S_{j_2}$, where j_1, j_2 are two distinct elements of J. Put $x = (0, (x_j)_{j\in J})$, where $x_j = 0$ for all $j \in J \setminus \{j_1\}$ and $y = (0, (y_j)_{j\in J})$, where $y_j = 0$ for all $j \in J \setminus \{j_2\}$. Then we have $I_x^P y = I_x^{(P,0)} y = \{r \in R | rM \subseteq (P, 0) + Rx\} y = (\cap_{j\neq j_1} \text{Ann}(S_j)) y = \{0\} \subseteq P$. Since P is a strongly prime submodule, we should have $x \in P$ or $y \in P$, which is a contradiction. □

A useful characterization of Artinian rings is that a ring is Artinian if and only if it is Noetherian and every prime ideal is maximal. In the following result we generalize this characterization for modules.

Corollary 2.7. *Let M be a finitely generated R-module. Then M is Artinian if and only if M is Noetherian and* $\text{S-Spec}_R(M) = \text{Max}_R(M)$.

Proof. Suppose that M is Artinian. Then by Theorem 2.6, we have $\text{S-Spec}_R(M) = \text{Max}_R(M)$. Further, by [21, Exercise 7.28], $R/\text{Ann}(M)$

is an Artinian ring. It follows that $R/\mathrm{Ann}(M)$ is a Noetherian ring and [21, Corollary 7.22(i)] implies that M is Noetherian as an $R/\mathrm{Ann}(M)$-module. Hence M is Noetherian as an R-module.

Conversely, suppose that M is Noetherian and every strongly prime submodule of M is maximal. We claim that every prime ideal of $R/\mathrm{Ann}(M)$ is maximal. Let $\mathfrak{p}$ be a prime ideal of $R/\mathrm{Ann}(M)$. Set

$$\Sigma = \{N \leq M | \mathrm{Ann}(M/N) = \mathfrak{p}\}.$$

By [11, Proposition 8], $\mathfrak{p}M \in \Sigma$ and hence $\Sigma \neq \emptyset$. By Zorn's Lemma Σ has maximal element P. From Theorem 2.3, we have that P is a strongly prime submodule of M and by the hypothesis P is a maximal submodule of M. Hence $\mathfrak{p}$ is a maximal ideal of $R/\mathrm{Ann}(M)$. Therefore every prime ideal of $R/\mathrm{Ann}(M)$ is maximal and hence $R/\mathrm{Ann}(M)$ is Artinian. Thus it follows from [21, Corollary 7.22(ii)] that M is Artinian as an $R/\mathrm{Ann}(M)$-module. Hence M is Artinian as an R-module. □

3. Strongly Dimension

In this section, we study the connections between strongly dimension and classical Krull dimension. The following lemma is used widely in the sequel.

Lemma 3.1. *Let M be an R-module and let $P_1 \subseteq P_2$ in* $\text{S-Spec}_R(M)$. *If $(P_1 : M) = (P_2 : M)$, then $P_1 = P_2$.*

Proof. On the contrary, suppose that $P_1 \neq P_2$. Choose $x \in P_2 \setminus P_1$. Thus

$$I_x^{P_1} M \subseteq Rx + P_1 \subseteq P_2.$$

It follows from the assumption that $I_x^{P_1} M \subseteq P_1$ and hence $x \in P_1$, which is a contradiction. □

Theorem 3.2. *Let M be an R-module. Then*

$$\text{s-dim}_R(M) \leq \text{cl. K. dim}_R(M).$$

Proof. If M is a strongly primeless module, then $\text{s-dim}_R(M) = -\infty$ and there is nothing to prove. If M is not a strongly primeless module, consider the following chain of distinct strongly prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n.$$

By the above lemma we have the following chain

$$(P_0 : M) \subsetneq (P_1 : M) \subsetneq \cdots \subsetneq (P_n : M)$$

of distinct prime ideals of R. It follows that $\text{s-dim}_R(M) \leq \text{cl. K. dim}_R(M)$. □

Theorem 3.3. *Let M be an R-module. Then* $\text{s-dim}_R(M) = \text{cl. K. dim}_R(M)$ *if one of the following conditions holds.*

1. *M is a multiplication module.*
2. *M is a finitely generated module.*

Proof. Assume that (1) holds. In view of the above theorem it is enough to show that $\text{cl. K. dim}_R(M) \le \text{s-dim}_R(M)$. Consider the following chain of distinct prime ideals of R

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,$$

where $\text{Ann}(M) \subseteq \mathfrak{p}_i$ for all i. By [22, Theorem 9] and [8, Corollary 2.11], we have the following chain of distinct prime submodules of M

$$\mathfrak{p}_0 M \subsetneq \mathfrak{p}_1 M \subsetneq \cdots \subsetneq \mathfrak{p}_n M.$$

Now the assertion follows from Proposition 2.4.

Now assume that (2) holds. By Theorem 3.2, it is enough to show that $\text{cl. K. dim}_R(M) \le \text{s-dim}_R(M)$. Consider the following chain of distinct prime ideals of R

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,$$

where $\text{Ann}(M) \subseteq \mathfrak{p}_i$ for all i. By Theorem 2.3(2), $\text{S-Spec}_{\mathfrak{p}_0}(M) \neq \emptyset$. Let $P_0 \in \text{S-Spec}_{\mathfrak{p}_0}(M)$, $\overline{M} = M/P_0$ and $\overline{R} = R/\mathfrak{p}_0$. Again, by Theorem 2.3(2), we obtain that $\text{S-Spec}_{\mathfrak{p}_1/\mathfrak{p}_0}(\overline{M}) \neq \emptyset$. Let $\overline{P}_1 \in \text{S-Spec}_{\mathfrak{p}_1/\mathfrak{p}_0}(\overline{M})$. By Proposition 2.1, there exists $P_1 \in \text{S-Spec}_{\mathfrak{p}_1}(M)$ such that $\overline{P}_1 = P_1/P_0$. Continuing this process we obtain the following chain of strongly prime submodules of M

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n.$$

Therefore $\text{cl. K. dim}_R(M) \le \text{s-dim}_R(M)$. □

Corollary 3.4. *Let $M \subseteq M'$ be R-modules. Then the following hold.*

1. *If M' is finitely generated, then* $\text{s-dim}_R M \le \text{s-dim}_R M'$.
2. *If M is finitely generated and* $\text{Ann}(M) \subseteq \sqrt{\text{Ann}(M')}$, *then* $\text{s-dim}_R M' \le \text{s-dim}_R M$.

Proof. Follows easily from Theorem 3.2 and Theorem 3.3(2). □

Corollary 3.5. *The following statements hold.*

1. *Let $M_1, M_2, \ldots, M_n$ be R-modules. If* $\text{s-dim}_R(M_i) = \text{cl. K. dim}_R(M_i)$ *for all $1 \le i \le n$, then*

$$\text{s-dim}_R\Big(\bigoplus_{i=1}^{n} M_i\Big) = \text{cl. K. dim}_R\Big(\bigoplus_{i=1}^{n} M_i\Big).$$

(2) *Let* $\{M_i\}_{i\in I}$ *be a family of* R*-modules. If there is a finite subset* J *of* I *such that* $\bigcap_{i\in I}\operatorname{Ann}(M_i)=\bigcap_{j\in J}\operatorname{Ann}(M_j)$ *and* $\text{s-dim}_R(M_j)=\text{cl. K. dim}_R(M_j)$ *for all* $j\in J$*, then*
  (a) $\text{s-dim}_R(\bigoplus_{i\in I}M_i)=\text{cl. K. dim}_R(\bigoplus_{i\in I}M_i)$.
  (b) $\text{s-dim}_R(\prod_{i\in I}M_i)=\text{cl. K. dim}_R(\prod_{i\in I}M_i)$.

(3) *Let* M *be a free* R*-module. Then*

$$\text{s-dim}_R(M)=\text{cl. K. dim}_R(M)=\text{cl. K. dim}(R).$$

(4) *Let* M *be an* R*-module, and let* $M[x]$ $(M[[x]])$ *be the set of all formal polynomials (power series) in indeterminate* x *with coefficients from* M*. If* $\text{s-dim}_R(M)=\text{cl. K. dim}_R(M)$*, then*

$$\text{s-dim}_R(M[x])=\text{s-dim}_R(M[[x]])=\text{cl. K. dim}_R(M).$$

Proof. (1): It is easy to see that $\text{s-dim}_R(M_i)\leq\text{cl. K. dim}_R(\bigoplus_{i=1}^{n}M_i)$ for each $1\leq i\leq n$. Hence

$$\begin{aligned}\text{cl. K. dim}_R(\bigoplus_{i=1}^{n}M_i) &= \max\{\text{cl. K. dim}_R(M_i)\mid 1\leq i\leq n\}\\ &= \max\{\text{s-dim}_R(M_i)\mid 1\leq i\leq n\}\\ &\leq \text{s-dim}_R(\bigoplus_{i=1}^{n}M_i).\end{aligned}$$

Now the assertion follows from Theorem 3.2.

(2): Since the proofs of (a) and (b) are similar, we only prove (a). By Theorem 3.2, it is enough to show that $\text{cl. K. dim}_R(\bigoplus_{i\in I}M_i)\leq\text{s-dim}_R(\bigoplus_{i\in I}M_i)$. From (1), we have

$$\begin{aligned}\text{cl. K. dim}_R(\bigoplus_{i\in I}M_i) &= \text{cl. K. dim}R/(\bigcap_{i\in I}\operatorname{Ann}(M_i))\\ &= \text{cl. K. dim}R/(\bigcap_{j\in J}\operatorname{Ann}(M_j))\\ &= \text{cl. K. dim}_R(\bigoplus_{j\in J}M_j)\\ &= \text{s-dim}_R(\bigoplus_{j\in J}M_j)\\ &\leq \text{s-dim}_R(\bigoplus_{i\in I}M_i).\end{aligned}$$

This completes part (a).

(3): The assertion follows easily from part 2(a).

(4): It is easy to see that $M[x]\cong\bigoplus_{i\in\mathbb{N}}M_i$ and $M[[x]]\cong\prod_{i\in\mathbb{N}}M_i$, where $M_i=M$. Now the assertion follows from part (2). □

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A. R. Naghipour
Department of Mathematics, Shahrekord University, P.O.Box 115, Shahrekord, Iran.
`Email: naghipour@sci.sku.ac.ir naghipourar@yahoo.com`