Journal of Algebraic Systems
Vol. 1, No. 2, (2013), pp 135-147

LATTICE OF WEAK HYPER K-IDEALS OF A HYPER K-ALGEBRA

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Abstract. In this note, we study the lattice structure of the class of all weak hyper K-ideals of a hyper K-algebra. We first introduce the notion of (left, right) scalar element in a hyper K-algebra which helps us to characterize the weak hyper K-ideals generated by a subset. In the sequel, using the notion of a closure operator, we study the lattice of all weak hyper K-ideals of a hyper K-algebra and prove that, under suitable conditions, a special subclass of this class forms a Boolean lattice.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki [9] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then, a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras)was introduced in 1934 by F. Marty [12] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [5], R. A. Borzooei et al. applied the hyperstructures to BCK-algebras, and

MSC(2010): Primary: 06F35; Secondary: 03G25,
Keywords: Hyper K-ideals, Weak hyper K-ideals, Boolean lattice.
Received: 4 September 2013, Revised: 24 October 2013.

introduced the notion of a hyper K-algebra which is a generalization of BCK-algebra, and investigated some related properties. They also introduced the notion of a hyper K-ideal and a weak hyper K-ideal and gave relations between hyper K-ideals and weak hyper K-ideals (see also [4, 6]). As we know, distributive lattices have played a many faceted role in the development of lattice theory and it is one of the most extensive and most satisfying chapters of lattice theory. Also, distributive lattices have provided the motivation for many results in general lattice theory. In [3], Borzooei et al. studied hyper BCK-algebras [11] from lattice theory point of view and investigated the properties of the class of all weak hyper BCK-ideals of a hyper BCK-algebra. So, in this paper we study the lattice structure of the class of all weak hyper K-ideals of a hyper K-algebra and we prove that a special subclass of this class together with the suitable operations forms a Boolean lattice.

2. Preliminaries

In this section, we first give some fundamental definitions and results from literature. For more details, we refer to the references [4, 5, 6, 10].

By a *hyper K-algebra* we mean a structure $(H; <, \circ, 0)$ in which $<$ is a binary relation in H, $\circ$ is a binary hyperoperation on H, and 0 is a fixed element of H, and the following are satisfied: $\forall x, y, z \in H$,

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ z) \circ y = (x \circ y) \circ z$,
(HK3) $x < x$,
(HK4) $x < y$ and $y < x$ imply $x = y$,
(HK5) $0 < x$,
(HK6) $x < y$ iff $0 \in x \circ y$.

and for $A, B \subseteq H$, $A < B$ means that there exists $a \in A$ and $b \in B$ such that $a < b$.

Example 2.1. (1) Let $(X; *, 0)$ be a BCK-algebra and $x \circ y = \{x * y\}$, for all $x, y \in X$ be a hyperoperation on X. Then $(X; \circ, 0)$ is a hyper K-algebra

(2) Let $n \in \mathbb{N} \cup \{0\}$ and define the hyperoperation " $\circ$ " on $H_n = [n, \infty)$ by

$$x \circ y = \begin{cases} [n, x] & , \quad x \leq y, \\ (n, y] & , \quad x > y \neq n, \\ \{x\} & , \quad y = n. \end{cases}$$

for all $x, y \in H_n$. Then $(H_n; \circ, n)$ is a hyper K-algebra.

Proposition 2.2. *In any hyper K-algebra H the following statements hold: for all $x, y, z \in H$,*

(i) $(x \circ z) \circ (x \circ y) < y \circ z$,
(ii) $x \circ y < x$,
(iii) $x \circ (x \circ y) < y$,
(iv) $x \in x \circ 0$,
(v) $0 \in x \circ (x \circ 0)$.

Definition 2.3. Let H be a hyper K-algebra.

(i) H is said to be *positive implicative* if
$$(x \circ y) \circ z = (x \circ z) \circ (y \circ z) \quad (\forall x, y, z \in H).$$
(ii) A non-empty subset I of H is called a *weak-hyper K-ideal* if $0 \in I$ and $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$.
(iii) An element a of H is called an *atom* if $x < a$ implies that $x = 0$ or $x = a$.

We denote the set of all atoms and the set of all weak hyper K-ideals of H, by $A(H)$ and $\mathscr{W}(H)$, respectively.

Example 2.4. (i) In any hyper K-algebra H, $\{0\}$ and H are weak hyper K-ideals of H.

(ii) Let $H = \{0, 1, 2\}$ and hyperoperation '$\circ$' be defined as shown in Table 1. Then $(H; \circ)$ is a hyper K-algebra (see [6]). Furthermore, it is easy to check that $\{0, 2\}$ is a weak hyper K-ideal of H.

TABLE 1. The action of '$\circ$' on H

$\circ$	0	1	2
0	{0}	{0}	{0}
1	{1}	{0}	{1}
2	{2}	{0,1}	{0,1,2}

Proposition 2.5. [5] *Let $\{I_\alpha : \alpha \in \Lambda\}$ be a nonempty family of weak hyper K-ideals of hyper K-algebra H. Then $\cap\{I_\alpha : \alpha \in \Lambda\}$ is a weak hyper K-ideal of H.*

Remark 2.6. Example 2.4 shows that for any nonempty subset A of a hyper K-algebra H, there always exists a weak hyper K-ideal of H that contains A and so by Propsition 2.5, it follows that the intersection of any family of weak hyper K-ideals of H containing A is the least weak hyper K-ideal of H containing A, called the weak hyper K-ideal of H generated by A and we denote it by $[A]_w$. In this case, for $A \subseteq \mathscr{W}(H)$, we let $\wedge A = \inf A = \bigcap_{I \in A} I$ and $\dot{\vee} A = \sup A = [A]_w$. Moreover, it is obvious that $\{0\}$ is the least element and H is the greatest element of $\mathscr{W}(H)$ with respect to the set-theoretic inclusion.

Definition 2.7. [8] A lattice L is said to be

(1) *complete* if for any subset A of L, $\bigwedge A = \inf A$ and $\bigvee A = \sup A$ exist in L,

(2) *infinitely distributive* if for any indexed set Λ and $z \in L$,

$$z \wedge \left(\bigvee_{\alpha \in \Lambda} x_\alpha \right) = \bigvee_{\alpha \in \Lambda} (z \wedge x_\alpha) \tag{2.1}$$

(3) *modular* if

$$x \geq z \ \Rightarrow \ (x \wedge y) \vee z = x \wedge (y \vee z) \ \ (\forall x, y, z \in L),$$

(4) *semimodular* if $x \prec y$ implies that $x \vee z \preceq y \vee z$, where $x \prec y$ means that $x < y$ and there is no $z \in L$ such that $x < z < y$.

Note 2.8. It must be noticed that any lattice which satisfies (2.1), for $\Lambda = \{1, 2\}$, is called a *distributive lattice*.

Definition 2.9. [8] (i) An element a of a complete lattice L is called *compact* if $a \leq \bigvee A$ implies that $a \leq \bigvee A_1$, for some finite subset A_1 of $A \subseteq L$.

(ii) An element a of a lattice L with zero is called an *atom* if $x \leq a$ implies that $x = 0$ or $x = a$.

Definition 2.10. [8] A lattice L is called

(1) *algebraic* if it is complete and every element of L is a join of compact elements of L,

(2) *geometric* if L is semimodular, algebraic and the compact elements of L are exactly the finite joins of atoms.

Definition 2.11. [8]

(1) For element a of a bounded lattice L, $b \in L$ is called a *complement* of a if and only if $a \wedge b = 0$ and $a \vee b = 1$.

(2) A bounded lattice in which every element has a complement is said to be a *complemented* lattice.

(3) Any complemented distributive lattice is said to be a *Boolean* lattice.

Definition 2.12. [7] For nonempty set A, mapping $C : 2^A \longrightarrow 2^A$ is called a *closure operator* if for all $X \in 2^A$,

- $X \subseteq C(X)$, (Extended)
- $C^2(X) = C(X)$, (Idempotent)
- $X \subseteq Y \ \Rightarrow \ C(X) \subseteq C(Y)$, (Isotone)

Nonempty subset X of A is said to be *closed* if $C(X) = X$. We denote the set of all closed subsets of A, by L_C. If for $X \in 2^A$ there exists $Y \in 2^A$ such that $C(Y) = X$, we say that X is generated by Y, and Y is a *generating set* for X. When Y is finite we say that X is *finitely generated*. A closure operator C on A is said to be *algebraic* if $C(X) = \cup\{C(Y) : Y \subseteq X \text{ is finite}\}$.

Theorem 2.13. [7] *If C is a closure operator on a set A, then L_C is a complete lattice with*

$$\bigwedge_{i\in I} C(A_i) = \bigcap_{i\in I} C(A_i), \ \bigvee_{i\in I} C(A_i) = C\left(\bigcup_{i\in I} A_i\right).$$

Theorem 2.14. [7] *If C is an algebraic closure operator on a set A, then L_C is an algebraic lattice, and the compact elements of L_C are precisely the closed sets $C(X)$, where X is a finite subset of A.*

3. Main results

Throughout the paper, H will denotes a hyper K-algebra, unless otherwise mentioned.

As a first result, from the previous section, we conclude that $\mathscr{W}(H)$ is a complete lattice. Indeed

Theorem 3.1. *$\mathscr{W}(H)$ is a complete lattice, where for all $I, J \in \mathscr{W}(H)$, $I \wedge J = I \cap J$ and $I \dot{\vee} J = [I \cup J]_w$.*

Proof. It follows from Proposition 2.5 and Remark 2.6. □

Definition 3.2. Let H be a hyper K-algebra.

(i) An element $a \in H$ is called a (*right*) *left scalar* if ($|x \circ a| = 1$) $|a \circ x| = 1$, for any $x \in H$.
(ii) An element $a \in H$ is called a *scalar* if a is both a left and a right scalar.
(iii) H is called (0_r or 0_l) 0-*scalar* if 0 is a (right or left) scalar element of H.

Notation. We denote the set of all (right) left scalars of H, by (resp. $R(H)$) $L(H)$.

Proposition 3.3. *Let $S(H) = \{a \in H : a \circ a = \{0\}\}$. Then $L(H) \subseteq R(H) \subseteq S(H)$.*

Proof. Let $a \in R(H)$. Then for each $x \in H$ we have $|x \circ a| = 1$. Taking $x = a$ we get $|a \circ a| = 1$. Since $0 \in a \circ a$, then $a \circ a = \{0\}$ and so $a \in S(H)$. The proof of other case is similar. □

Definition 3.4. In a hyper K-algebra H, we say that $<$ is *isotone* to the right if $x < y$ implies that $x \circ z < y \circ z$, for all $z \in H$. In this case, we say that H is $<_r$-*isotone*.

Example 3.5. Let $H = \{0, 1, 2\}$ and hyperoperation '$\circ$' be defined as shown in Table 2. Then $(H; \circ)$ is a hyper K-algebra (see [6]). Furthermore, it is easy to check that $<$ is isotone to the right.

TABLE 2. The action of '$\circ$' on H

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$

Lemma 3.6. *Let H be $<_r$-isotone and $|x \circ y| < \infty$, for all $x, y \in H$.*

(i) *If $A \subseteq R(H)$, then*

$$[A]_w = \left\{ \begin{array}{lr} x \in H : (\cdots(x \circ a_1) \circ \cdots) \circ a_n = \{0\}, & \textit{for some } n \in \mathbb{N}, \\ & a_1, \cdots, a_n \in A \end{array} \right\}.$$

(ii) *Assume that H is a positive implicative hyper K-algebra and $a \in R(H)$. Then*

$$[a]_w = \{x \in H : x \circ a = \{0\}\}.$$

(iii) *If H is a positive implicative hyper K-algebra and $a \in R(H) \cap A(H)$, then $[a]_w = \{0, a\}$. Moreover, $[a]_w$ is an atom of $\mathscr{W}(H)$.*

Proof. (i) Let

$$B = \{x \in H : (\cdots(x \circ a_1) \circ \cdots) \circ a_n = \{0\}, n \in \mathbb{N}, a_1, \cdots, a_n \in A\}.$$

First we have to prove that B is a weak hyper K-ideal of H. Since $A \subseteq R(H)$, then for any $a \in A$ we have $0 \circ a = \{0\}$ whence $0 \in B$. Now, let $x \circ y \subseteq B$ and $y \in B$, for $x, y \in H$. Hence, for all $z \in x \circ y$ there exist $n_z \in \mathbb{N}$ and $a_1, a_2, \cdots, a_{n_z} \in A$ such that

$$(\cdots((z \circ a_1) \circ \cdots) \circ a_{n_z} = \{0\}.$$

Since $|x \circ y| < \infty$, we can take $n = \max\{n_z : z \in x \circ y\}$ and so without loss of generality,

$$\begin{aligned} (\cdots((x \circ a_1) \circ \cdots) \circ a_n) \circ y &= (\cdots((x \circ y) \circ a_1) \circ \cdots) \circ a_n \\ &= \bigcup_{z \in x \circ y} (\cdots((z \circ a_1) \circ \cdots) \circ a_{n_z} = \{0\}. \end{aligned}$$

This implies that

$$(\cdots((x \circ a_1) \circ \cdots) \circ a_n) < y$$

i.e, $a < y$, for some $a \in (\cdots((x \circ a_1) \circ \cdots) \circ a_n$. Also, since $y \in B$, then there exist $m \in \mathbb{N}$ and $b_1, b_2, \ldots, b_m \in A$ such that

$$(\cdots((y \circ b_1) \circ b_2) \circ \cdots) \circ b_m = \{0\}.$$

Since $<$ is isotone to the right, $a \circ b_1 < y \circ b_1$, which implies that

$$((\cdots(x \circ a_1) \circ \cdots) \circ a_n) \circ b_1 < y \circ b_1.$$

By continuing this process, we obtain

$$(\cdots(((\cdots(x \circ a_1) \circ \cdots) \circ a_n) \circ b_1) \circ \cdots) \circ b_m < (\cdots((y \circ b_1) \circ b_2) \circ \cdots) \circ b_m = \{0\}.$$

Since

$$|(\cdots(((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n) \circ b_1) \circ \cdots) \circ b_m| = 1,$$

so

$$(\cdots(((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n) \circ b_1) \circ \cdots) \circ b_m = \{0\},$$

proving $x \in B$. Consequently, B is a weak hyper K-ideal of H. Moreover, since $a \circ a = \{0\}$ for all $a \in A$, then $a \in B$ means that $A \subseteq B$. Now, let C be a weak hyper K-ideal of H containing A and $x \in B$. Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \cdots, a_n \in A$ such that

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\} \subseteq C.$$

Since C is a weak hyper K-ideal and $a_n \in A \subseteq C$, then

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{n-1} \subseteq C.$$

By continuing this process we get that $x \in C$. Therefore, $B = [A]_w$.

(ii) Obviously, $x \circ a = \{0\}$ implies that $x \in [a]_w$, that is, $\{x \in H : x \circ a = \{0\}\} \subseteq [a]_w$. Conversely, since $a \in R(H)$ and $R(H) \subseteq S(H)$ and H is positive implicative, then

$$(x \circ a) \circ a = (x \circ a) \circ (a \circ a) = (x \circ a) \circ 0 = x \circ a.$$

By continuing this process after n steps (with $n \in \mathbb{N}$) we get

$$(\cdots((x \circ \underbrace{a) \circ a) \circ \cdots) \circ a}_{n\ times} = x \circ a.$$

Hence, if $x \in [a]_w$, then there exists $n \in \mathbb{N}$ such that

$$x \circ a = (\cdots((x \circ \underbrace{a) \circ a) \circ \cdots) \circ a}_{n\ times} = \{0\}.$$

Therefore, $[a]_w = \{x \in H : x \circ a = \{0\}\}$.

(iii) Let $a \in R(H) \cap A(H)$. Then $a \circ a = \{0\} = 0 \circ a$, which imply that $\{0, a\} \subseteq [a]_w$. Now, let $x \in [a]_w$. Then by (ii), $x \circ a = \{0\}$ and so $x < a$. Since $a \in A(H)$, then $x = 0$ or $x = a$ whence $x \in \{0, a\}$. Hence, $[a]_w \subseteq \{0, a\}$. Therefore, $[a]_w = \{0, a\}$. $\square$

Note. Hereafter, in this paper, we suppose that $|x \circ y| < \infty$, for all $x, y \in H$.

Example 3.7. (i) Consider the hyper K-algebra H given in Example 3.5. Obviously, $\{1,2\} \nsubseteq R(H)$. Moreover,

$$\{x \in H : (\cdots(x \circ a_1)\circ \cdots)\circ a_n = \{0\}, n \in \mathbb{N}, a_1, \ldots, a_n \in \{1,2\}\} = \{0,1\}$$

which is a weak hyper K-ideal of H, does not contain $\{1,2\}$. So, $[\{1,2\}]_w \neq \{0,1\}$. Moreover, $\{x \in H : x \circ 2 = \{0\}\} = \{0\} \neq [2]_w$. This example shows that the condition "inclusion in $R(H)$", in Lemma 3.6(i), and "positive implicativity", in Lemma 3.6(ii), are necessary.

(ii) Let $(H; \circ)$ be a hyper K-algebra (see [4]), where $H = \{0,1,2,3\}$ and Cayley table of $\circ$ is below (see Table 3). It is seen that 3 is a right scalar and $[3]_w = \{0,1,3\}$, while H is not $<_r$-isotone and positive implicative because, $2 < 1$ whereas $2 \circ 3 = \{2\} \nless \{0\} = 1 \circ 3$ and $(3 \circ 1) \circ 1 = \{0\} \neq \{1\} = (3 \circ 1) \circ (1 \circ 1)$. This shows that "positive implicativity" and "$<_r$-isotonicity" in Lemma 3.6(ii), are sufficient conditions.

TABLE 3. The action of '$\circ$' on H

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
2	$\{2\}$	$\{0,2\}$	$\{0,2\}$	$\{2\}$
3	$\{3\}$	$\{1\}$	$\{3\}$	$\{0\}$

Notation. We denote a hyper K-algebra in which $<$ is isotone to the right, by $H^{<_r}$. Let $PR(H^{<_r})$ be the set of all nonempty subsets of H contained in $R(H^{<_r})$, $\mathscr{WR}(H^{<_r})$ be the set of all weak hyper K-ideals of H that is contained in $R(H)$, and $[A]_w^r$ be the intersection of all weak hyper K-ideals of H contained in $R(H)$ which contain A. To ensure that this situation can be exists we consider the next example.

Example 3.8. Let $H = \{0,1,2,3\}$ and hyperoperation '$\circ$' be given as shown in Table 4. Then, $(H, \circ)$ is a hyper K-algebra [6]. It is easy to see that $<$ is isotone to the right, $R(H^{<_r}) = \{0,2\}$, and $\mathscr{WR}(H^{<_r}) = \{\{0\},\{0,2\}\}$. So, the definitions given in the above notation are well-defined.

Lemma 3.9. *The mapping $C_w^r : PR(H^{<_r}) \longrightarrow PR(H^{<_r})$ which is defined by $C_w^r(A) = [A]_w^r$, for all $A \in PR(H^{<_r})$, is a closure operator provided that $[A]_w^r$ exists.*

TABLE 4. The action of '$\circ$' on H

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{0,1\}$	$\{3\}$	$\{0,1,3\}$

Proof. Obviously, $A \subseteq [A]^r_w$ and $[[A]^r_w]^r_w = [A]^r_w$, for all $A \in PR(H^{<_r})$, proving C^r_w is extended and idempotent, respectively. Now, let $A, B \in PR(H^{<_r})$ be such that $A \subseteq B$ and $x \in [A]^r_w$. Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n \in A$ such that $(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}$ and since $A \subseteq B$, $a_1, \dots, a_n \in B$. Hence, $x \in [B]^r_w$, proving that C^r_w is isotone. $\square$

Remark 3.10. In Lemma 3.9, we assumed that for all nonempty subsets A of $PR(H^{<_r})$, $[A]^r_w$ always exists. In such a case, $\mathscr{W}\mathscr{R}(H^{<_r})$ has the greatest element, denoted by $\mathbf{1}$. Also, if H is 0_r-scalar, then $\mathbf{0}$ is a weak hyper K-ideal contained in $R(H^{<_r})$; i.e, $\mathbf{0} \in \mathscr{W}\mathscr{R}(H^{<_r})$ means that $\mathscr{W}\mathscr{R}(H^{<_r})$ is a lattice with zero. Consequently, $\mathscr{W}\mathscr{R}(H^{<_r})$ is bounded. Moreover, $\mathscr{W}\mathscr{R}(H^{<_r})$ is the set of all closed subsets of $PR(H^{<_r})$, which is a poset under set inclusion as the partial ordering.

By Theorem 2.13, we have

Theorem 3.11. *$\mathscr{W}\mathscr{R}(H^{<_r})$ is a complete lattice in which $\wedge I_i = \cap I_i$ and $\dot{\vee} I_i = [\cup I_i]^r_w$. Indeed, it is a complete sublattice of $\mathscr{W}(H)$.*

Lemma 3.12. *C^r_w is algebraic.*

Proof. Let $A \in PR(H)$ and $B \subseteq A$ be finite. Then $[B]^r_w \subseteq [A]^r_w$ and so $\cup\{[B]^r_w : B \subseteq A \text{ is finite}\} \subseteq [A]^r_w$. Now, let $x \in [A]^r_w$. Then there exist $n \in \mathbb{N}$ and $a_1, \dots, a_n \in A$ such that $(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\}$. Let $B = \{a_1, a_2, \dots, a_n\}$. Then $x \in [B]^r_w$ and so

$$x \in \cup\{[B]^r_w : B \subseteq A \text{ is finite}\}.$$

Thus, $[A]^r_w = \cup\{[B]^r_w : B \subseteq A \text{ is finite}\}$, showing that C^r_w is algebraic. $\square$

Now, by Lemma 3.12 and Theorem 2.14 we get

Theorem 3.13. *$\mathscr{W}\mathscr{R}(H^{<_r})$ is an algebraic lattice in which the compact elements are precisely the weak hyper K-ideals $[A]^r_w$, where A is a finite set in $PR(H^{<_r})$.*

Corollary 3.14. *The finitely generated subsets of $PR(H^{<_r})$ are precisely the compact elements of $\mathscr{W}\mathscr{R}(H^{<_r})$.*

Theorem 3.15. *$\mathscr{W}\mathscr{R}(H^{<_r})$ is an infinitely distributive lattice.*

Proof. Let $J_\alpha, K \in \mathscr{W}\mathscr{R}(H^{<_r})$, $\alpha \in \Lambda$. We prove that

$$K \cap \left(\dot{\bigvee}_{\alpha\in\Lambda} J_\alpha\right) = \dot{\bigvee}_{\alpha\in\Lambda}(K \cap J_\alpha).$$

Obviously,

$$\dot{\bigvee}_{\alpha\in\Lambda}(K \cap J_\alpha) \subseteq K \cap \left(\dot{\bigvee}_{\alpha\in\Lambda} J_\alpha\right).$$

Let $x \in K \cap (\dot{\bigvee}_{\alpha\in\Lambda} J_\alpha)$. Then $x \in K$ and $x \in \dot{\bigvee}_{\alpha\in\Lambda} J_\alpha = \left[\bigcup_{\alpha\in\Lambda} J_\alpha\right]_w$, which implies that

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\},$$

for some $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \bigcup_{\alpha\in\Lambda} J_\alpha$. Observe that since $a_i \in R(H)$, then $|(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_i| = 1$, for $i \in \{1, 2, \ldots, n\}$. Let

$$\{j_r\} = (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{r-1}$$

and $a_r \in J_r$, for $r \in \{1, 2, \ldots, n\}$. Then

$$(\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_n = \{0\} \subseteq J_n,$$

which implies that

$$\{j_n\} = (\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{n-1} \subseteq J_n,$$

because $a_n \in J_n$ and J_n is a weak hyper K-ideal. Then,

$$\begin{aligned}(\cdots((x \circ j_n) \circ a_1) \circ \cdots) \circ a_{n-1} &= ((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{n-1}) \circ j_n \\ &= j_n \circ j_n = \{0\} \subseteq J_{n-1}.\end{aligned}$$

By continuing this process we get

$$(\cdots((x \circ j_1) \circ j_2) \circ \cdots) \circ j_n = \{0\} \tag{3.1}$$

where $j_i \in J_i$, for $i \in \{1, 2, \ldots, n\}$. Now, since $x \in R(H)$ and $0 \in x \circ x$, then $x \circ x = \{0\}$ and so

$$\begin{aligned} j_r \circ x &= ((\cdots((x \circ a_1) \circ a_2) \circ \cdots) \circ a_{r-1}) \circ x \\ &= (\cdots((x \circ x) \circ a_1) \circ \cdots) \circ a_{r-1} \\ &= \{0\} \subseteq K.\end{aligned}$$

Since K is a weak hyper K-ideal and $x \in K$, then $j_r \in K$ whence $j_r \in K \cap J_r$, for $r \in \{1, 2, \ldots, n\}$. This together with (3.1) imply that

$$x \in \left[\bigcup_{\alpha\in\Lambda}(K \cap J_\alpha)\right]_w = \dot{\bigvee}_{\alpha\in\Lambda}(K \cap J_\alpha).$$

$\square$

Proposition 3.16. *For every $I \in \mathscr{W}\mathscr{R}(H^{<r})$ we have $I = \dot{\bigvee}_{a\in I}[a]_w^r$.*

Proof. Let I be a weak hyper K-ideal of H contained in $R(H^{<r})$. It is clear that $\dot{\bigvee}_{a\in I}[a]_w^r \subseteq I$. Now, let $x \in I$. Since $I \subseteq R(H^{<r})$, then $x \circ x = \{0\}$ and so $x \in [x]_w^r \subseteq \dot{\bigvee}_{a\in I}[a]_w^r$. Hence, $I \subseteq \dot{\bigvee}_{a\in I}[a]_w^r$ and so $I = \dot{\bigvee}_{a\in I}[a]_w^r$. $\square$

Notation. Let $\mathscr{W}\mathscr{R}\mathscr{A}(H^{<r})$ denote the set of all weak hyper K-ideals of $H^{<r}$ contained in $R(H^{<r}) \cap A(H^{<r})$.

Theorem 3.17. *If $H^{<r}$ is positive implicative, $\mathscr{W}\mathscr{R}\mathscr{A}(H^{<r})$ is a geometric lattice.*

Proof. The only thing remains to prove is to verify that every compact element in $\mathscr{W}\mathscr{R}(H^{<r})$ is exactly the finite join of atoms of $\mathscr{W}\mathscr{R}(H^{<r})$. By Proposition 3.16, we know that $I = \dot{\bigvee}_{a\in I}[a]_w$. Then, if I is compact we get $I = \dot{\bigvee}_{i=1}^n[a_i]_w$, for some $a_1, a_2, \ldots, a_n \in I$. Now, by Lemma 3.6(iii), the proof is complete. $\square$

Definition 3.18. [8] For lattice L with zero and $a \in L$, a^* is called the *pseudocomplement* of a if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$, for all $x \in L$.

Let $\mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r})) = \{I^* : I \in \mathscr{W}\mathscr{R}(H^{<r})\}$. Then $\mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$ with respect to the partial ordering of $\mathscr{W}\mathscr{R}(H^{<r})$ is a lattice. Furthermore, we have the following.

Lemma 3.19. *In $\mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$ the following hold:*

(1) $I \wedge J = \inf\limits_{I,J\in\mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))}\{I, J\} = \inf\limits_{I,J\in\mathscr{W}\mathscr{R}(H^{<r})}\{I, J\}$.
(2) $I \tilde{\vee} J = \sup\limits_{I,J\in\mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))}\{I, J\} = (I^* \wedge J^*)^*$.
(3) $I \subseteq I^{**}$.
(4) $I \subseteq J$ *implies that* $I^* \supseteq J^*$.
(5) $I^* = I^{***}$.
(6) $I \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$ *if and only if* $I = I^{**}$.
(7) $I, J \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$ *implies that* $I \wedge J = (I \wedge J)^{**} \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$.
(8) *For* $I, J \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$ *we have* $(I \dot{\vee} J)^{**} = (I^* \wedge J^*)^*$.
(9) *For* $I, J \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$, $I \tilde{\vee} J = (I \dot{\vee} J)^{**}$.

Proof. The proof of (1)-(6) is similar to that of [8, Theorem 100].

(7) Let $I, J \in \mathcal{S}(\mathscr{W}\mathscr{R}(H^{<r}))$. Hence, there exist $L, K \in \mathscr{W}\mathscr{R}(H^{<r})$ such that $I = L^*$ and $J = K^*$. Then by (5),

$$I^{**} = L^{***} = L^* = I.$$

Similarly, we can prove that $J^{**} = J$. Since $I \wedge J \subseteq I, J$, then

$$I = I^{**} \supseteq (I \wedge J)^{**}\ ,\ J = J^{**} \supseteq (I \wedge J)^{**}.$$

Hence, $I \wedge J \supseteq (I \wedge J)^{**}$. But by (3), $I \wedge J \subseteq (I \wedge J)^{**}$ and so $I \wedge J = (I \wedge J)^{**} \in \mathcal{S}(\mathscr{WR}(H^{<_r}))$. Now, if $K \in \mathcal{S}(\mathscr{WR}(H^{<_r}))$ be such that $K \subseteq I$ and $K \subseteq J$, then $K \subseteq I \wedge J = (I \wedge J)^{**}$. Therefore,

$$I \wedge J = (I \wedge J)^{**} = \inf_{I,J \in \mathcal{S}(\mathscr{WR}(H^{<_r}))} \{I, J\}$$

(8) Since by (3), $I, J \subseteq I \dot{\vee} J \subseteq (I \dot{\vee} J)^{**}$, then $(I \dot{\vee} J)^{**}$ is an upper bound for $\{I, J\}$. Now, let $K \in \mathcal{S}(\mathscr{WR}(H^{<_r}))$ be such that $I \subseteq K$ and $J \subseteq K$. Then $I \dot{\vee} J \subseteq K$ and so by (4) and (6), $(I \dot{\vee} J)^{**} \subseteq K^{**} = K$. This implies that

$$I \tilde{\vee} J = (I \dot{\vee} J)^{**} = \sup_{I,J \in \mathcal{S}(\mathscr{WR}(H^{<_r}))} \{I, J\}.$$

(9) Let $I, J \in \mathcal{S}(\mathscr{WR}(H^{<_r}))$. Since $I^*, J^* \supseteq I^* \wedge J^*$, then by (3) and (4), $I \subseteq I^{**} \subseteq (I^* \wedge J^*)^*$. Similarly, $J \subseteq (I^* \wedge J^*)^*$. Hence, in $\mathscr{WR}(H^{<_r})$, $I \dot{\vee} J \subseteq (I^* \wedge J^*)^*$ and so by (5),

$$(I \dot{\vee} J)^{**} \subseteq (I^* \wedge J^*)^{***} = (I^* \wedge J^*)^*.$$

Moreover, since $I, J \subseteq I \dot{\vee} J$ in $\mathscr{WR}(H^{<_r})$, then $I^*, J^* \supseteq (I \dot{\vee} J)^*$. Thus $I^* \wedge J^* \supseteq (I \dot{\vee} J)^*$ and so by (4), $(I^* \wedge J^*)^* \subseteq (I \dot{\vee} J)^{**}$. Therefore,

$$(I \dot{\vee} J)^{**} = (I^* \wedge J^*)^*.$$

□

By Glivenko's theorem [1, Theorem 7.2] we have

Theorem 3.20. $(\mathcal{S}(\mathscr{WR}(H^{<_r}), \wedge, \dot{\vee}, {}^*, \mathbf{0}, \mathbf{1})$ *is a Boolean algebra.*

Acknowledgments

The author would like to thank the anonymous referees for the valuable suggestions and comments for improving of this paper.

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