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QUASI-PRIMARY DECOMPOSITION IN MODULES OVER PRÜFER DOMAINS

M. BEHBOODI*, R. JAHANI-NEZHAD AND M. H. NADERI

Abstract. In this paper we investigate decompositions of submodules in modules over a Prüfer domain into intersections of quasi-primary and classical quasi-primary submodules. In particular, existence and uniqueness of quasi-primary decompositions in modules over a Prüfer domain of finite character are proved.

1. Introduction

Throughout this paper all rings are commutative with identity elements, and all modules are unital. Let M be an R-module. For every nonempty subset X of M and every submodule N of M, the ideal $\{r \in R \mid rX \subseteq N\}$ will be denoted by $(N : X)$. Note that $(N : M)$ is the annihilator of the module M/N. Also we denote the classical Krull dimension of R by $\dim(R)$, and for an ideal I of R, $\sqrt{I} := \{r \in R \mid r^k \in I \text{ for some } k \in \mathbb{N}\}$.

We recall that a proper ideal $\mathcal{Q}$ of the ring R is called a *primary* ideal if $ab \in \mathcal{Q}$ where $a, b \in R$, implies that either $a \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see for example [2]). The notion of primary ideal was generalized by Fuchs [6] by defining an ideal $\mathcal{Q}$ of a ring R to be quasi-primary if its radical is a prime ideal, i.e., if $ab \in \mathcal{Q}$ where a, $b \in R$, then either $a^k \in \mathcal{Q}$ or $b^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$ (see also [7]). There are some extensions of these notions to modules. For instance, a proper submodule Q of M is called a *primary submodule* if $am \in Q$, where

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*Corresponding author .

$a \in R$, $m \in M \setminus Q$, then $a^k M \subseteq Q$ for some $k \in \mathbb{N}$ (see for example [9, 10]). Also, Q is called *quasi-primary* if $\sqrt{(Q : M)}$ is a prime ideal of R (see [1]). Moreover, Q is called a *classical primary* (resp. *classical quasi-primary*) submodule of M if $abN \subseteq Q$, where $a, b \in R$ and N is a submodule of M, then either $aN \subseteq Q$ or $b^k N \subseteq Q$ (resp. $a^k N \subseteq Q$ or $b^k N \subseteq Q$) for some $k \in \mathbb{N}$ (see [3, 4]). We note that if Q is a primary, quasi-primary, classical primary or a classical quasi-primary submodule of M, then $\mathcal{P} := \sqrt{(Q : M)}$ is a prime ideal of R, and hence, we say that Q is a $\mathcal{P}$-primary, $\mathcal{P}$-quasi-primary, classical $\mathcal{P}$-primary or a classical $\mathcal{P}$-quasi-primary submodule; respectively.

Let $K, N, N_1, \cdots, N_l$, for some $l \in \mathbb{N}$, be submodules of an R-module M. We say that N and K are *co-maximal* (resp. *with incomparable radicals*) when $N + K = M$ (resp. when $\sqrt{(N : M)}$ and $\sqrt{(K : M)}$ are not comparable); also we say that the submodules $N_1, ..., N_l$ are *pairwise co-maximal* (resp. *with pairwise incomparable radicals*) if and only if for every $i, j \in \{1, 2, ..., l\}$ such that $i \neq j$, $N_i + N_j = M$ (resp. $\sqrt{(N_i : M)}$ and $\sqrt{(K_j : M)}$ are not comparable). An R-module M is called a *multiplication* module if, for each submodule N of M, there exists an ideal I of R such that $N = IM$; In this case we can take $I = (N : M)$ (see for example [5]). For an integral domain R, we say that R is *of finite character*, if every nonzero element of R is contained but in a finite number of maximal ideals.

In a Prüfer domain of finite character, Fuchs and Mosteig [7] established the decomposition of an ideal as (shortest) intersections of a finite number of quasi-primary ideals. In particular, they proved that every nonzero ideal I in a Prüfer domain of finite character is a finite intersection of quasi-primary ideals with incomparable radicals, and the components in such a decomposition are uniquely determined by I (see [7, Theorem 5.6]). In Section 1, some results on quasi-primary and classical quasi-primary submodules are given. For instance, it is shown that if R is a domain, then for each R-module M, every classical quasi-primary submodule of M is a quasi-primary submodule if and only if every proper ideal of R is (classical) quasi-primary, if and only if, the set of prime ideals, Spec(R), is a chain (see Proposition 1.5). In Section 2, we generalize some main results of [7] to modules over a Prüfer domain of finite character. In particular, we prove that over a Prüfer domain of finite character, every submodule N of a module M such that $(N : M) \neq (0)$, can be shown as an (minimal) intersection of finite number of (classical) quasi-primary submodules (see Theorem 2.7). Also we prove that the components in the decomposition of N

into quasi-primary submodules are uniquely determined by N (see Theorem 2.10). If M is also a multiplication module, such decomposition into quasi-primary submodules exists for every nonzero submodule of M (see Theorem 2.11).

2. Some results on (classical) quasi-primary submodules

We begin this section with two Propositions 1.1 and 1.2, which give many examples of classical primary submodules; so many examples of classical quasi-primary submodules; that are not primary submodules.

Proposition 2.1. *Let R be an integral domain and $\mathcal{P}$ be a nonzero prime ideal of R. Let for a non-empty set I, $Q = \oplus_{i\in I} A_i$ be a submodule of a free R-module $F = \oplus_{i\in I} R$ such that for every $i \in I$, $A_i = (0)$ or A_i is a $\mathcal{P}$-primary ideal of R. If the set $\Gamma := \{A_i \mid i \in I$ and A_i is a $\mathcal{P}$-primary ideal of $R\}$ is a finite set, then Q is a classical primary submodule of F. In addition, if $Q \neq (0)$ and for some $i \in I$, $A_i = (0)$, then Q is not a primary submodule of F.*

Proof. Let $r, s \in R$ and N be a submodule of F such that $rN \nsubseteq Q$ and $rsN \subseteq Q$. Then there is $y = \{y_i\}_{i\in I} \in N$ such that $ry \notin Q$. We can assume that r and s are nonzero; so $rs \neq 0$, because R is an integral domain. Since $rsy \in Q$, $rsy_i \in A_i$, for every $i \in I$. But $ry \notin Q$, so there is an $i_0 \in I$ that $ry_{i_0} \notin A_{i_0}$. Clearly A_{i_0} is nonzero, so A_{i_0} is a $\mathcal{P}$-primary ideal of R. Now since $rsy_{i_0} \in A_{i_0}$ and $ry_{i_0} \notin A_{i_0}$, we conclude that $s \in \sqrt{A_{i_0}} = \mathcal{P}$. Evidently for every $z = \{z_i\}_{i\in I} \in N$, if $A_j = 0$, for some $j \in I$, then $z_j = 0$, so since the set Γ is finite, there is a positive integer k such that $s^k N \subseteq Q$; on the other word, Q is a classical primary submodule of F.

Now, suppose that $Q \neq (0)$ and for some $i \in I$, $A_i = (0)$. So there are $i_1, i_2 \in I$ such that $A_{i_1} \neq (0)$ and $A_{i_2} = (0)$. Set $f = \{f_i\}_{i\in I}$ where $f_{i_1} = 1$ and for every $i \in I\setminus\{i_1\}$, $f_i = 0$. Evidently $f \notin Q$ and for every nonzero element $p \in \mathcal{P}$, there is a positive integer k that $p^k f \in Q$. Now if for a positive integer l, $(p^k)^l F \subseteq Q$, then $p^{lk} \in A_{i_2} = (0)$, i.e., $p^{lk} = 0$. But R is an integral domain, so $p = 0$, a contradiction. On the other word, Q is not a primary submodule of F. □

Proposition 2.2. *Let $\mathcal{P}$ be a prime ideal of an integral domain R and $\mathcal{Q}$ be a $\mathcal{P}$-primary ideal of R. Let $Q = \mathcal{Q}\{x_i\}_{i\in I}$, for a non-empty set I, be a submodule of free R-module $F = \oplus_{i\in I} R$ such that for an $j \in I$, x_j is a unit of R. Then Q is a classical primary submodule of F. In addition, if $\mathcal{Q}$ is nonzero and I has at least two elements, then Q is not a primary submodule of F.*

Proof. Set $x = \{x_i\}_{i\in I}$, and let x_j be a unit of R, for an $j \in I$. Let $r, s \in R$ and N be a submodule of F that $rsN \subseteq Q$ and $rN \nsubseteq Q$; so there is $y = \{y_i\}_{i\in I} \in N$ such that $rsy \in Q$ and $ry \notin Q$. We can assume that r and s are nonzero; so $rs \neq 0$, because R is an integral domain. Then for every $i \in I$, $rsy_i = qx_i$, that $q \in \mathcal{Q}$; especially, $rsy_j = qx_j$. Since x_j is a unit of R, $rsy_j x_j^{-1} x_i = qx_i$, and since $rsy_i = qx_i$, $rsy_j x_j^{-1} x_i = rsy_i$. Therefore $y_i = y_j x_j^{-1} x_i$, because R is an integral domain. Then $y = \{y_j x_j^{-1} x_i\}_{i\in I} = y_j x_j^{-1} x$. Thus for every $z \in N\backslash Q$, there is $r_z \in R$ such that $z = r_z x$. On the other hand, since $ry \notin Q$, then $ry_j x_j^{-1} \notin \mathcal{Q}$, so $ry_j \notin \mathcal{Q}$. Also, since $rsy_j = qx_j \in \mathcal{Q}$, and $\mathcal{Q}$ is a $\mathcal{P}$-primary ideal of R, $s \in \mathcal{P}$, i.e., $s^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$. Then for every $z \in N\backslash Q$, $s^k z = s^k r_z x \in Q$, so $s^k N \subseteq Q$. Thus Q is a classical primary submodule of R.

Now suppose that $\mathcal{Q}$ is nonzero and I has at least two elements. Evidently, there exists a subset $J = \{i_1, \cdots, i_t\}$, where $t \geq 2$ and $i_1 < i_2 < \cdots < i_t$, of I such that for every $i \in I \setminus J$, $x_i = 0$. Let $e = \{e_i\}_{i\in I}$ such that for every $i \in J, e_i = 1$, and for every $i \in I \setminus J, e_i = 0$. Also let $f = \{f_i\}_{i\in I}$ such that $f_{i_1} = 1$ and for every $i \in I \setminus \{i_1\}, f_i = 0$. Obviously, $x \notin Q$ and for every nonzero $q \in \mathcal{Q}$, $qx \in Q$. Now if for a positive integer k, $q^k F \subseteq Q$, then $q^k e \in Q$, so $q^k e = q_1 x$ for some $q_1 \in \mathcal{Q}$. Then for every $i \in J$, $q^k = q_1 x_i$, therefore $q_1 x_i = q_1 x_j$. Since R is an integral domain and $q \neq 0$, $x_i = x_j$ for every $i \in J$, so $x = x_j e$. On the other hand, $q^k f = q_2 x$, for some $q_2 \in \mathcal{Q}$. Then $q^k f = q_2 x_j e$, so $q^k f_{i_1} = q^k f_{i_2}$, i.e., $q^k = 0$. Now since R is an integral domain we conclude that $q = 0$, a contradiction. Therefore Q is not a primary submodule of F. □

Proposition 2.3. *Let $\mathcal{P}$ be a prime ideal of an integral domain R and $\mathcal{Q}$ be a $\mathcal{P}$-primary ideal of R. Let $F = \oplus_{i=1}^{n} R$ and $x = (x_1, x_2, \cdots, x_n) \in F$ such that for some $i, 1 \leq i \leq n$, x_i is invertible. If $Q = \mathcal{Q}x$, then Q is a classical primary submodule of F. In addition, if $\mathcal{Q}$ is nonzero and $n \geq 2$, then Q is not a primary submodule of F.*

Proof. Set $x = \{x_i\}_{i\in I}$, and let x_j be a unit of R, for an $j \in I$. Let $r, s \in R$ and N be a submodule of F that $rsN \subseteq Q$ and $rN \nsubseteq Q$; so there is $y = \{y_i\}_{i\in I} \in N$ such that $rsy \in Q$ and $ry \notin Q$. We can assume that r and s are nonzero; so $rs \neq 0$, because R is an integral domain. Then for every $i \in I$, $rsy_i = qx_i$, that $q \in \mathcal{Q}$; especially, $rsy_j = qx_j$. Since x_j is a unit of R, $rsy_j x_j^{-1} x_i = qx_i$, and since $rsy_i = qx_i$, $rsy_j x_j^{-1} x_i = rsy_i$. Therefore $y_i = y_j x_j^{-1} x_i$, because R is an integral domain. Then $y = \{y_j x_j^{-1} x_i\}_{i\in I} = y_j x_j^{-1} x$. Thus for every $z \in N\backslash Q$, there is $r_z \in R$ such that $z = r_z x$. On the other hand, since $ry \notin Q$,

then $ry_jx_j^{-1} \notin \mathcal{Q}$, so $ry_j \notin \mathcal{Q}$. Also, since $rsy_j = qx_j \in \mathcal{Q}$, and $\mathcal{Q}$ is a $\mathcal{P}$-primary ideal of R, $s \in \mathcal{P}$, i.e., $s^k \in \mathcal{Q}$ for some $k \in \mathbb{N}$. Then for every $z \in N \backslash Q$, $s^k z = s^k r_z x \in Q$, so $s^k N \subseteq Q$. Thus Q is a classical primary submodule of R.

Now suppose that $\mathcal{Q}$ is nonzero and I has at least two elements. Evidently, there exists a subset $J = \{i_1, \cdots, i_t\}$, where $t \geq 2$ and $i_1 < i_2 < \cdots < i_t$, of I such that for every $i \in I \setminus J$, $x_i = 0$. Let $e = \{e_i\}_{i \in I}$ such that for every $i \in J, e_i = 1$, and for every $i \in I \setminus J, e_i = 0$. Also let $f = \{f_i\}_{i \in I}$ such that $f_{i_1} = 1$ and for every $i \in I \setminus \{i_1\}, f_i = 0$. Obviously, $x \notin Q$ and for every nonzero $q \in \mathcal{Q}$, $qx \in Q$. Now if for a positive integer k, $q^k F \subseteq Q$, then $q^k e \in Q$, so $q^k e = q_1 x$ for some $q_1 \in \mathcal{Q}$. Then for every $i \in J$, $q^k = q_1 x_i$, therefore $q_1 x_i = q_1 x_j$. Since R is an integral domain and $q \neq 0$, $x_i = x_j$ for every $i \in J$, so $x = x_j e$. On the other hand, $q^k f = q_2 x$, for some $q_2 \in \mathcal{Q}$. Then $q^k f = q_2 x_j e$, so $q^k f_{i_1} = q^k f_{i_2}$, i.e., $q^k = 0$. Now since R is an integral domain we conclude that $q = 0$, a contradiction. Therefore Q is not a primary submodule of F. □

Even in a ring R, the classical quasi-primary ideals and primary ideals are not the same, see the following example.

Example 2.4.

(a): Let R be valuation domain. It is easy to see that every ideal of R is a quasi-primary ideal (see for example [8, Theorem 5.10]). Then every ideal of R is a classical quasi-primary ideal by [4, Proposition 1.3]. Since every ideal of R need not to be a primary ideal, then there are non-primary ideals of R that are classical quasi-primary.

(b): Let R be an integral domain and $\mathcal{I}$ be a valuation ideal of R (an ideal $\mathcal{I}$ of integral domain R with quotient filed K is a valuation ideal if there is a valuation ring V of K containing R such that $\mathcal{I} = \mathcal{J} \cap R$ for some ideal $\mathcal{J}$ of V). By [8, Exercise V13-page 122], every valuation ideal of R is a (classical) quasi-primary ideal, but there are valuation ideals of R that are not primary ideals. For example, if K is a filed and $\mathcal{I}$ is the ideal generated by x^2 and y^2 in $K[x, y]$, for indeterminates x and y, then $\mathcal{I}$ is a (classical) quasi-primary ideal that is not a primary ideal.

Following [3, 4], we call an R-module M *(quasi) primary compatible* if its (quasi) primary and its classical (quasi) primary submodules are the same. A ring R is said to be (quasi) primary compatible if every

R-module is (quasi) primary compatible. Some results about quasi-primary compatible rings were proved in [4]; for example it was shown that if $\dim(R) = 0$, then R is a quasi-primary compatible ring, and if R is a Noetherian quasi-primary compatible ring, then $\dim(R) \leq 1$. In the sequel of this section, we will prove some other results about quasi-primary compatible rings.

The next proposition gives some equivalent conditions for a ring that is a quasi-primary compatible ring:

Proposition 2.5. *Let R be an integral domain. Then the following statements are equivalent:*

(1) $\operatorname{Spec}(R)$ *is a chain of prime ideals;*
(2) *Every proper ideal of R is quasi-primary;*
(3) *Every proper ideal of R is classical quasi-primary;*
(4) *R is a quasi-primary compatible ring.*

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{I}$ be a proper ideal of R. It is well-known that $\sqrt{\mathcal{I}} = \bigcap_{\mathcal{P}\in Var(\mathcal{I})} \mathcal{P}$; where $Var(\mathcal{I}) = \{\mathcal{P} \in \operatorname{Spec}(R) | \mathcal{I} \subseteq \mathcal{P}\}$ (see for example [2, Proposition 1.14]). Since $\operatorname{Spec}(R)$ is a chain, $\sqrt{\mathcal{I}} = \mathcal{P}_0$ for some $\mathcal{P}_0 \in Var(\mathcal{I})$; on the other word, $\mathcal{I}$ is a quasi-primary ideal of R.

(2) $\Rightarrow$ (3) follows from [4, Proposition 2.3].

(3) $\Rightarrow$ (4) is evident.

(4) $\Rightarrow$ (1) follows from [4, Proposition 2.11]. □

Corollary 2.6. *Let R be a quasi-primary compatible ring. Then for every $\mathcal{P} \in \operatorname{Spec}(R)$, $\operatorname{Spec}(R/\mathcal{P})$ is a chain of prime ideals.*

Proof. Evidently, every factor ring of a quasi-primary compatible ring is quasi-primary compatible. Then for every $\mathcal{P} \in \operatorname{Spec}(R)$, $R/\mathcal{P}$ is a quasi-primary compatible integral domain; therefore $\operatorname{Spec}(R/\mathcal{P})$ is a chain of prime ideals by Proposition 1.5. □

Lemma 2.7. *Let R be an integral domain. If R is a quasi-primary compatible ring, then any two prime elements of R are associated.*

Proof. It is clear from the definition of a prime element, for $p \in R$, pR is a nonzero prime ideal of R if and only if p is a prime element of R. Now assume that $p_1, p_2 \in R$ are prime elements. Since by Propositions 2.5, $\operatorname{Spec}(R)$ is a chain, $p_1R \subseteq p_2R$ or $p_2R \subseteq p_1R$. It follows that $p_1R = p_2R$, i.e., p_1 and p_2 are associated. □

Theorem 2.8. *Let R be a unique factorization domain. Then R is quasi-primary compatible if and only if R is a field.*

Proof. By Lemma 2.7, any two prime elements of R are associated. Now if R is not a filed, then $\dim(R) \geq 1$ and there is a prime element p of R. Since R is an unique factorization domain, every nonzero non-unit element $r \in R$, is a finite multiple of prime elements; then $r = up^k$, for some unit $u \in R$, and some positive integer k. Now, if we define $\theta(r) = k$, for every nonzero element $r = up^k$ of R, then it is easy to check that θ is an Euclidean valuation. Then R is an Euclidean domain; so, R is a principle ideal domain. Since $\dim(R) \geq 1$, R has one nonzero prime ideal Rp; so any nonzero ideal of R is of the form Rp^k, for some positive integer k. Thus every ideal of R is a primary ideal. This implies that R is a primary compatible ring, so by [4, Theorem 1.14], $\dim(R) = 0$, a contradiction. Therefore R is a filed. The converse is clear. □

3. Decomposition into quasi-primary submodules

The decomposition into classical quasi-primary submodules in Noetherian modules was introduced in detail in [4]. The purpose of this section is to investigate decomposition of submodules into quasi-primary submodules in non-Noetherian modules over a Prüfer domain.

Definition 3.1. Let R be a commutative ring and N be a proper submodule of an R-module M. A *quasi-primary* (resp., *classical quasi-primary*) *decomposition* of N is an expression $N = \bigcap_{i=1}^{n} Q_i$, where each Q_i is a quasi-primary (resp., classical quasi-primary) submodule of M(see also [4, Definition 2.6]). The decomposition is called *reduced* if it satisfies the following two conditions:

(1) no $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is a quasi-primary (resp., classical quasi-primary) submodule, where $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, n\}$ for $t \geq 2$ with $i_1 < i_2 < \cdots < i_t$.

(2) for each j, $Q_j \not\supseteq \bigcap_{i \neq j} Q_i$.

Corresponding to the above definition, by the definition of (classical) quasi-primary submodules, we have a *list* of prime ideals $\sqrt{(Q_1 : M)}, \cdots, \sqrt{(Q_n : M)}$. Among reduced quasi-primary (resp., classical quasi-primary) decompositions, any one that has the least number of distinct primes will be called *minimal*.

Let R be a commutative ring, N a non-zero submodule of an R-module M, $N_{\mathcal{P}} = N \otimes_R R_{\mathcal{P}}$ the localization of N by a maximal ideal $\mathcal{P}$ and $N_{(\mathcal{P})} := f^{-1}(N_{\mathcal{P}})$, that $f : M \to M_{\mathcal{P}}$ is the canonical map with $f(m) = m/1$, for every $m \in M$. First of all note that $N = \bigcap_{\mathcal{P} \in \mathrm{Max}(R)} N_{(\mathcal{P})}$, that $\mathrm{Max}(R)$ is the set of maximal ideals of R. Because it is evident that $N \subseteq \bigcap_{\mathcal{P} \in \mathrm{Max}(R)} N_{(\mathcal{P})}$. Now if $m \in \bigcap_{\mathcal{P} \in \mathrm{Max}(R)} N_{(\mathcal{P})}$, then $m/1 \in N_{\mathcal{P}}$ for every $\mathcal{P} \in \mathrm{Max}(R)$, so there is an $s_{\mathcal{P}} \in R \setminus \mathcal{P}$ such that $s_{\mathcal{P}}m \in N$. Suppose $\mathcal{I}$ is the ideal generated by all such $s_{\mathcal{P}}$. If $\mathcal{I} \neq R$, then there is a maximal ideal $\mathcal{P}_0$ of R such that $\mathcal{I} \subseteq \mathcal{P}_0$, therefore $s_{\mathcal{P}_0} \in \mathcal{P}_0$, that is contradicts with choosing $s_{\mathcal{P}_0}$. Then $\mathcal{I} = R$, so for some positive integer k, there are $r_j \in R$, $1 \leq j \leq k$, such that $1 = \sum_{j=1}^{k} r_j s_{\mathcal{P}_j}$. Therefore $m = \sum_{j=1}^{k} r_j s_{\mathcal{P}_j} m \in N$, this implies that $\bigcap_{\mathcal{P} \in \mathrm{Max}(R)} N_{(\mathcal{P})} \subseteq N$. Thus $N = \bigcap_{\mathcal{P} \in \mathrm{Max}(R)} N_{(\mathcal{P})}$.

Over an integral domain of finite character, the number of proper components of this intersection can be finite, but for proving this fact, first note the following lemma:

Lemma 3.2. *Let $\mathcal{P}$ be a maximal ideal of a commutative ring R and N be a submodule of an R-module M. Then the following statements hold:*

(1) *$M_{\mathcal{P}} = N_{\mathcal{P}}$ if and only if $(N : m) \nsubseteq \mathcal{P}$ for every $m \in M$.*
(2) *If R is an integral domain of finite character and M/N is torsion, then N is a finite intersection of submodules of the form $N_{(\mathcal{P})}$, for maximal ideals $\mathcal{P}$ of R.*

Proof. (1) Set $S = R \setminus \mathcal{P}$. Clearly, $M_{\mathcal{P}} = N_{\mathcal{P}}$ if and only if for every $m \in M$, there exists $s \in S$ such that $sm \in N$, i.e., $s \in (N : m)$. On the other word, $M_{\mathcal{P}} = N_{\mathcal{P}}$ if and only if for every $m \in M$, $S \cap (N : m) \neq \emptyset$, i.e., $(N : m) \nsubseteq \mathcal{P}$.

(2) Since R is of finite character and $(N : M) \neq (0)$, there are a finite number of maximal ideals of R, say $\mathcal{P}_1, ..., \mathcal{P}_k$, containing $(N : M)$. Obviously for every $m \in M$, $(N : M) \subseteq (N : m)$, so for every $\mathcal{P} \in \mathrm{Max}(R) \setminus \{\mathcal{P}_1, ..., \mathcal{P}_k\}$, $(N : m) \nsubseteq \mathcal{P}$. Then by (1), for every $\mathcal{P} \in \mathrm{Max}(R) \setminus \{\mathcal{P}_1, ..., \mathcal{P}_k\}$, $M_{\mathcal{P}} = N_{\mathcal{P}}$. Therefore $N = \bigcap_{i=1}^{k} N_{(\mathcal{P}_i)}$. □

Lemma 3.3. *Let S be a multiplicatively closed subset of a commutative ring R. Let M be an R-module, and Q be a (classical) quasi-primary submodule of R_S-module M_S. Then $Q \cap M$ is a (classical) quasi-primary submodule of M.*

Proof. Let Q be a classical quasi-primary submodule of R_S-module M_S. Suppose N is a submodule of M such that $N \nsubseteq Q \cap M$ and $abN \subseteq Q \cap M$ for some $a, b \in R$. Then $\frac{ab}{1} N_S \subseteq (Q \cap M)_S = Q$. Since Q is a classical quasi-primary submodule, $\frac{a^k}{1} N_S \subseteq Q$ or $\frac{b^k}{1} N_S \subseteq Q$ for some positive integer k. Then $a^k N \subseteq (\frac{a^k}{1} N_S) \cap M \subseteq Q \cap M$ or $b^k N \subseteq (\frac{b^k}{1} N_S) \cap M \subseteq Q \cap M$. Consequently, $Q \cap M$ is a classical quasi-primary submodule of M.

In the same way one can easily show that if Q is a quasi-primary submodule of M_S, then $Q \cap M$ is a quasi-primary submodule of M. □

Lemma 3.4. *Let for every $i, 1 \leq i \leq n$, $\mathcal{P}_i$ be a prime ideal of a ring R, Q_i be a submodule of an R-module M, and $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. For each submodule N of M and each $i, 1 \leq i \leq n$, set $\mathcal{P}_{i,N} = \sqrt{(Q_i : N)}$. Then the following statements hold:*

(1) *If for every $i, 1 \leq i \leq n$, Q_i is a classical $\mathcal{P}_i$-quasi-primary submodule, then Q is a classical quasi-primary submodule if and only if the set $\{\mathcal{P}_{1,N}, \cdots, \mathcal{P}_{n,N}\}$ has the least element (with respect to the relation $\subseteq$) for every submodule N of M.*

(2) *If for every $i, 1 \leq i \leq n$, Q_i is a $\mathcal{P}_i$-quasi-primary submodule, then Q is a quasi-primary submodule if and only if the set $\{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$ has the least element (with respect to the relation $\subseteq$).*

Proof. We only prove (1), the proof of (2) is similar.

(1) For every submodule N of M, set $\mathcal{P}_N = \sqrt{(Q_1 \cap Q_2 \cap \cdots \cap Q_n : N)}$. Clearly, $\mathcal{P}_N = \mathcal{P}_{1,N} \cap \mathcal{P}_{2,N} \cap \cdots \cap \mathcal{P}_{n,N}$. By [4, Lemma 1.3(2)], $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if for every submodule N of M such that $N \nsubseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $\mathcal{P}_N$ is a prime ideal of R, i.e., $\mathcal{P}_N = \mathcal{P}_{j,N}$ for some $j, 1 \leq j \leq n$. But if for a submodule N of M, $N \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_n$, then $\mathcal{P}_N = \mathcal{P}_{i,N} = R$ for every $i, 1 \leq i \leq n$. Thus $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if for every submodule N of M, there exists an $j, 1 \leq j \leq n$, such that $\mathcal{P}_N = \mathcal{P}_{j,N}$. On the other words, $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a classical quasi-primary submodule if and only if the set $\{\mathcal{P}_{1,N}, \cdots, \mathcal{P}_{n,N}\}$ has the least element (with respect to the relation $\subseteq$). □

By using the fact that every classical quasi-primary submodule is a quasi-primary submodule, we can get the following corollary:

Corollary 3.5. *Let for every $i, 1 \leq i \leq n$, $\mathcal{P}_i$ be a prime ideal of a ring R, Q_i be a $\mathcal{P}_i$-quasi-primary submodule of an R-module M, and $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. If Q is a classical quasi-primary submodule, then the set $\{\mathcal{P}_1, \cdots, \mathcal{P}_n\}$ has the least element (with respect to the relation $\subseteq$).*

The following example shows that the converse of Corollary 3.5 is not necessarily true (even if the decomposition $Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a minimal primary decomposition).

Example 3.6. (see [3, Example 2.2]). Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$, $Q_1 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (0)$, $Q_2 = \mathbb{Z}_2 \oplus (0) \oplus \mathbb{Z}$, and $Q_3 = (0) \oplus \mathbb{Z}_3 \oplus \mathbb{Z}$. Clearly,

Q_1, Q_2, and Q_3 are primary submodules of M with $\sqrt{(Q_1 : M)} = (0)$, $\sqrt{(Q_2 : M)} = 3\mathbb{Z}$, and $\sqrt{(Q_3 : M)} = 2\mathbb{Z}$. On the other hand, $(0) = Q_1 \cap Q_2 \cap Q_3$ is a (minimal) primary decomposition of (0). Now, the set $\{(0), 2\mathbb{Z}, 3\mathbb{Z}\}$ has the least element (with respect to the relation $\subseteq$), but (0) is not a classical quasi-primary submodule of M.

Let R be a Prüfer domain of finite character and N be a proper submodule of an R-module M such that $(N : M) \neq (0)$. In the next theorem, the existence of a minimal classical quasi-primary decomposition of N are proved.

Theorem 3.7. *Let R be a Prüfer domain of finite character and N be a proper submodule of an R-module M such that $(N : M) \neq (0)$. Then N has a minimal classical quasi-primary decomposition. In particular N has a minimal quasi-primary decomposition.*

Proof. It is well-known that every proper ideal in a valuation domain is a quasi-primary ideal (see for example [8]). Then by [4, Proposition 1.3], N is a classical quasi-primary submodule of M. Therefore by Lemmas 3.2 and 3.3, we obtain a decomposition of N as $N = \bigcap_{i=1}^{k'} Q_i$ where each Q_i, $1 \leq i \leq k'$, is a classical quasi-primary submodule of M. If $Q_0 := Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_t}$ is a classical quasi-primary submodule of M, where $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, k'\}$ for $t \geq 2$ with $i_1 < i_2 < \cdots < i_t$, then we can replace $Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_t}$ with the single component Q_0. Now by using this argument, we can get the decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ such that no $Q_{i_1} \cap \cdots \cap Q_{i_t}$ is a classical quasi-primary submodule, where $\{i_1, \cdots, i_t\} \subseteq \{1, \cdots, n\}$ for $t \geq 2$ with $i_1 < i_2 < \cdots < i_t$. If there is some $j, 1 \leq j \leq n$ such that $Q_j \supseteq \bigcap_{i \neq j} Q_i$, then we can exclude the Q_j from the decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. By using this argument, we can get the decomposition $N = Q_1 \cap Q_2 \cap \cdots \cap Q_k$ such that no component is abundant, so the decomposition is reduced. Obviously, among such reduced decompositions, we can get a minimal classical quasi-primary decomposition of N. □

Recall that any two incomparable primary ideals of a Prüfer domain are co-maximal (see for example [8, page 131]). Also by [7, Lemma 5.5], any two quasi-primary ideals with incomparable radicals of a prüfer domain are co-maximal. The next lemma proves a similar result for quasi-primary submodules.

Lemma 3.8. *Let R be a Prüfer domain, Q_1 and Q_2 be two quasi-primary submodules of an R-module M, and N be a submodule of M such that $Q_1 + Q_2 \subseteq N$. If $\sqrt{(Q_1 : N)}$ and $\sqrt{(Q_2 : N)}$ are incomparable,* *then $Q_1 + Q_2 = N$. In particular, any two quasi-primary submodules of M with incomparable radicals are co-maximal.*

Proof. It suffices to prove that $(Q_1+Q_2 : N) = R$. We can assume that $N \nsubseteq Q_1$ and $N \nsubseteq Q_2$, so $\sqrt{(Q_1 : N)}$ and $\sqrt{(Q_2 : N)}$ are prime ideals of R. Since R is a Prüfer domain, $\sqrt{(Q_1 : N)} + \sqrt{(Q_2 : N)} = R$. Finally, because $\sqrt{(Q_1 : N)}+\sqrt{(Q_2 : N)} \subseteq \sqrt{(Q_1 + Q_2 : N)}$, we conclude that $(Q_1 + Q_2 : N) = R$. □

One can easily see that a proper submodule N of an R-module M has a minimal quasi-primary decomposition if N can be shown as an intersection of finite number of quasi-primary submodules with pairwise incomparable radicals where no component can be omitted. So by Theorem 3.7 and Lemma 3.8, we can get the following corollary:

Corollary 3.9. *Let R be a Prüfer domain of finite character and N be a submodule of an R-module M such that $(N : M) \neq (0)$. Then N can be shown as an intersection of finite number of co-maximal submodules of M.*

The next theorem proves uniqueness of the decomposition of submodules into quasi-primary submodules of modules over a Prüfer domain of finite character.

Theorem 3.10. [Uniqueness Theorem]. *Let R be a Prüfer domain of finite character, $\mathcal{P}_i$, $1 \leq i \leq k$, be prime ideals of R, and N be a submodule of an R-module M. If $N = \bigcap_{i=1}^{k} Q_i$ is a minimal decomposition of N to $\mathcal{P}_i$-quasi-primary submodules $Q_i, 1 \leq i \leq k$, then k is independent of any such decompositions of N and*

$$\{\mathcal{P}_1, ..., \mathcal{P}_k\} = \mathrm{Min}(N : M).$$

Proof. First note that $\sqrt{(N : M)} = \bigcap_{i=1}^{k} \sqrt{(Q_i : M)} = \bigcap_{i=1}^{k} \mathcal{P}_i$. Since $\mathcal{P}_i$'s are incomparable prime ideals, then $\mathcal{P}_i$'s are minimal prime ideals of the ideal $(N : M)$ and so $\{\mathcal{P}_1, ..., \mathcal{P}_k\} = \mathrm{Min}(N : M)$. On the other word, k and the set $\{\mathcal{P}_1, ..., \mathcal{P}_k\}$ are independent of any such decompositions of N. □

Theorem 3.11. *Let R be a Prüfer domain of finite character and M be a multiplication R-module. Then every nonzero submodule N of M is the intersection of finite number of quasi-primary submodules with pairwise incomparable radicals, uniquely determined by N.*

Proof. Since M is a multiplication module, $N = (N : M)M$; so, $(N : M) \neq (0)$. Then the result follows form Theorems 3.7 and 3.11 (compare with [1, Theorem 3.4]). □

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References

1. S. E. Atani and A. Y. Darani, On quasi-primary submodules, *Chiang Mai J. Science* **33**(3) (2006), 249-254.
2. M. F. Atiyah and I. G. MacDonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, 1969.
3. M. Baziar and M. Behboodi, Classical primary submodules and decomposition theory of modules, *J. Algebra Appl.* **8**(3) (2009), 351-362.
4. M. Behboodi, R. Jahani-Nezhad, and M. H. Naderi, Classical quasi-primary submodules, *Bull. Iranian Math. Soc.* (to appear).
5. Z. El-Bast and P. F. Smith, Multiplication modules, *Comm. Algebra* **16**(4) (1998), 755-779.
6. L. Fuchs, On quasi-primary ideals, *Acta Sci. Math.* (Szeged) **11** (1947), 174-183.
7. L. Fuchs and E. Mosteig, Ideal theory in Prüfer domains, *J. Algebra* **252** (2002), 411-430.
8. M. D. Larsen and P. J. McCarthy, *Multiplicative ideal theory*, Academic press, 1971.
9. R. Y. Sharp, *Steps in commutative algebra*, London Math. Soc. Cambridge University Press, Cambridge, 1990.
10. P. F. Smith, Primary modules over commutative rings, *Glasg. Math. J.* **43**(1) (2001), 103-111.

Mahmood Behboodi

Department of Mathematical Sciences, Isfahan University of Technology, P.O.Box 84156-83111, Isfahan, Iran, and
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5746, Tehran, Iran.
Email: mbehbood@cc.iut.ac.ir

Reza Jahani-Nezhad

Department of Mathematics, Faculty of Science, University of Kashan, Kashan, Iran.
Email: jahanian@kashanu.ac.ir

Mohammad Hasan Naderi

Department of Mathematics, Faculty of Science, University of Qom, Qom, Iran.
Email: mh.naderi@qom.ac.ir