QUASI-PRIMARY DECOMPOSITION IN MODULES OVER PRÜFER DOMAINS

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Abstract. In this paper we investigate decompositions of submodules in modules over a Prüfer domain into intersections of quasi-primary and classical quasi-primary submodules. In particular, existence and uniqueness of quasi-primary decompositions in modules over a Prüfer domain of finite character are proved.

1. Introduction

Throughout this paper all rings are commutative with identity elements, and all modules are unital. Let $M$ be an $R$-module. For every nonempty subset $X$ of $M$ and every submodule $N$ of $M$, the ideal $\{ r \in R \mid rX \subseteq N \}$ will be denoted by $(N : X)$. Note that $(N : M)$ is the annihilator of the module $M/N$. Also we denote the classical Krull dimension of $R$ by $\dim(R)$, and for an ideal $I$ of $R$, $\sqrt{I} := \{ r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N} \}$.

We recall that a proper ideal $Q$ of the ring $R$ is called a primary ideal if $ab \in Q$ where $a, b \in R$, implies that either $a \in Q$ or $b^k \in Q$ for some $k \in \mathbb{N}$ (see for example [2]). The notion of primary ideal was generalized by Fuchs [6] by defining an ideal $Q$ of a ring $R$ to be quasi-primary if its radical is a prime ideal, i.e., if $ab \in Q$ where $a, b \in R$, then either $a^k \in Q$ or $b^k \in Q$ for some $k \in \mathbb{N}$ (see also [7]). There are some extensions of these notions to modules. For instance, a proper submodule $Q$ of $M$ is called a primary submodule if $am \in Q$, where

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a \in R, \ m \in M \setminus Q, \ then \ a^kM \subseteq Q \ for \ some \ k \in \mathbb{N} \ (see \ for \ example [9, 10]). \ Also, \ Q \ is \ called \ quasi-primary \ if \ \sqrt{(Q : M)} \ is \ a \ prime \ ideal \ of \ R \ (see \ [1]). \ Moreover, \ Q \ is \ called \ a \ classical \ primary \ submodule \ of \ M \ if \ abN \subseteq Q, \ where \ a, b \in R \ and \ N \ is \ a \ submodule \ of \ M, \ then \ either \ aN \subseteq Q \ or \ b^kN \subseteq Q \ (resp. \ a^kN \subseteq Q \ or \ b^kN \subseteq Q) \ for \ some \ k \in \mathbb{N} \ (see \ [3, 4]). \ We \ note \ that \ if \ Q \ is \ a \ primary, \ quasi-primary, \ classical \ primary \ or \ a \ classical \ quasi-primary submodule \ of \ M, \ then \ \mathcal{P} := \sqrt{(Q : M)} \ is \ a \ prime \ ideal \ of \ R, \ and \ hence, \ we \ say \ that \ Q \ is \ a \ \mathcal{P}-primary, \ \mathcal{P}-quasi-primary, \ classical \ \mathcal{P}-primary \ or \ a \ classical \ \mathcal{P}-quasi-primary submodule; \ respectively.

Let K, N, N_1, \ldots, N_l, \ for \ some \ l \in \mathbb{N}, \ be \ submodules \ of \ an \ R-module \ M. \ We \ say \ that \ N \ and \ K \ are \ co-maximal \ (resp. \ with \ incomparable radicals) \ when \ N + K = M \ (resp. \ when \ \sqrt{(N : M)} \ and \ \sqrt{(K : M)} \ are \ not \ comparable); \ also \ we \ say \ that \ the \ submodules \ N_1, \ldots, N_l \ are \ pairwise \ co-maximal \ (resp. \ with \ pairwise \ incomparable radicals) \ if \ and only \ if \ for \ every \ i, j \in \{1, 2, \ldots, l\} \ such \ that \ i \neq j, \ N_i + N_j = M \ (resp. \ \sqrt{(N_i : M)} \ and \ \sqrt{(K_j : M)} \ are \ not \ comparable). \ An \ R-module \ M \ is \ called \ a \ multiplication module \ if, \ for \ each \ submodule \ N \ of \ M, \ there \ exists \ an \ ideal \ I \ of \ R \ such \ that \ N = IM; \ In \ this \ case \ we \ can \ take \ I = (N : M) \ (see \ for \ example [5]). \ For \ an \ integral \ domain \ R, \ we \ say \ that \ R \ is \ of \ finite \ character, \ if \ every \ nonzero \ element \ of \ R \ is \ contained \ but \ in \ a \ finite \ number \ of \ maximal \ ideals.

In a Prüfer domain of finite character, Fuchs and Mosteig [7] established the decomposition of an ideal as (shortest) intersections of a finite number of quasi-primary ideals. In particular, they proved that every nonzero ideal I in a Prüfer domain of finite character is a finite intersection of quasi-primary ideals with incomparable radicals, and the components in such a decomposition are uniquely determined by I (see [7, Theorem 5.6]). In Section 1, some results on quasi-primary and classical quasi-primary submodules are given. For instance, it is shown that if R is a domain, then for each R-module M, every classical quasi-primary submodule of M is a quasi-primary submodule if and only if every proper ideal of R is (classical) quasi-primary, if and only if, the set of prime ideals, Spec(R), is a chain (see Proposition 1.5). In Section 2, we generalize some main results of [7] to modules over a Prüfer domain of finite character. In particular, we prove that over a Prüfer domain of finite character, every submodule N of a module M such that (N : M) \neq (0), can be shown as an (minimal) intersection of finite number of (classical) quasi-primary submodules (see Theorem 2.7). Also we prove that the components in the decomposition of N.
into quasi-primary submodules are uniquely determined by $N$ (see Theorem 2.10). If $M$ is also a multiplication module, such decomposition into quasi-primary submodules exists for every nonzero submodule of $M$ (see Theorem 2.11).

2. Some results on (classical) quasi-primary submodules

We begin this section with two Propositions 1.1 and 1.2, which give many examples of classical primary submodules; so many examples of classical quasi-primary submodules; that are not primary submodules.

Proposition 2.1. Let $R$ be an integral domain and $\mathcal{P}$ be a nonzero prime ideal of $R$. Let for a nonempty set $I$, $Q = \oplus_{i \in I} A_i$ be a submodule of a free $R$-module $F = \oplus_{i \in I} R$ such that for every $i \in I$, $A_i = (0)$ or $A_i$ is a $\mathcal{P}$-primary ideal of $R$. If the set $\Gamma := \{ A_i \mid i \in I \text{ and } A_i \text{ is a } \mathcal{P}\text{-primary ideal of } R \}$ is a finite set, then $Q$ is a classical primary submodule of $F$. In addition, if $Q \neq (0)$ and for some $i \in I$, $A_i = (0)$, then $Q$ is not a primary submodule of $F$.

Proof. Let $r, s \in R$ and $N$ be a submodule of $F$ such that $rN \not\subseteq Q$ and $rsN \subseteq Q$. Then there is $y = \{y_i\}_{i \in I} \subseteq N$ such that $ry \not\subseteq Q$. We can assume that $r$ and $s$ are nonzero; so $rs \neq 0$, because $R$ is an integral domain. Since $rsy \in Q$, $rsy_i \in A_i$, for every $i \in I$. But $ry \not\subseteq Q$, so there is an $i_0 \in I$ that $ry_{i_0} \not\in A_{i_0}$. Clearly $A_{i_0}$ is nonzero, so $A_{i_0}$ is a $\mathcal{P}$-primary ideal of $R$. Now since $rsy_{i_0} \in A_{i_0}$ and $ry_{i_0} \not\in A_{i_0}$, we conclude that $s \in \sqrt{A_{i_0}} = \mathcal{P}$. Evidently for every $z = \{z_i\}_{i \in I} \subseteq N$, if $A_j = 0$, for some $j \in I$, then $z_j = 0$, so since the set $\Gamma$ is finite, there is a positive integer $k$ such that $s^kN \subseteq Q$; on the other word, $Q$ is a classical primary submodule of $F$.

Now, suppose that $Q \neq (0)$ and for some $i \in I$, $A_i = (0)$. So there are $i_1, i_2 \in I$ such that $A_{i_1} \neq (0)$ and $A_{i_2} = (0)$. Set $f = \{f_i\}_{i \in I}$ where $f_{i_1} = 1$ and for every $i \in I \setminus \{i_1\}$, $f_i = 0$. Evidently $f \not\subseteq Q$ and for every nonzero element $p \in \mathcal{P}$, there is a positive integer $k$ that $p^k f \in Q$. Now if for a positive integer $l$, $(p^k)^lF \subseteq Q$, then $p^{lk} \in A_{i_2} = (0)$, i.e., $p^{lk} = 0$. But $R$ is an integral domain, so $p = 0$, a contradiction. On the other word, $Q$ is not a primary submodule of $F$. □

Proposition 2.2. Let $\mathcal{P}$ be a prime ideal of an integral domain $R$ and $Q$ be a $\mathcal{P}$-primary ideal of $R$. Let $Q = Q\{x_i\}_{i \in I}$, for a nonempty set $I$, be a submodule of free $R$-module $F = \oplus_{i \in I} R$ such that for an $j \in I$, $x_j$ is a unit of $R$. Then $Q$ is a classical primary submodule of $F$. In addition, if $Q$ is nonzero and $I$ has at least two elements, then $Q$ is not a primary submodule of $F$.\[\]
Proof. Let \( k \) be a positive integer, and let \( x \) be a unit of \( R \), for an \( j \in I \). Let \( r, s \in R \) and \( N \) be a submodule of \( F \) that \( rsN \subseteq Q \) and \( rN \nsubseteq Q \); so there is \( y = \{y_i\}_{i \in I} \in N \) such that \( rsy \in Q \) and \( ry \notin Q \). We can assume that \( r \) and \( s \) are nonzero; so \( rs \neq 0 \), because \( R \) is an integral domain. Then for every \( i \in I \), \( rsy_i = qx_i \), that \( q \in Q \); especially, \( rsy_j = qx_j \). Since \( x_j \) is a unit of \( R \), \( rsy_jx_j^{-1}x_i = qx_i \), and since \( rsy_i = qx_i \), \( rsy_jx_j^{-1}x_i = rsy_i \). Therefore \( y_i = y_jx_j^{-1}x_i \), because \( R \) is an integral domain. Then \( y = \{y_jx_j^{-1}x_i\}_{i \in I} = y_jx_j^{-1}x \). Thus for every \( z \in N \setminus Q \), there is \( x \in R \) such that \( z = xz \). On the other hand, since \( ry \notin Q \), then \( ry_jx_j^{-1} \notin Q \), so \( ry_j \notin Q \). Also, since \( rsy_j = qx_j \), and \( Q \) is a \( \mathcal{P} \)-primary ideal of \( R \), \( s \in \mathcal{P} \), i.e., \( s^k \in Q \) for some \( k \in \mathbb{N} \). Then for every \( z \in N \setminus Q \), \( s^kz = s^krx \in Q \), so \( s^kN \subseteq Q \). Thus \( Q \) is a classical primary submodule of \( R \).

Now suppose that \( Q \) is nonzero and \( I \) has at least two elements. Evidently, there exists a subset \( J = \{i_1, \ldots, i_t\} \), where \( t \geq 2 \) and \( i_1 < i_2 < \cdots < i_t \), of \( I \) such that for every \( i \in I \setminus J \), \( x_i = 0 \). Let \( e = \{e_i\}_{i \in I} \) such that for every \( i \in J, e_i = 1 \), and for every \( i \in I \setminus J, e_i = 0 \). Also let \( f = \{f_i\}_{i \in I} \) such that \( f_{i_1} = 1 \) and for every \( i \in I \setminus \{i_1\}, f_i = 0 \). Obviously, \( x \notin Q \) and for every nonzero \( q \in Q \), \( qx \notin Q \). Now if for a positive integer \( k \), \( q^kF \subseteq Q \), then \( q^k e \in Q \), so \( q^k e = qx \) for some \( q \in Q \). Then for every \( i \in J \), \( q^k = q^kx_i \), therefore \( q^kx_i = q^kx_j \). Since \( R \) is an integral domain and \( q \neq 0 \), \( x_i = x_j \) for every \( i \in J \), so \( x = xje \). On the other hand, \( q^k f = q_2x \), for some \( q_2 \in Q \). Then \( q^k f = q_2x \), so \( q^k f = q^k f_{i_2} \), i.e., \( q^k = 0 \). Now since \( R \) is an integral domain we conclude that \( q = 0 \), a contradiction. Therefore \( Q \) is not a primary submodule of \( F \). \( \square \)

Proposition 2.3. Let \( \mathcal{P} \) be a prime ideal of an integral domain \( R \) and \( Q \) be a \( \mathcal{P} \)-primary ideal of \( R \). Let \( F = \bigoplus_{i=1}^n R \) and \( x = (x_1, x_2, \ldots, x_n) \in F \) such that for some \( i, 1 \leq i \leq n \), \( x_i \) is invertible. If \( Q = Qx \), then \( Q \) is a classical primary submodule of \( F \). In addition, if \( Q \) is nonzero and \( n \geq 2 \), then \( Q \) is not a primary submodule of \( F \).

Proof. Set \( x = \{x_i\}_{i \in I} \), and let \( x_j \) be a unit of \( R \), for an \( j \in I \). Let \( r, s \in R \) and \( N \) be a submodule of \( F \) that \( rsN \subseteq Q \) and \( rN \nsubseteq Q \); so there is \( y = \{y_i\}_{i \in I} \in N \) such that \( rsy \in Q \) and \( ry \notin Q \). We can assume that \( r \) and \( s \) are nonzero; so \( rs \neq 0 \), because \( R \) is an integral domain. Then for every \( i \in I \), \( rsy_i = qx_i \), that \( q \in Q \); especially, \( rsy_j = qx_j \). Since \( x_j \) is a unit of \( R \), \( rsy_jx_j^{-1}x_i = qx_i \), and since \( rsy_i = qx_i \), \( rsy_jx_j^{-1}x_i = rsy_i \). Therefore \( y_i = y_jx_j^{-1}x_i \), because \( R \) is an integral domain. Then \( y = \{y_jx_j^{-1}x_i\}_{i \in I} = y_jx_j^{-1}x \). Thus for every \( z \in N \setminus Q \), there is \( z \in R \) such that \( z = rzx \). On the other hand, since \( ry \notin Q \),
then \( ry_j x_j^{-1} \notin \mathcal{Q} \), so \( ry_j \notin \mathcal{Q} \). Also, since \( rsy_j = qx_j \in \mathcal{Q} \), and \( \mathcal{Q} \) is a \( \mathcal{P} \)-primary ideal of \( R \), \( s \in \mathcal{P} \), i.e., \( s^k \in \mathcal{Q} \) for some \( k \in \mathbb{N} \). Then for every \( z \in \mathbb{N}\backslash \mathcal{Q} \), \( s^k z = s^k r x \in \mathcal{Q} \), so \( s^k \mathcal{N} \subseteq \mathcal{Q} \). Thus \( \mathcal{Q} \) is a classical primary submodule of \( R \).

Now suppose that \( \mathcal{Q} \) is nonzero and \( I \) has at least two elements. Evidently, there exists a subset \( J = \{ i_1, \ldots, i_t \} \), where \( t \geq 2 \) and \( i_1 < i_2 < \cdots < i_t \), of \( I \) such that for every \( i \in I \backslash J \), \( x_i = 0 \). Let \( e = \{ e_i \}_{i \in I} \) such that for every \( i \in J, e_i = 1 \), and for every \( i \in I \backslash J, e_i = 0 \). Also let \( f = \{ f_i \}_{i \in I} \) such that \( f_{i_1} = 1 \) and for every \( i \in I \backslash \{ i_1 \}, f_i = 0 \). Obviously, \( x \notin \mathcal{Q} \) and for every nonzero \( q \in \mathcal{Q}, q x \in \mathcal{Q} \). Now if for a positive integer \( k, q^k F \subseteq \mathcal{Q} \), then \( q^k e \in \mathcal{Q} \), so \( q^k e = q_1 x \) for some \( q_1 \in \mathcal{Q} \). Then for every \( i \in J, q^k = q_1 x_i \), therefore \( q_1 x_i = q_1 x_1 \). Since \( R \) is an integral domain and \( q \neq 0 \), \( x_i = x_j \) for every \( i \in J \), so \( x = x j e \).

On the other hand, \( q^k f = q_2 x \), for some \( q_2 \in \mathcal{Q} \). Then \( q^k f = q_2 x j e \), so \( q^k f_{i_1} = q^k f_{i_2} \), i.e., \( q^k = 0 \). Now since \( R \) is an integral domain we conclude that \( q = 0 \), a contradiction. Therefore \( \mathcal{Q} \) is not a primary submodule of \( F \).

\[ \square \]

Even in a ring \( R \), the classical quasi-primary ideals and primary ideals are not the same, see the following example.

**Example 2.4.**

(a): Let \( R \) be valuation domain. It is easy to see that every ideal of \( R \) is a quasi-primary ideal (see for example [8, Theorem 5.10]). Then every ideal of \( R \) is a classical quasi-primary ideal by [4, Proposition 1.3]. Since every ideal of \( R \) need not to be a primary ideal, then there are non-primary ideals of \( R \) that are classical quasi-primary.

(b): Let \( R \) be an integral domain and \( \mathcal{I} \) be a valuation ideal of \( R \) (an ideal \( \mathcal{I} \) of integral domain \( R \) with quotient filed \( K \) is a valuation ideal if there is a valuation ring \( V \) of \( K \) containing \( R \) such that \( \mathcal{I} = \mathcal{J} \cap R \) for some ideal \( \mathcal{J} \) of \( V \)). By [8, Exercise V13-page 122], every valuation ideal of \( R \) is a (classical) quasi-primary ideal, but there are valuation ideals of \( R \) that are not primary ideals. For example, if \( K \) is a filed and \( \mathcal{I} \) is the ideal generated by \( x^2 \) and \( y^2 \) in \( K[x, y] \), for indeterminates \( x \) and \( y \), then \( \mathcal{I} \) is a (classical) quasi-primary ideal that is not a primary ideal.

Following [3, 4], we call an \( R \)-module \( M \) (quasi) primary compatible if its (quasi) primary and its classical (quasi) primary submodules are the same. A ring \( R \) is said to be (quasi) primary compatible if every
$R$-module is (quasi) primary compatible. Some results about quasi-primary compatible rings were proved in [4]; for example it was shown that if $\dim(R) = 0$, then $R$ is a quasi-primary compatible ring, and if $R$ is a Noetherian quasi-primary compatible ring, then $\dim(R) \leq 1$. In the sequel of this section, we will prove some other results about quasi-primary compatible rings.

The next proposition gives some equivalent conditions for a ring that is a quasi-primary compatible ring:

**Proposition 2.5.** Let $R$ be an integral domain. Then the following statements are equivalent:

1. $\text{Spec}(R)$ is a chain of prime ideals;
2. Every proper ideal of $R$ is quasi-primary;
3. Every proper ideal of $R$ is classical quasi-primary;
4. $R$ is a quasi-primary compatible ring.

**Proof.** (1) $\Rightarrow$ (2) Let $\mathcal{I}$ be a proper ideal of $R$. It is well-known that $\sqrt{\mathcal{I}} = \bigcap_{P \in \text{Var}(\mathcal{I})} P$; where $\text{Var}(\mathcal{I}) = \{P \in \text{Spec}(R) | \mathcal{I} \subseteq P\}$ (see for example [2, Proposition 1.14]). Since $\text{Spec}(R)$ is a chain, $\sqrt{\mathcal{I}} = P_0$ for some $P_0 \in \text{Var}(\mathcal{I})$; on the other word, $\mathcal{I}$ is a quasi-primary ideal of $R$.

(2) $\Rightarrow$ (3) follows from [4, Proposition 2.3].

(3) $\Rightarrow$ (4) is evident.

(4) $\Rightarrow$ (1) follows from [4, Proposition 2.11]. □

**Corollary 2.6.** Let $R$ be a quasi-primary compatible ring. Then for every $P \in \text{Spec}(R)$, $\text{Spec}(R/P)$ is a chain of prime ideals.

**Proof.** Evidently, every factor ring of a quasi-primary compatible ring is quasi-primary compatible. Then for every $P \in \text{Spec}(R)$, $R/P$ is a quasi-primary compatible integral domain; therefore $\text{Spec}(R/P)$ is a chain of prime ideals by Proposition 1.5. □

**Lemma 2.7.** Let $R$ be an integral domain. If $R$ is a quasi-primary compatible ring, then any two prime elements of $R$ are associated.

**Proof.** It is clear from the definition of a prime element, for $p \in R$, $pR$ is a nonzero prime ideal of $R$ if and only if $p$ is a prime element of $R$. Now assume that $p_1, p_2 \in R$ are prime elements. Since by Propositions 2.5, $\text{Spec}(R)$ is a chain, $p_1 R \subseteq p_2 R$ or $p_2 R \subseteq p_1 R$. It follows that $p_1 R = p_2 R$, i.e., $p_1$ and $p_2$ are associated. □

**Theorem 2.8.** Let $R$ be a unique factorization domain. Then $R$ is quasi-primary compatible if and only if $R$ is a field.
Proof. By Lemma 2.7, any two prime elements of $R$ are associated. Now if $R$ is not a field, then $\dim(R) \geq 1$ and there is a prime element $p$ of $R$. Since $R$ is an unique factorization domain, every nonzero non-unit element $r \in R$, is a finite multiple of prime elements; then $r = up^k$, for some unit $u \in R$, and some positive integer $k$. Now, if we define $\theta(r) = k$, for every nonzero element $r = up^k$ of $R$, then it is easy to check that $\theta$ is an Euclidean valuation. Then $R$ is an Euclidean domain; so, $R$ is a principal ideal domain. Since $\dim(R) = 1$, $R$ has one nonzero prime ideal $P$; so any nonzero ideal of $R$ is of the form $P^k$, for some positive integer $k$. Thus every ideal of $R$ is a primary ideal. This implies that $R$ is a primary compatible ring, so by [4, Theorem 1.14], $\dim(R) = 0$, a contradiction. Therefore $R$ is a field. The converse is clear. □

3. Decomposition into quasi-primary submodules

The decomposition into classical quasi-primary submodules in Noetherian modules was introduced in detail in [4]. The purpose of this section is to investigate decomposition of submodules into quasi-primary submodules in non-Noetherian modules over a Prüfer domain.

Definition 3.1. Let $R$ be a commutative ring and $N$ be a proper submodule of an $R$-module $M$. A quasi-primary (resp., classical quasi-primary) decomposition of $N$ is an expression $N = \bigcap_{i=1}^n Q_i$, where each $Q_i$ is a quasi-primary (resp., classical quasi-primary) submodule of $M$ (see also [4, Definition 2.6]). The decomposition is called reduced if it satisfies the following two conditions:

(1) no $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is a quasi-primary (resp., classical quasi-primary) submodule, where $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$ for $t \geq 2$ with $i_1 < i_2 < \cdots < i_t$.

(2) for each $j$, $Q_j \nsubseteq \bigcap_{i \neq j} Q_i$.

Corresponding to the above definition, by the definition of (classical) quasi-primary submodules, we have a list of prime ideals $\sqrt{(Q_1 : M)}, \ldots, \sqrt{(Q_n : M)}$. Among reduced quasi-primary (resp., classical quasi-primary) decompositions, any one that has the least number of distinct primes will be called minimal.

Let $R$ be a commutative ring, $N$ a non-zero submodule of an $R$-module $M$, $N_P = N \otimes_R R_P$ the localization of $N$ by a maximal ideal $P$ and $N_P := f^{-1}(N_P)$, that $f : M \rightarrow M_P$ is the canonical map with $f(m) = m/1$, for every $m \in M$. First of all note that $N = \bigcap_{P \in \operatorname{Max}(R)} N_P$, that $\operatorname{Max}(R)$ is the set of maximal ideals of $R$. Because it is evident that $N \subseteq \bigcap_{P \in \operatorname{Max}(R)} N_P$. Now if $m \in \bigcap_{P \in \operatorname{Max}(R)} N_P$,
then $m/1 \in N_{\mathcal{P}}$ for every $\mathcal{P} \in \text{Max}(R)$, so there is an $s_\mathcal{P} \in R \setminus \mathcal{P}$ such that $s_\mathcal{P}m \in N$. Suppose $\mathcal{I}$ is the ideal generated by all such $s_\mathcal{P}$. If $\mathcal{I} \neq R$, then there is a maximal ideal $\mathcal{P}_0$ of $R$ such that $\mathcal{I} \subseteq \mathcal{P}_0$, therefore $s_\mathcal{P}_0 \in \mathcal{P}_0$, that is contradicts with choosing $s_\mathcal{P}_0$. Then $\mathcal{I} = R$, so for some positive integer $k$, there are $r_j \in R$, $1 \leq j \leq k$, such that $1 = \sum_{j=1}^k r_j s_\mathcal{P}_j$. Therefore $m = \sum_{j=1}^k r_j s_\mathcal{P}_j m \in N$, this implies that $\bigcap_{\mathcal{P} \in \text{Max}(R)} N(\mathcal{P}) \subseteq N$. Thus $N = \bigcap_{\mathcal{P} \in \text{Max}(R)} N(\mathcal{P})$.

Over an integral domain of finite character, the number of proper components of this intersection can be finite, but for proving this fact, first note the following lemma:

**Lemma 3.2.** Let $\mathcal{P}$ be a maximal ideal of a commutative ring $R$ and $N$ be a submodule of an $R$-module $M$. Then the following statements hold:

1. $M_\mathcal{P} = N_\mathcal{P}$ if and only if $(N : m) \not\subseteq \mathcal{P}$ for every $m \in M$.
2. If $R$ is an integral domain of finite character and $M/N$ is torsion, then $N$ is a finite intersection of submodules of the form $N(\mathcal{P})$, for maximal ideals $\mathcal{P}$ of $R$.

**Proof.** (1) Set $S = R \setminus \mathcal{P}$. Clearly, $M_\mathcal{P} = N_\mathcal{P}$ if and only if for every $m \in M$, there exists $s \in S$ such that $sm \in N$, i.e., $s \in (N : m)$. On the other word, $M_\mathcal{P} = N_\mathcal{P}$ if and only if for every $m \in M$, $S \cap (N : m) \neq \emptyset$, i.e., $(N : m) \not\subseteq \mathcal{P}$.

(2) Since $R$ is of finite character and $(N : M) \neq (0)$, there are a finite number of maximal ideals of $R$, say $\mathcal{P}_1, \ldots, \mathcal{P}_k$, containing $(N : M)$. Obviously for every $m \in M$, $(N : M) \subseteq (N : m)$, so for every $\mathcal{P} \in \text{Max}(R) \setminus \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$, $(N : m) \not\subseteq \mathcal{P}$. Then by (1), for every $\mathcal{P} \in \text{Max}(R) \setminus \{\mathcal{P}_1, \ldots, \mathcal{P}_k\}$, $M_\mathcal{P} = N_\mathcal{P}$. Therefore $N = \bigcap_{i=1}^k N(\mathcal{P}_i)$. $\square$

**Lemma 3.3.** Let $S$ be a multiplicatively closed subset of a commutative ring $R$. Let $M$ be an $R$-module, and $Q$ be a (classical) quasi-primary submodule of $R_S$-module $M_S$. Then $Q \cap M$ is a (classical) quasi-primary submodule of $M$.

**Proof.** Let $Q$ be a classical quasi-primary submodule of $R_S$-module $M_S$. Suppose $N$ is a submodule of $M$ such that $N \not\subseteq Q \cap M$ and $abN \subseteq Q \cap M$ for some $a, b \in R$. Then $\frac{ab}{1} N_S \subseteq (Q \cap M)_S = Q$. Since $Q$ is a classical quasi-primary submodule, $\frac{a^k}{1} N_S \subseteq Q$ or $\frac{b^k}{1} N_S \subseteq Q$ for some positive integer $k$. Then $a^k N \subseteq (\frac{a^k}{1} N_S) \cap M \subseteq Q \cap M$ or $b^k N \subseteq (\frac{b^k}{1} N_S) \cap M \subseteq Q \cap M$. Consequently, $Q \cap M$ is a classical quasi-primary submodule of $M$. 
Let for every $i, 1 \leq i \leq n$, $P_i$ be a prime ideal of a ring $R$, $Q_i$ be a submodule of an $R$-module $M$, and $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. For each submodule $N$ of $M$ and each $i, 1 \leq i \leq n$, set $P_{i,N} = \sqrt{(Q_i : N)}$. Then the following statements hold:

1. If for every $i, 1 \leq i \leq n$, $Q_i$ is a classical $P_i$-quasi-primary submodule, then $Q$ is a classical quasi-primary submodule if and only if the set $\{P_{1,N}, \ldots, P_{n,N}\}$ has the least element (with respect to the relation $\subseteq$) for every submodule $N$ of $M$.

2. If for every $i, 1 \leq i \leq n$, $Q_i$ is a $P_i$-quasi-primary submodule, then $Q$ is a quasi-primary submodule if and only if the set $\{P_1, \ldots, P_n\}$ has the least element (with respect to the relation $\subseteq$).

Proof. We only prove (1), the proof of (2) is similar.

(1) For every submodule $N$ of $M$, set $P_N = \sqrt{(Q_1 \cap Q_2 \cap \cdots \cap Q_n : N)}$. Clearly, $P_N = P_{1,N} \cap P_{2,N} \cap \cdots \cap P_{n,N}$. By [4, Lemma 1.3(2)], $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if for every submodule $N$ of $M$ such that $N \not\subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $P_N$ is a prime ideal of $R$, i.e., $P_N = P_{j,N}$ for some $j, 1 \leq j \leq n$. But if for a submodule $N$ of $M$, $N \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$, then $P_N = P_{i,N} = R$ for every $i, 1 \leq i \leq n$. Thus $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if for every submodule $N$ of $M$, there exists an $j, 1 \leq j \leq n$, such that $P_N = P_{j,N}$. On the other words, $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if the set $\{P_{1,N}, \ldots, P_{n,N}\}$ has the least element (with respect to the relation $\subseteq$).

By using the fact that every classical quasi-primary submodule is a quasi-primary submodule, we can get the following corollary:

Corollary 3.5. Let for every $i, 1 \leq i \leq n$, $P_i$ be a prime ideal of a ring $R$, $Q_i$ be a $P_i$-quasi-primary submodule of an $R$-module $M$, and $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. If $Q$ is a classical quasi-primary submodule, then the set $\{P_1, \ldots, P_n\}$ has the least element (with respect to the relation $\subseteq$).

The following example shows that the converse of Corollary 3.5 is not necessarily true (even if the decomposition $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a minimal primary decomposition).

Example 3.6. (see [3, Example 2.2]). Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$, $Q_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (0)$, $Q_2 = \mathbb{Z}_2 \oplus (0) \oplus \mathbb{Z}$, and $Q_3 = (0) \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Clearly,
\[Q_1, Q_2, \text{ and } Q_3 \text{ are primary submodules of } M \text{ with } \sqrt{(Q_1 : M)} = (0), \]
\[\sqrt{(Q_2 : M)} = 3\mathbb{Z}, \text{ and } \sqrt{(Q_3 : M)} = 2\mathbb{Z}. \] On the other hand, \((0) = Q_1 \cap Q_2 \cap Q_3\) is a (minimal) primary decomposition of \((0)\). Now, the set \(\{(0), 2\mathbb{Z}, 3\mathbb{Z}\}\) has the least element (with respect to the relation \(\subseteq\)), but \((0)\) is not a classical quasi-primary submodule of \(M\).

Let \(R\) be a Prüfer domain of finite character and \(N\) be a proper submodule of an \(R\)-module \(M\) such that \((N : M) \neq (0)\). In the next theorem, the existence of a minimal classical quasi-primary decomposition of \(N\) are proved.

**Theorem 3.7.** Let \(R\) be a Prüfer domain of finite character and \(N\) be a proper submodule of an \(R\)-module \(M\) such that \((N : M) \neq (0)\). Then \(N\) has a minimal classical quasi-primary decomposition. In particular \(N\) has a minimal quasi-primary decomposition.

**Proof.** It is well-known that every proper ideal in a valuation domain is a quasi-primary ideal (see for example [8]). Then by [4, Proposition 1.3], \(N\) is a classical quasi-primary submodule of \(M\). Therefore by Lemmas 3.2 and 3.3, we obtain a decomposition of \(N\) as \(N = \bigcap_{i=1}^{k'} Q_i\), where each \(Q_i, 1 \leq i \leq k'\), is a classical quasi-primary submodule of \(M\). If \(Q_0 := Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_t}\) is a classical quasi-primary submodule of \(M\), where \(\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, k'\}\) for \(t \geq 2\) with \(i_1 < i_2 < \cdots < i_t\), then we can replace \(Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_t}\) with the single component \(Q_0\). Now by using this argument, we can get the decomposition \(N = Q_1 \cap Q_2 \cap \cdots \cap Q_n\), such that no \(Q_{i_1} \cap \cdots \cap Q_{i_t}\) is a classical quasi-primary submodule, where \(\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, n\}\) for \(t \geq 2\) with \(i_1 < i_2 < \cdots < i_t\). If there is some \(j, 1 \leq j \leq n\) such that \(Q_j \supseteq \bigcap_{i \neq j} Q_i\), then we can exclude the \(Q_j\) from the decomposition \(N = Q_1 \cap Q_2 \cap \cdots \cap Q_n\). By using this argument, we can get the decomposition \(N = Q_1 \cap Q_2 \cap \cdots \cap Q_k\) such that no component is abundant, so the decomposition is reduced. Obviously, among such reduced decompositions, we can get a minimal classical quasi-primary decomposition of \(N\). \(\Box\)

Recall that any two incomparable primary ideals of a Prüfer domain are co-maximal (see for example [8, page 131]). Also by [7, Lemma 5.5], any two quasi-primary ideals with incomparable radicals of a prüfer domain are co-maximal. The next lemma proves a similar result for quasi-primary submodules.

**Lemma 3.8.** Let \(R\) be a Prüfer domain, \(Q_1\) and \(Q_2\) be two quasi-primary submodules of an \(R\)-module \(M\), and \(N\) be a submodule of \(M\) such that \(Q_1 + Q_2 \subseteq N\). If \(\sqrt{(Q_1 : N)}\) and \(\sqrt{(Q_2 : N)}\) are incomparable,
then $Q_1 + Q_2 = N$. In particular, any two quasi-primary submodules of $M$ with incomparable radicals are co-maximal.

Proof. It suffices to prove that $(Q_1 + Q_2 : N) = R$. We can assume that $N \not\subseteq Q_1$ and $N \not\subseteq Q_2$, so $\sqrt{(Q_1 : N)}$ and $\sqrt{(Q_2 : N)}$ are prime ideals of $R$. Since $R$ is a Prüfer domain, $\sqrt{(Q_1 : N)} + \sqrt{(Q_2 : N)} = R$. Finally, because $\sqrt{(Q_1 : N)} + \sqrt{(Q_2 : N)} \subseteq \sqrt{(Q_1 + Q_2 : N)}$, we conclude that $(Q_1 + Q_2 : N) = R$. □

One can easily see that a proper submodule $N$ of an $R$-module $M$ has a minimal quasi-primary decomposition if $N$ can be shown as an intersection of finite number of quasi-primary submodules with pairwise incomparable radicals where no component can be omitted. So by Theorem 3.7 and Lemma 3.8, we can get the following corollary:

**Corollary 3.9.** Let $R$ be a Prüfer domain of finite character and $N$ be a submodule of an $R$-module $M$ such that $(N : M) \neq (0)$. Then $N$ can be shown as an intersection of finite number of co-maximal submodules of $M$.

The next theorem proves uniqueness of the decomposition of submodules into quasi-primary submodules of modules over a Prüfer domain of finite character.

**Theorem 3.10.** [Uniqueness Theorem]. Let $R$ be a Prüfer domain of finite character, $\mathcal{P}_i, 1 \leq i \leq k$, be prime ideals of $R$, and $N$ be a submodule of an $R$-module $M$. If $N = \bigcap_{i=1}^{k} Q_i$ is a minimal decomposition of $N$ to $\mathcal{P}_i$-quasi-primary submodules $Q_i, 1 \leq i \leq k$, then $k$ is independent of any such decompositions of $N$ and

$$\{\mathcal{P}_1, ..., \mathcal{P}_k\} = \text{Min}(N : M).$$

Proof. First note that $\sqrt{(N : M)} = \bigcap_{i=1}^{k} \sqrt{(Q_i : M)} = \bigcap_{i=1}^{k} \mathcal{P}_i$. Since $\mathcal{P}_i$'s are incomparable prime ideals, then $\mathcal{P}_i$'s are minimal prime ideals of the ideal $(N : M)$ and so $\{\mathcal{P}_1, ..., \mathcal{P}_k\} = \text{Min}(N : M)$. On the other word, $k$ and the set $\{\mathcal{P}_1, ..., \mathcal{P}_k\}$ are independent of any such decompositions of $N$. □

**Theorem 3.11.** Let $R$ be a Prüfer domain of finite character and $M$ be a multiplication $R$-module. Then every nonzero submodule $N$ of $M$ is the intersection of finite number of quasi-primary submodules with pairwise incomparable radicals, uniquely determined by $N$.

Proof. Since $M$ is a multiplication module, $N = (N : M)M$; so, $(N : M) \neq (0)$. Then the result follows form Theorems 3.7 and 3.11 (compare with [1, Theorem 3.4]). □
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