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ON COMULTIPLICATION AND R-MULTIPLICATION MODULES

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ABSTRACT. We state several conditions under which comultiplication and weak comultiplication modules are cyclic and study strong comultiplication modules and comultiplication rings. In particular, we will show that every faithful weak comultiplication module having a maximal submodule over a reduced ring with a finite indecomposable decomposition is cyclic. Also we show that if M is an strong comultiplication R-module, then R is semilocal and M is finitely cogenerated. Furthermore, we define an R-module M to be p-comultiplication, if every nontrivial submodule of M is the annihilator of some prime ideal of R containing the annihilator of M and give a characterization of all cyclic p-comultiplication modules. Moreover, we prove that every p-comultiplication module which is not cyclic, has no maximal submodule and its annihilator is not prime. Also we give an example of a module over a Dedekind domain which is not weak comultiplication, but all of whose localizations at prime ideals are comultiplication and hence serves as a counterexample to [11, Proposition 2.3] and [12, Proposition 2.4].

1. Introduction

In this paper all rings are commutative with identity, all modules are unitary, R denotes a ring and M denotes an R-module. Also by $\mathbb{N}$ we mean the set of positive integers. Furthermore, Z(M), Ann(M) and N(R) denote the set of zero divisors of M, the annihilator of M and the nilradical of R, respectively.

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It is said that M is a *multiplication module*, if for each submodule N of M, there is an ideal I of R, such that $N = IM$. It is easy to see that, in this case $N = (N : M)M$, where $(N : M) = \mathrm{Ann}\left(\frac{\mathrm{M}}{\mathrm{N}}\right)$ (see [13]).

In [6] the notion of a comultiplication module was introduced as a dual of the concept of a multiplication module. An R-module M is called *comultiplication* (*co-m* for short), if for every submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$. For example, the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$ is a co-m module since all of its proper submodules are of the form $(0 :_M 2^k\mathbb{Z})$ for $k = 0, 1, \ldots$. It is clear that M is co-m if and only if for every submodule N of M, we have $\mathrm{Ann_M(Ann_R(N)) = N}$.

In [12], a dual of the notion of weak multiplication modules (see [9]) is introduced and studied. A proper submodule N of M is said to be prime if from $rm \in N$ for $r \in R$ and $m \in M$ we can deduce that $r \in (N : M)$ or $m \in N$ (see for example [10]). An R-module M is said to be a *weak comultiplication* module, when for each prime submodule N of M, there is an ideal I of R, such that $N = (0 :_M I)$.

It should be mentioned that another dual notion of weak multiplication modules is defined in [4]. A submodule N of M is said to be a second submodule if $rN = N$ or $rN = 0$ for each $r \in R$. In [4], M is called weak comultiplication when for each second submodule N of M, there is an ideal I of R, such that $N = (0 :_M I)$. Here we use the term weak comultiplication module exclusively in the sense of [12].

In this paper, we find several conditions under which co-m modules are cyclic and study rings which are co-m modules over themselves (called *comultiplication rings*). In particular, we show that if M is a faithful weak co-m R-module with a maximal submodule N and R is a reduced ring (recall that a *reduced* ring is one with no nilpotents) with a finite indecomposable decomposition, then $M \cong R$ and R is semisimple.

Proposition 2.3 of [11] and Proposition 2.4 of [12] state that a module over a Dedekind domain is [weak] comultiplication, if and only if all of its localizations at prime ideals are [weak] comultiplication. Here in (5.4), we give a counterexample to both of these statements and hence show that they are inaccurate.

Moreover, we state some properties of co-m R-modules, for every submodule N of which there is exactly one ideal I of R with $(0 :_M I) = N$. Such modules are called strong comultiplication modules (see [5]). Particularly, we show that if there exists an strong comultiplication module M over R, then R is semilocal (that is, it has finitely many maximal ideals) and M is finitely cogenerated.

Finally, we study modules such as M, such that for every nontrivial submodule of them, say N, there can be found a prime ideal I of R containing $\mathrm{Ann(M)}$ such that $N = (0 :_M I)$. We call such modules *p-comultiplication (p-co-m)* modules. In fact, we conjecture that every such module is cyclic and give several conditions under which, this conjecture is true, for example if $\mathrm{Ann(M)}$ is not prime or if M has a maximal submodule. Also we give a characterization of cyclic p-co-m modules.

2. Comultiplication Modules

Multiplication modules have been studied by many mathematicians from different points of view. A suitable reference is [13]. In recent years, some authors have tried to find the dual of some of the interesting results concerning multiplication modules. Particularly in [6], comultiplication modules were introduced as a dual to multiplication modules. An R-module M is a [weak] comultiplication module, abbreviated as [weak] co-m module, when for every [prime] submodule N of M, there exists an ideal I of R such that $N = (0 :_M I)$. In this section we show that under various (not much strong) conditions co-m modules are cyclic. First we need a lemma.

Lemma 2.1. **(i)** *If a submodule N of M equals $(0 :_M I)$ for some ideal I of R, then $(N : M) = (\mathrm{Ann(M)} :_{\mathrm{R}} \mathrm{I})$.*

(ii) *M is a [weak] co-m R-module if and only if it is a [weak] co-m $\frac{R}{\mathrm{Ann(M)}}$-module.*

(iii) *Suppose that M is an R-module and $R = R_1 \times R_2$, where R_1 and R_2 are nontrivial rings. Then $M = M_1 \oplus M_2$ where M_1 is an R_1-module and M_2 is an R_2-module. Also in this case, M is [weak] co-m if and only if both M_1 and M_2 are so.*

Proof. (i): Let J be an arbitrary ideal of R, then:

$$J \subseteq (N : M) \;\Leftrightarrow\; JM \subseteq N \Leftrightarrow\; JM \subseteq (0 :_M I)$$
$$\Leftrightarrow\; IJM = 0 \;\Leftrightarrow\; IJ \subseteq \mathrm{Ann(M)} \;\Leftrightarrow\; \mathrm{J} \subseteq (\mathrm{Ann(M)} :_{\mathrm{R}} \mathrm{I}),$$

and the result follows.

(ii): Easy.

(iii): Set $M_1 = (1, 0)M$ and $M_2 = (0, 1)M$. Then M_1 and M_2 have the required properties. Also clearly every submodule of M is of the form $N_1 \oplus N_2$ where N_i is a submodule of M_i $(i = 1, 2)$. Assume that M_1 and M_2 are co-m, $N_1 = (0 :_{M_1} I)$ and $N_2 = (0 :_{M_2} J)$ where I and J are ideals of R_1 and R_2, respectively. Now $N_1 \times N_2 = (0 :_M I \times J)$, that is, M is co-m. Conversely if M is co-m and N_1 is a submodule of M_1, then $N_1 \times 0 = (0 :_M I \times J)$ for ideals I and J of R_1 and R_2, respectively. Therefore, $N_1 = (0 :_{M_1} I)$ as required.

Also note that prime submodules of M are exactly submodules of the form $N_1 \oplus M_2$ or $M_1 \oplus N_2$, where N_1 and N_2 are prime submodules of M_1 and M_2, respectively. Thus the result for weak co-m modules follows similar to the proof for co-m modules. □

According to (2.1), when M is a co-m module, it may be reduced to a faithful co-m module.

Theorem 2.2. *If M is a faithful weak co-m R-module with a maximal submodule N and R is a reduced ring with a decomposition as a finite direct product of indecomposable rings, then $M \cong R$ and R is semisimple.*

Proof. By (2.1), we can assume that R is indecomposable and show that R is a field. Suppose that $A = \mathrm{Ann(N)}$. Then $A \neq 0$ else $N = (0 :_M A) = M$ a contradiction. Suppose that $r \in A \cap (N : M)$. Then $r^2 M \subseteq rN = 0$. Therefore, $r^2 \in \mathrm{Ann(M)} = 0$ and since R is reduced, $r = 0$. Consequently, $A \cap (N : M) = 0$. On the other hand, since $A \neq 0$ and $(N : M)$ is a maximal ideal of R, we get $A + (N : M) = R$. Thus $R \cong \frac{R}{A} \times \frac{R}{(N:M)}$ and since $A \neq 0$ and R is indecomposable, we must have $(N : M) = 0$. This means that 0 is a maximal ideal of R, whence R is a field. So the only submodules of M are $M = (0 :_M 0)$ and $0 = (0 :_M R)$, that is, M is a simple vector space over the field R, as claimed. □

Immediate from (2.1)(ii) and (2.2) we have:

Corollary 2.3. *Assume that M is a weak co-m module having a maximal submodule (for example, if M is finitely generated) and $\mathfrak{M} = \mathrm{Ann(M)}$. Then $\mathfrak{M}$ is a prime ideal if and only if $\mathfrak{M}$ is a maximal ideal and $M \cong \frac{R}{\mathfrak{M}}$ is a simple module.*

Corollary 2.4. *If M is a finitely generated co-m module with* $\mathrm{Ann(M)}$ *a radical ideal, then M is cyclic and $\frac{R}{\mathrm{Ann(M)}}$ is a semisimple ring.*

Proof. We can assume that M is faithful and hence R is reduced. In [1], it is proved that if there exists a finitely generated faithful co-m R-module, then R is semilocal. Clearly a semilocal ring has an indecomposable decomposition as finite direct product of rings. Thus the result follows by (2.2). □

A *chained ring* is a ring in which every two ideals are comparable. For example, localization of $\mathbb{Z}$ at any prime ideal or more generally every valuation domain is a chained ring.

Lemma 2.5. *If R is a chained ring and M is co-m having a maximal submodule N, then M is cyclic.*

Proof. Choose an $m \in M \setminus N$. Set $A = \mathrm{Ann(Rm)}$ and $A' = \mathrm{Ann(N)}$. Then either $A \subseteq A'$ or $A' \subseteq A$. Therefore, either $Rm = (0 :_M A) \subseteq (0 :_M A') = N$ or $N \subseteq Rm$. Since the former is not possible, we deduce that $N \subseteq Rm$. On the other hand, since N is maximal and $m \notin N$, we have $Rm = Rm + N = M$. □

A ring, in which every nonzero proper ideal is a product of prime ideals, is called a ZPI-ring. Theorem 9.10 of [16] states that a ZPI-ring is a finite direct product of SPIRs (that is, principal ideal rings with exactly one prime ideal) and Dedekind domains.

Corollary 2.6. *Suppose that R is a ZPI-ring and M is a finitely generated R-module, then M is co-m if and only if M is cyclic and $\frac{R}{\mathrm{Ann(M)}}$ is a finite direct product of SPIRs.*

Proof. Since every quotient of a ZPI-ring is itself a ZPI-ring, using (2.1)(ii) and (iii), we can assume that M is faithful.

($\Rightarrow$): Suppose that M is co-m. According to [16, Theorem 9.10], $R \cong R_1 \times \cdots \times R_n$, where each R_i is either an SPIR or a Dedekind domain. By (2.1)(iii), $M \cong M_1 \oplus \cdots \oplus M_n$ where each M_i is a (clearly faithful and finitely generated) co-m R_i-module. If R_i is an SPIR, then by (2.5), $M_i \cong R_i$ (note that every SPIR is a chained ring). If R_i is a Dedekind domain, then according to (2.3), $M_i \cong R_i$ and R_i is a field, hence an SPIR. Consequently, each R_i is an SPIR and $M \cong R$.

($\Leftarrow$): Using (2.1)(iii), we assume that R is indecomposable and hence an SPIR. If Rp is the unique prime ideal of R and k is the minimum nonnegative integer with $Rp^k = 0$, then $Rp^i = (0 : Rp^{k-i})$ for $i = 0, 1, \ldots, k$ and hence R is co-m. □

In what follows, by a *semi-non-torsion* R-module M, we mean a module which is non-torsion over $\frac{R}{\mathrm{Ann(M)}}$. The following remark states some other conditions under which a co-m module must be cyclic.

Remark 2.7.
- **(i)** If M is a semi-non-torsion co-m R-module, then $\mathrm{Ann(m)} = \mathrm{Ann(M)}$ for some $m \in M$ and hence $Rm = (0 :_M \mathrm{Ann(Rm)}) = (0 :_\mathrm{M} \mathrm{Ann(M)}) = \mathrm{M}$. That is, M is cyclic.
- **(ii)** If M is a finitely generated co-m R-module and $\mathrm{Ann(M)}$ is irreducible, then $\mathrm{Ann(M)} = \bigcap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Ann(m_i)}$, where $\{m_1, \ldots, m_n\}$ is a generating set of M. Hence $\mathrm{Ann(M)} = \mathrm{Ann(m_i)}$ for some i and $M = Rm_i$ is cyclic.
- **(iii)** Suppose that R is a finitely cogenerated ring with irreducible zero ideal and M is a faithful co-m R-module. Then $0 =$

$\bigcap_{m\in M} \mathrm{Ann(m)}$. Hence $\mathrm{Ann(M)} = 0 = \mathrm{Ann(m)}$ for some $m \in M$ and $M = Rm$ is cyclic.

As we saw under various conditions, finitely generated co-m modules are cyclic (although there are noncyclic finitely generated co-m modules, see for example (2.18)). Also in (5.5) we shall give some other criteria under which a co-m module is cyclic. This proposes to consider rings, which as modules over themselves are [weak] co-m. Such rings are called [weak] co-m rings. As an example one can readily check that every SPIR is a co-m ring. It is easy to see that a ring is co-m if and only if it satisfies the double annihilator condition, that is, $\mathrm{Ann(Ann(I))} = \mathrm{I}$ for every ideal I of R.

Remark 2.8. If R is a co-m ring, then for every $r \in R \setminus \mathrm{Z(R)}$ we have $rR = R$ by [6, Lemma 3.15]. Thus every co-m ring is a total quotient ring. Also according to (2.2), a reduced indecomposable ring (such as an integral domain) is a weak co-m ring if and only if it is a field.

Remark 2.9. Assume that M is a finitely generated weak co-m R-module and let $\mathfrak{P}$ be a prime ideal of R containing $\mathrm{Ann(M)}$. Then by [10, Lemma 4], M has a prime submodule N with $(N : M) = \mathfrak{P}$. If $I = \mathrm{Ann(N)}$, then since M is weak co-m and N is prime, by applying (2.1), we get that $(\mathrm{Ann(M)} :_{\mathrm{R}} \mathrm{I}) = \mathfrak{P}$. Therefore, $\frac{R}{\mathrm{Ann(M)}}$ is a weak co-m ring.

Proposition 2.10. *The following are equivalent for a ring R.*

- **(i)** *R is a [weak] co-m ring.*
- **(ii)** *Every faithful [weak] multiplication R-module is a [weak] co-m R-module.*
- **(iii)** *There exists a finitely generated faithful multiplication [weak] co-m R-module.*

Proof. (i) $\Rightarrow$ (ii): Let M be a faithful multiplication R-module. Assume that $N = IM$ is an arbitrary submodule of M where I is an ideal of R (if N is prime, we can choose $I = (N : M)$ which is a prime ideal of R). By assumption there is an ideal J of R such that $I = (0 :_R J)$. Set $K = (0 :_M J)$, then by (2.1)(i), $(K : M) = (0 :_R J) = I$. Thus $(0 :_M J) = K = (K : M)M = IM = N$, as required.

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (i): Let I be an ideal of R and set $N = IM$, where M is a finitely generated faithful co-m multiplication module. Then by [13, Theorem 3.1], $I = (N : M)$ which, according to (2.1)(i), is equal to $(0 :_R J)$, where J is the ideal of R such that $N = (0 :_M J)$. For the weak version, just note that if I is a prime ideal, then N is a prime submodule by [13, Corollary 2.11]. □

We can apply (2.1)(iii) on [weak] co-m modules over $R = \prod_{a\in A} R_a$ only when A is finite. But the following theorem shows that at least for the faithful co-m modules we have a similar result even if A is infinite.

Theorem 2.11. *Suppose that $R = \prod_{a\in A} R_a$, where R_a's are nontrivial rings. A faithful R-module M is co-m if and only if $M = \bigoplus_{a\in A} M_a$, where each M_a is a co-m R_a-module and $R_b M_a = 0$ for $a \neq b \in A$. In particular, R is a co-m ring, if and only if $|A| < \infty$ and each R_a is a co-m ring.*

Proof. Let e_a be the element of R with 1 in the a'th component and zero at others. First assume that M is a co-m R-module. Set $M_a = e_a M$. Then $\mathrm{Ann(M_a)} = \mathrm{R'_a}$, where R'_a is the set of all elements of R with zero a'th component and hence M_a is an R_a-module. Suppose that $m_1 + m_2 + \cdots + m_n = 0$ where $m_i \in M_{a_i}$ for some a_i's in A. Then by multiplying both sides by e_{a_i} we see that each $m_i = 0$. Therefore, $N = \bigoplus_{a\in A} M_a$ is a submodule of M. But $\mathrm{Ann(N)} = \bigcap_{\mathrm{a\in A}} \mathrm{Ann(M_a)} = \bigcap_{\mathrm{a\in A}} \mathrm{R'_a} = 0$. Thus $N = (0 :_M \mathrm{Ann(N)}) = (0 :_{\mathrm{M}} 0) = \mathrm{M}$.

If K_a is a submodule of M_a, then

$$\mathrm{Ann_R(K_a)} = \mathrm{Ann_{R_a}(K_a)} \times \prod_{\mathrm{a\neq b\in A}} \mathrm{R_b}.$$

Because M is co-m we get

$$K_a = (0 :_M \mathrm{Ann_R(K_a)}) = (0 :_{\mathrm{M_a}} \mathrm{Ann_{R_a}(K_a)}) \oplus \left(\bigoplus_{\mathrm{a\neq b\in A}} (0 :_{\mathrm{M_b}} \mathrm{R_b})\right) = (0 :_{M_a} \mathrm{Ann_{R_a}(K_a)}).$$

Consequently, M_a is a co-m R_a-module.

Conversely, suppose that $M = \bigoplus_{a\in A} M_a$ for co-m R_a-modules M_a's. Now let K be a submodule of $\bigoplus M_a$. If $K_a = K \cap M_a$, then clearly $\bigoplus K_a \leq K$. Also, since $k = k_1 + \cdots + k_n$ for some k_i's in M_{a_i}'s and $e_{a_i} k = e_{a_i} k_i = k_i \in K \cap M_a$, so $k \in \bigoplus K_a$, that is, $K = \bigoplus K_a$. Consider the ideal $I = \prod \mathrm{Ann(K_a)}$. Then $(0 :_M I) = \bigoplus (0 :_{M_a} \mathrm{Ann(K_a)}) = \bigoplus \mathrm{K_a} = \mathrm{K}$ as required. For the last statement, just note that if R is a direct sum of some rings, then only finitely many of these rings can be nonzero. □

Corollary 2.12. *Let R be a reduced Noetherian ring. Then R is co-m if and only if it is a finite direct product of fields.*

Proof. This follows from the fact that every reduced Noetherian ring is a finite direct product of indecomposable reduced rings, (2.11) and (2.2). □

Thus we see that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a co-m ring although it is not a co-m $\mathbb{Z}$-module or co-m $\mathbb{Z}_2$-module.

The proof of the following can be found in [1].

Proposition 2.13. *Suppose that M is co-m. Then M is of finite length if and only if it is Noetherian.*

For weak co-m modules, the similar statement does not hold, as the next example shows.

Example 2.14. Set $R_0 = K[X,Y]_{\mathfrak{M}}$, where $\mathfrak{M} =< X, Y >$, K is a field and let x, y denote the images of X, Y in R_0, respectively. Suppose that $I =< x^3, x^2y >$ and $R = \frac{R_0}{I}$. Then R is a Noetherian weak co-m R-module, which is not of finite length. Hence R is not a co-m R-module.

Proof. Clearly $\dim R = 1$ and the only prime ideals of R are $R\bar{x}$ and $R\bar{x} + R\bar{y}$, where $\bar{x}, \bar{y}$ are the images of x, y in R. Thus R is Noetherian but not of finite length. Now $R\bar{x} = (0 : \bar{x}\bar{y})$ (if $\frac{f}{1}xy \in< x^3, x^2y >$ for some $f \in K[X,Y]$, then for some $t \in K[X,Y] \setminus \mathfrak{M}$, we get that $tfXY \in< X^3, X^2Y >$, which implies $f \in< X >$). Also $R\bar{x} + R\bar{y} = (0 : R\bar{x}^2)$. So R is a weak co-m R-module, which according to (2.13), is not a co-m R-module. □

Using (2.13) one can see that if a Noetherian ring is a co-m ring, then it is Artinian. Since for a ring being co-m is equivalent to satisfying the double annihilator condition, we see that Noetherian (or equivalently Artinian) co-m rings are exactly quasi-Frobenius rings, which are well studied (see, for example [2, §30]). Because an Artinian ring is a finite direct product of some Artinian local rings, to know which Noetherian rings are co-m, it suffices to consider Artinian local rings. Not all Artinian local rings are co-m, as the following example shows.

Example 2.15. Set $R_0 = K[X,Y]$, where K is a field and $\mathfrak{M} =< X, Y >$. Let $R = \frac{R_0}{\mathfrak{M}^2}$. Then clearly R is an Artinian local ring. But R is not a co-m ring, else its submodule $\frac{\mathfrak{M}}{\mathfrak{M}^2}$ also must be a co-m R-module and by (2.1)(ii) a co-m $\frac{R}{\mathfrak{M}}$-module. According to (2.2), this means that $\frac{\mathfrak{M}}{\mathfrak{M}^2}$ is cyclic, a contradiction.

In [9], it is proved that although not every weak multiplication module is multiplication but at least for finitely generated modules being weak multiplication and being multiplication are equivalent. It can easily be checked that if R is an integral domain and $Q \neq R$ is its field of fractions, then Q is not a co-m R-module but the only prime submodule of Q is $0 = (0 :_M R)$, hence Q is weak co-m. But the following remark shows that even there are Artinian cyclic weak co-m modules, which are not co-m.

Remark 2.16. If $(R, \mathfrak{M})$ is an Artinian local ring, then $\mathfrak{M}^n = 0$ for some $n \in \mathbb{N}$. If n is the least such number, then obviously $\mathfrak{M} = (0 : \mathfrak{M}^{n-1})$. Therefore, every Artinian local ring (and by (2.1)(iii), every Artinian ring) is a weak co-m ring. Hence (2.15) shows that, even under the strong conditions of being cyclic and Artinian, a weak co-m module need not be co-m.

The argument in (2.15) shows that if $(R, \mathfrak{M})$ is a quasi-local (that is, $\mathfrak{M}$ is the only maximal ideal of R) co-m ring with $\mathfrak{M}^2 = 0$, then $\mathfrak{M}$ is cyclic and hence R is an SPIR by [18, Lemma 15.41]. This might cause one to guess that every Artinian local co-m ring is an SPIR, but the following example shows that this is not the case.

Example 2.17. Set $R_0 = \mathbb{Z}_3[X, Y]$ and $R = \frac{R_0}{I}$, where $I =< XY, X^2 - Y^2 >$. Then R is an Artinian local co-m ring, but not an SPIR.

Proof. Let x, y denote the images of X, Y in R, respectively and $\mathfrak{M} =< x, y >$. First note that $x^3 = x(x^2) = x(y^2) = y(xy) = 0$. Similarly $y^3 = 0$, that is, $X^3, Y^3 \in I$ and hence $(R, \mathfrak{M})$ is a zero dimensional local ring, with $\mathfrak{M}^3 = 0$. Also $\mathfrak{M}^2 =< x^2, y^2, xy >=< x^2 >$ is principal. Clearly every element of R can be written uniquely in the form of $f = ax^2 + bx + cy + d$, where $a, b, c, d \in \mathbb{Z}_3$. Thus it is impossible that $x = fy = cx^2 + dy$, whence $x \notin Ry$ and similarly $y \notin Rx$. So $\{x, y\}$ is a minimal generating set of $\mathfrak{M}$ and $\mathfrak{M}$ is not cyclic. Thus R is a non-SPIR Artinian local ring.

Suppose that I is a proper ideal of R not contained in $\mathfrak{M}^2$, say $z = ax + by + cx^2 \in I$ with not both of $a, b = 0$. Therefore, either yz or xz is a nonzero element in $\mathfrak{M}^2$ and $\mathfrak{M}^2 \cap I \neq 0$. But every element of $\mathfrak{M}^2$ is of the form ux^2 where $u \in \mathbb{Z}_3$ is a unit. Therefore, $\mathfrak{M}^2$ is simple and hence $\mathfrak{M}^2 \subseteq I$. Thus $\frac{I}{\mathfrak{M}^2}$ is a nonzero subspace of the two dimensional vector space $\frac{\mathfrak{M}}{\mathfrak{M}^2}$ over $\frac{R}{\mathfrak{M}}$. Consequently, either $I = \mathfrak{M}$ or I is generated by some element $f \in \mathfrak{M} \setminus \mathfrak{M}^2$. Using this one can check that nontrivial ideals of R are: $\mathfrak{M}$, $\mathfrak{M}^2$, $< x >$, $< y >$, $< x + y >$ and $< x - y >$. Now $\mathfrak{M} = (0 : \mathfrak{M}^2)$, $\mathfrak{M}^2 = (0 : \mathfrak{M})$, $< y >= (0 : x)$, $< x >= (0 : y)$, $< x + y >= (0 : x - y)$ and $< x - y >= (0 : x + y)$, as required. ∎

Example 2.18. The maximal ideal $\mathfrak{M}$ of R in (2.17), being a submodule of a co-m module, is itself a finitely generated co-m module which is not cyclic.

3. r-Multiplication Modules

Before continuing our investigation of co-m modules, we pay some attentions to another concept which is related to multiplication modules and will be used in the next section. An R-module M is multiplication, when for every proper submodule N of M, there is a proper ideal I of R with $N = IM$. Thus it is natural to consider R-modules such as M with the "reverse" property: for every proper ideal I of R there exists a proper submodule N of M with $N = IM$. Clearly this is equivalent to the property that for every proper ideal I of R, $IM \neq M$.

Definition 3.1. We say that an R-module M is an r-multiplication (r-m, for short) module, when $IM \neq M$ for every proper ideal I of R.

For example by [15, Theorem 76], every finitely generated faithful module is r-m. Also clearly $\mathbb{Q}$ as a $\mathbb{Z}$-module or more generally every divisible module over an integral domain which is not a field, is not r-m.

Proposition 3.2. *The following are equivalent.*

- **(i)** *M is an r-m R-module.*
- **(ii)** *$M_{\mathfrak{M}}$ is an r-m $R_{\mathfrak{M}}$-module, for every maximal ideal $\mathfrak{M}$ of R.*
- **(iii)** *$\frac{M_{\mathfrak{M}}}{\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}}} \neq 0$, for every maximal ideal $\mathfrak{M}$ of R.*

Proof. (i) ⇒ (ii): Suppose that I is a proper ideal of $R_{\mathfrak{M}}$. If $\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}}$ is not a proper submodule of $M_{\mathfrak{M}}$, then $(\mathfrak{M}M)_{\mathfrak{P}} = M_{\mathfrak{P}}$ for every maximal ideal $\mathfrak{P}$ of R (if $\mathfrak{P} \neq \mathfrak{M}$ then both sides equal $M_{\mathfrak{P}}$, and the case $\mathfrak{P} = \mathfrak{M}$ is the assumption), whence $\mathfrak{M}M = M$. Thus M is not r-m against (i). So $IM_{\mathfrak{M}} \subseteq \mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}} < M$, whence is proper.

(ii) ⇒ (iii): Since $\mathfrak{M}_{\mathfrak{M}}$ is a proper ideal of $R_{\mathfrak{M}}$ and $M_{\mathfrak{M}}$ is r-m, we have $\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}} \neq M_{\mathfrak{M}}$ whence the result.

(iii) ⇒ (i): For every maximal ideal $\mathfrak{M}$ of R, we have $\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}} \neq M_{\mathfrak{M}}$ by (iii). Consequently, $\mathfrak{M}M \neq M$. But every proper ideal I of R is contained in a maximal ideal $\mathfrak{M}$ of R and therefore, $IM \subseteq \mathfrak{M}M$ is proper. □

In the following J(M) means the Jacobson radical of M.

Proposition 3.3. *Suppose that $(R, \mathfrak{M})$ is a quasi-local ring. Then M is r-m if and only if it has a maximal submodule if and only if* $\mathrm{J(M)} \neq \mathrm{M}$ *if and only if there is an $m \in M$ such that Rm is not superfluous in M.*

Proof. If M is r-m, then by (3.2), $\frac{M}{\mathfrak{M}M}$ is a nonzero $\frac{R}{\mathfrak{M}}$ vector space and hence has a maximal subspace $\frac{N}{\mathfrak{M}M}$, where $\mathfrak{M}M \subseteq N$. Thus N is a maximal submodule of M. Conversely, if N is a maximal submodule of M, then $(N : M) = \mathfrak{M}$ and hence for every proper ideal I of R we have $IM \subseteq \mathfrak{M}M \subseteq N \neq M$.

Since by [2, Proposition 9.13],

$$\mathrm{J(M)} = \bigcap\{\mathrm{N|N\ is\ a\ maximal\ submodule\ of\ M}\} =$$
$$\sum\{N|N \text{ is a superfluous submodule of } M\}$$

the other equivalencies follow. □

Recall that a ring is perfect if and only if it satisfies DCC on its principal ideals.

Corollary 3.4. *Every nonzero R-module is r-m, if and only if R is a perfect quasi-local ring.*

Proof. If R is a perfect quasi-local ring, then every nonzero R-module has a maximal submodule by [2, Theorem 28.4] and hence by (3.3), every nonzero module is r-m. Conversely, if every nonzero R-module is r-m and $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are maximal ideals of R and $M = \frac{R}{\mathfrak{M}_2}$, then $\mathfrak{M}_1 M \neq M$. From this we conclude that $\mathfrak{M}_1 = \mathfrak{M}_2$, that is, R is quasi-local. Now the result follows by [2, Theorem 28.4] and (3.3). □

Example 3.5. Suppose that F is a field, $R_0 = F[\{x_a\}_{a\in A}]$, $I = \langle\{x_a^2|a \in A\}\rangle$ and $R = \frac{R_0}{I}$. If $|A| < \infty$, then clearly R is an Artinian local ring and according to (3.4), every nonzero R-module is r-m. But if A is infinite, say $\mathbb{N} \subseteq A$, then there is no $n \in \mathbb{N}$ such that $\bar{x}_1\bar{x}_2\cdots\bar{x}_n = 0$, therefore by [2, Theorem 28.4] and (3.4), there exists a nonzero R-module which is not r-m, although R is zero dimensional quasi-local.

In [17] a nonzero module was called *weakly présimplifiable*, when for any $r \in R$ if $rm = m$ for all $m \in M$, then r is a unit. As an example, it is not hard to see that for any ring R, $\frac{R}{\mathrm{J(R)}}$ is a weakly présimplifiable R-module.

Proposition 3.6. *Suppose that M is a finitely generated R-module. Then the following are equivalent.*

- **(i)** *M is r-m.*
- **(ii)** *M is weakly présimplifiable.*
- **(iii)** $\mathrm{Ann(M)} \subseteq \mathrm{J(R)}$.
- **(iv)** *$M_{\mathfrak{M}} \neq 0$ for every maximal ideal $\mathfrak{M}$ of R.*

Proof. (i) ⇔ (ii): By [17, Proposition 2.12], since M is finitely generated, M is weakly présimplifiable if and only if for every ideal I of R, $IM = M$ implies $I = R$. But clearly this means that M is r-m.

(ii) $\Leftrightarrow$ (iii): This is [17, Proposition 2.8(1)].

(i) $\Leftrightarrow$ (iv): Since $M_{\mathfrak{M}}$ is finitely generated, $\mathfrak{M}_{\mathfrak{M}} M_{\mathfrak{M}} = M_{\mathfrak{M}}$ if and only if $M_{\mathfrak{M}} = 0$. Therefore, the result follows by (3.2). □

Note that the $\mathbb{Z}$-module $\mathbb{Q}$ shows that finitely generated condition in (3.6) is necessary.

Corollary 3.7. *A multiplication R-module M is r-m, if and only if it is weakly présimplifiable and finitely generated.*

Proof. By [13, Theorem 3.1], for a multiplication module M, if $IM \neq M$ for each ideal I of R containing $\mathrm{Ann(M)}$, then M is finitely generated. Therefore every r-m multiplication module is finitely generated. Now the result follows by (3.6). □

At the end of this section, we give a comparison of the three properties of being an r-m, a co-m or a multiplication module. Clearly $\mathbb{Q}$ as a $\mathbb{Z}$-module has neither of these properties and as a $\mathbb{Q}$-module has all of them. Every vector space over a field with $\dim \leq 2$ is an r-m module which is not co-m or multiplication. Also $\mathbb{Z}_{2^\infty}$ is a co-m $\mathbb{Z}$-module but not r-m or multiplication. If R is a two dimensional integral domain with $\mathrm{J(R)} = 0$ (say, $R = \mathbb{Z}[x]$) and $\mathfrak{P} \subsetneqq \mathfrak{Q}$ are nonzero prime ideals of R, then $\frac{R}{\mathfrak{P}}$ is a multiplication R-module, but is neither r-m (by (3.7)) nor co-m (by (2.3)). The $\mathbb{Z}$-module $\mathbb{Z}$ is not co-m but is r-m and multiplication. A simple module over a non-local ring is clearly co-m and multiplication but not r-m by (3.7). Finally, in (2.18), we presented a finitely generated noncyclic co-m module over a local ring which is r-m by (3.6) but it is not multiplication (since every multiplication module over a local ring is cyclic by [13, Theorem 1.2]).

4. Strong Comultiplication Modules

It is possible for a co-m R-module, say M, to have a submodule N for which there exist two ideals $I \neq J$ with the property $(0 :_M I) = N = (0 :_M J)$. For example, if $M = \mathbb{Z}_{2^\infty}$, then $(0 :_M 2\mathbb{Z}) = (0 :_M 6\mathbb{Z})$. It is easy to see that for each submodule N of M there exists a unique ideal of R such that $N = (0 :_M I)$ if and only if M is co-m and satisfies the double annihilator condition (that is, $\mathrm{Ann_R(Ann_M(I))} = \mathrm{I}$ for each ideal I of R). In [5] modules with this property are called *strong comultiplication* (abbreviated as s-co-m). For example, if $(R, \mathfrak{M})$ is a complete Noetherian local ring and $M = E\left(\frac{R}{\mathfrak{M}}\right)$ is the injective envelope of $\frac{R}{\mathfrak{M}}$, then M is an s-co-m R-module (see[5, Example 2.2]). Also every co-m ring is an s-co-m module over itself. In this section we turn our attention to this uniqueness condition.

Proposition 4.1. *Assume that M is a co-m R-module. The following are equivalent for any pair of ideals I and J of R.*

(i) $(0 :_M I) = (0 :_M J)$.
(ii) $(0 :_M I + K) = (0 :_M J + K)$ *for every ideal K of R.*
(iii) $(0 :_M IK) = (0 :_M JK)$ *for every ideal K of R.*
(iv) $IN = JN$ *for every submodule N of M.*

Proof. (ii) and (iii) $\Rightarrow$ (i): Set $K = R$ or $K = 0$. (iv) $\Rightarrow$ (i) and also (i) $\Rightarrow$ (ii): Easy. (i) $\Rightarrow$ (iii): Just note that $(0 :_M IK) = ((0 :_M I) :_M K)$. (i) $\Rightarrow$(iv):

$$\mathrm{Ann(IN)} = ((0 :_{\mathrm{M}} \mathrm{I}) :_{\mathrm{R}} \mathrm{N}) = ((0 :_{\mathrm{M}} \mathrm{J}) :_{\mathrm{R}} \mathrm{N}) = \mathrm{Ann(JN)}.$$

Thus $IN = JN$, because M is co-m. □

Corollary 4.2. *Assume that M is a finitely generated co-m module. If $(0 :_M I) = (0 :_M J)$ for some ideals I and J of R, then $\sqrt{I + \mathrm{Ann(M)}} = \sqrt{J + \mathrm{Ann(M)}}$. Hence if $\mathfrak{P}$ is a prime ideal of R containing* $\mathrm{Ann(M)}$*, then* $\mathrm{Ann}(0 :_{\mathrm{M}} \mathfrak{P}) = \mathfrak{P}$.

Proof. If $(0 :_M I) = (0 :_M J)$, then according to (4.1), $IM = JM$ and the result follows by [8, Proposition 2.4]. Now the final assertion is clear. □

Proposition 4.3. *A nonzero multiplication R-module M is s-co-m if and only if it is finitely generated faithful and R is a co-m ring.*

Proof. If M is s-co-m, then $\mathrm{Ann(M)} = \mathrm{Ann_R(Ann_M(0))} = 0$. Now if $IM = M$ for some ideal I of R, then $(0 :_R I)M = 0$, whence $(0 :_R I) = 0$. Now set $N = (0 :_M I)$. By (2.1) $(N : M) = (0 :_R I) = 0$ and $N = (N : M)M = 0$. Therefore, $I = \mathrm{Ann_R(Ann_M(I))} = \mathrm{Ann_R(0)} = \mathrm{R}$. This shows that M is r-m and by (3.7) is finitely generated. Consequently, R is a co-m ring by (2.10). Using (2.10) and its proof the converse can easily be established. □

Suppose that M is an s-co-m R-module. Consider the mapping $\phi : l(R) \to l(M)$, where $l(N)$ denotes the lattice of submodules of N for any R-module N, defined by $\phi(I) = \mathrm{Ann_M(I)}$. Clearly ϕ is one-to-one, onto and order reversing with the order reversing inverse $\phi^{-1}(N) = \mathrm{Ann(N)}$ for each submodule N of M. That is, ϕ is a lattice anti-isomorphism . As the following result shows, using this mapping we can easily establish some properties of s-co-m modules.

Recall that if A, A' and B are submodules of M such that $A' \subseteq B$, $M = A + A'$ and A' is minimal with respect to this property, then A' is said to be a *supplement* of A in B (this is the dual notion of a complement of a submodule). In [14] M is said to be *amply supplemented* when for each pair of submodules A, B of M with $M = A + B$, A has a supplement in B.

Theorem 4.4. *Assume that M is an s-co-m R-module. Then:*

(i) *M is finitely cogenerated and both M and R are amply supplemented R-modules.*
(ii) *R is semilocal.*

Proof. (i): Note that being finitely generated and finitely cogenerated are lattice properties which are mapped to each other under anti-isomorphisms. Thus since R is finitely generated, M is finitely cogenerated. Now consider submodules A, B of M with $M = A + B$ and let ϕ be as above. Then $\phi^{-1}(A) \cap \phi^{-1}(B) = \phi^{-1}(M) = 0$. Thus $\phi^{-1}(A)$ has a complement, say I, in R which contains $\phi^{-1}(B)$ (see [2, p. 75]). Now it is easy to see that $\phi(I)$ is a supplement in B of A. A similar argument shows that R is amply supplemented.

(ii): Suppose that N is the socle of M. Since by (i) M is finitely cogenerated, N is also a finitely cogenerated and hence a finitely generated R-module. But ϕ^{-1} maps minimals to maximals and summations to intersections. Thus $\phi^{-1}(N) = \mathrm{J(R)}$ and the image of $l(N)$ under ϕ^{-1} is the set of those ideals of R which contain $\mathrm{J(R)}$, that is, $l\left(\frac{R}{\mathrm{J(R)}}\right)$. Therefore, since N is finitely generated, $\frac{R}{\mathrm{J(R)}}$ is finitely cogenerated. So $\mathrm{J(R)} = \bigcap_{\mathrm{i=1}}^{\mathrm{n}} \mathfrak{M}_{\mathrm{i}}$ for a finite set of maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ of R. Hence R is semilocal. □

Corollary 4.5. *If M is an s-co-m module having a maximal submodule over a reduced ring R, then $M \cong R$ and R is semisimple.*

Proof. Similar to the proof of (2.4) □

5. p-Comultiplication Modules

In this section we study co-m modules M, in which every nontrivial submodule is the annihilator of a prime ideal containing $\mathrm{Ann(M)}$. (5.5) shows that under various (weak) conditions, such modules are cyclic. In fact, we conjecture that every such module is cyclic.

Definition 5.1. We call M a p-comultiplication R-module, if for each nontrivial submodule N of M, there is a prime ideal $\mathfrak{P}$ of R containing $\mathrm{Ann(M)}$, such that $N = (0 :_M \mathfrak{P})$.

For example, it is easy to see that $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are p-co-m $\mathbb{Z}$-modules. It is obvious that M is a p-co-m R-module if and only if it is a p-co-m $\frac{R}{\mathrm{Ann(M)}}$-module and that every p-co-m module is co-m. Note that the converse of the last statement is not correct as $\mathbb{Z}_{p^\infty}$ is easily seen to be a co-m $\mathbb{Z}$-module which is not p-co-m.

Lemma 5.2. *Suppose that $R = R_0 \times R_1$ and M is an R-module. Then M is p-co-m if and only if either (1) for some $i = 0, 1$, $R_iM = 0$ and M is a p-co-m R_{1-i}-module or (2) $M = M_0 \oplus M_1$ where M_i is a simple R_i-module such that $R_{1-i}M_i = 0$.*

Proof. Noting that prime ideals of R are exactly those of the form $\mathfrak{P}_0 \times R_1$ or $R_0 \times \mathfrak{P}_1$ where $\mathfrak{P}_i$ is a prime ideal of R_i, one can easily check that in both cases (1) and (2), M is p-co-m. Conversely, assume that M is p-co-m. As in (2.1)(iii) $M = M_0 \oplus M_1$, where M_i is an R_i module and $R_{1-i}M_i = 0$. If one of the M_i's is zero, then clearly case (1) happens.

Suppose that no $M_i = 0$ and A is a nontrivial submodule of M_0. Then $N = A \oplus 0$ is a nontrivial submodule of M and hence for some prime ideal $\mathfrak{P}$ of R containing Ann(M), we have $N = (0 :_M \mathfrak{P})$. But since $\mathfrak{P}$ is one of the two forms stated above, we conclude that either $N = N_0 \oplus M_1$ or $N = M_0 \oplus N_1$, where N_i is a submodule of M_i. Therefore, either $M_1 = 0$ or $A = M_0$, both of which are contradictions. Thus M_0 and similarly M_1 have no nontrivial submodules and so are simple, as asserted. □

In [1], it is proved that a Noetherian module is co-m if and only if all of its localizations at prime ideals are co-m. Similarly one can easily prove the following lemma. For case (iii) just note that when M is Artinian, $\frac{M}{N}$ is finitely cogenerated for every submodule N of M and hence by [7, Theorem 2.4(a)], $N = (0 :_M I)$ for some finitely generated ideal I of R.

Lemma 5.3. **(i)** *Assume that M is a co-m [resp. weak co-m, p-co-m] module over a Noetherian ring R. Then $S^{-1}M$ is co-m [resp. weak co-m, p-co-m] for every multiplicatively closed subset $S \subseteq R$.*

(ii) *If M is Noetherian and $M_{\mathfrak{M}}$ is a weak co-m $R_{\mathfrak{M}}$-module for every maximal ideal $\mathfrak{M}$ of R, then M is weak co-m.*

(iii) *If M is Artinian and co-m, then $S^{-1}M$ is co-m for every multiplicatively closed subset $S \subseteq R$.*

The following example shows that it is possible that every localization of M be co-m, without M being so. Also note that the following serves as a counterexample to [11, Proposition 2.3] and [12, Proposition 2.4] which state that if R is a Dedekind domain, then M is [weak] co-m R-module if and only if $M_{\mathfrak{P}}$ is a [weak] co-m $R_{\mathfrak{P}}$-module, for all prime ideals $\mathfrak{P}$ of R.

Example 5.4. Let $R = \mathbb{Z}$ and $M = \bigoplus_{p \in \mathcal{P}} \mathbb{Z}_p$ where $\mathcal{P}$ is the set of positive prime integers. Clearly $N = \bigoplus_{2 \neq p \in \mathcal{P}} \mathbb{Z}_p$ is a maximal (and hence prime) submodule of M and $N \neq (0 :_M \mathrm{Ann(N)})$. Therefore, M is not a weak co-m R-module. But for each maximal ideal of R such as $\mathfrak{M} = Rp$ ($p \in \mathcal{P}$), $M_{\mathfrak{M}} \cong \mathbb{Z}_p$ as $R_{\mathfrak{M}}$-module and hence is a simple and p-co-m $R_{\mathfrak{M}}$-module. Notice that $M_0 = 0$ is trivially a p-co-m $\mathbb{Q}$-module.

Theorem 5.5. *Assume that M is a p-co-m module. In either of the following cases, M is cyclic.*

- **(i)** *M has a maximal submodule.*
- **(ii)** $\mathrm{Ann(M)}$ *is not a prime ideal of R.*
- **(iii)** $\frac{R}{\mathrm{Ann(M)}}$ *is an integral domain with finitely many primes of hight one such that every nonzero prime ideal of* $\frac{R}{\mathrm{Ann(M)}}$ *contains a height one prime ideal; in particular if* $\frac{R}{\mathrm{Ann(M)}}$ *is a valuation domain.*
- **(iv)** $\frac{R}{\mathrm{Ann(M)}}$ *is a Noetherian domain with Krull dimension ≤ 1, for example a Dedekind domain.*

Proof. Clearly we can assume that M is faithful. First we will show that if M is not cyclic, then R is reduced and indecomposable. So suppose that M is not cyclic. Then for each $0 \neq m \in M$, Rm is a nontrivial submodule of M, whence $Rm = (0 :_M \mathfrak{P})$ for some prime ideal $\mathfrak{P}$ of R. Therefore, $\mathrm{N(R)} \subseteq \mathfrak{P} \subseteq \mathrm{Ann(m)}$. Hence $\mathrm{N(R)} \subseteq \bigcap_{0 \neq \mathrm{m} \in \mathrm{M}} \mathrm{Ann(m)} = \mathrm{Ann(M)} = 0$, that is, R is a reduced ring. Suppose that R is decomposable, say $R = R_0 \times R_1$ for nontrivial rings R_1 and R_2, then according to (5.2), either $R_i M = 0$ for some $i = 0, 1$, which is impossible because M is faithful, or $M \cong \frac{R_1}{\mathfrak{M}_1} \oplus \frac{R_2}{\mathfrak{M}2} \cong \frac{R}{\mathfrak{M}_1 \times \mathfrak{M}_2}$ for some maximal ideals $\mathfrak{M}_i$'s of R_i's. But this implies that M is cyclic, against our assumption and so R is indecomposable. Therefore, we assume that M is a faithful and R is reduced and indecomposable.

(i) The result in this case follows by (2.2).

(ii) Assume that R is not a domain. We will show that M has a maximal submodule, then the result follows by case (i). On the contrary, suppose that M has no maximal submodule. Let $\mathfrak{P}$ be a minimal prime ideal of R and $N = (0 :_M \mathfrak{P})$. Since M is faithful and $\mathfrak{P} \neq 0$, we have $N \neq M$, therefore there is a proper submodule $N' = (0 :_M \mathfrak{P}')$ of M properly containing N.

Now according to (2.1)(i), $(0 :_R \mathfrak{P}) = (N : M) \subseteq (N' : M) = (0 :_R \mathfrak{P}')$. Clearly $\mathfrak{P}' \neq \mathfrak{P}$, thus there is an $x \in \mathfrak{P}' \setminus \mathfrak{P}$, for $\mathfrak{P}$ is minimal. Let $r \in (0 :_R \mathfrak{P}) \subseteq (0 :_R \mathfrak{P}')$, then $rx = 0 \in \mathfrak{P}$ and since $x \notin \mathfrak{P}$, we see that $r \in \mathfrak{P}$. Consequently, $(0 :_R \mathfrak{P}) \subseteq \mathfrak{P}$ and hence,

$(0:_R \mathfrak{P})^2 \subseteq (0:_R \mathfrak{P})\mathfrak{P} = 0$. But R is reduced, whence $(0:_R \mathfrak{P}) = 0$, for every minimal prime ideal $\mathfrak{P}$ of R.

Let $K = (0:_M \mathfrak{P}_1)$ be an arbitrary nontrivial submodule of M. There is a minimal prime ideal $\mathfrak{P}$ of R, contained in $\mathfrak{P}_1$, hence $(K : M) = (0:_R \mathfrak{P}_1) \subseteq (0:_R \mathfrak{P}) = 0$. So for each $0 \neq r \in R$, we have $rM = M$ (else $0 \neq r \in (rM : M)$). Now R is not an integral domain, say $r_1r_2 = 0$ for some $0 \neq r_1, r_2 \in R$. Thus $M = r_1M = r_1(r_2M) = 0$ is cyclic, yielding a contradiction which completes the proof of case (ii).

(iii) Note that if $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$ are prime ideals of R, then $(0:_M \mathfrak{P}_2) \subseteq (0:_M \mathfrak{P}_1)$. Therefore, if $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are the height one primes of R, then by the assumption of this case, every submodule of M is contained in some $N_i = (0:_M \mathfrak{P}_i)$. Consequently, at least one of the N_i's is a maximal submodule of M, hence the result follows by case (i).

(iv) Assume that R is a one dimensional Noetherian domain and M is not cyclic. Then by (2.2), M has no maximal submodule (in particular, R is not a field). Since R is Noetherian, (5.3) shows that $M_{\mathfrak{M}}$ is a p-co-m $R_{\mathfrak{M}}$-module, for every maximal ideal $\mathfrak{M}$ of R. Also $M_{\mathfrak{M}}$ has no maximal submodule (else one can readily show that its contraction in M is a maximal submodule of M). But by case (iii) $M_{\mathfrak{M}}$ is cyclic and hence has a maximal submodule, a contradiction. □

Every finitely generated nonzero module and every multiplication module has a maximal submodule. Thus:

Corollary 5.6. *A multiplication or finitely generated R-module M is p-co-m if and only if it is cyclic and $\frac{R}{\mathrm{Ann(M)}}$ is p-co-m.*

Question 5.7. Is there any p-co-m module, which is not cyclic?

(5.5) shows that an important part of characterizing faithful p-co-m modules, is to characterize rings which are p-co-m modules over themselves. In what follows we make use of the well-known fact that every proper ideal of a ring R is prime if and only if R is a field.

Theorem 5.8. *A ring R is a p-co-m module over itself if and only if either R is a field or $R = F_1 \times F_2$, where F_i's are fields or R is an SPIR with a unique prime ideal Rp and $p^2 = 0$.*

Proof. ($\Leftarrow$): Easy.

($\Rightarrow$): First note that since R is a co-m ring, it satisfies the double annihilator condition. Noting that every nontrivial ideal is the annihilator of a nontrivial prime ideal, it is easy to see that every nontrivial ideal of R is prime. If $R = R_1 \times R_2$, where R_i's are nontrivial rings, then by (5.2), each R_i is a simple R_i-module and hence is a field. Thus assume that R is indecomposable. Let $\mathfrak{P}$ be a minimal prime ideal of

R. Then $\mathfrak{P}$ is a simple R-module, because every nonzero ideal of R contained in $\mathfrak{P}$ is prime and equals $\mathfrak{P}$ by minimality of $\mathfrak{P}$. If $\mathfrak{P} = 0$, then every proper ideal of R is prime and hence R is a field.

Suppose $\mathfrak{P} \neq 0$. Then every proper ideal of $\frac{R}{\mathfrak{P}}$ is prime and thus $\frac{R}{\mathfrak{P}}$ is a field, that is, $\mathfrak{P}$ is maximal. Suppose that R has another minimal prime ideal $\mathfrak{Q} \neq \mathfrak{P}$. On the same lines, it follows that $\mathfrak{Q}$ is maximal and a simple R-module. Then $\mathfrak{P} \cap \mathfrak{Q} = 0$ (else, both being simple R-modules, one must contain the other, which is against both being minimal). So $R \cong \frac{R}{\mathfrak{P}} \times \frac{R}{\mathfrak{Q}}$ is decomposable, a contradiction. Thus R has a unique minimal prime ideal which is maximal and simple, whence principal, which means that R is an SPIR. Let $\mathfrak{P} = Rp$, then its submodule Rp^2 equals either zero or Rp. But by Nakayama's lemma, since $\mathfrak{P} \neq 0$, $Rp^2 \neq Rp$, whence $Rp^2 = 0$, as asserted. □

Two immediate corollaries to this theorem and its proof are:

Corollary 5.9. *A ring R is a p-co-m module over itself if and only if every nontrivial ideal of R is prime.*

Corollary 5.10. *A cyclic R-module M is p-co-m if and only if either* Ann(M) *is a maximal ideal or an intersection of two maximal ideals or* $\mathrm{Ann(M)} = \mathfrak{M}^2$ *for some maximal ideal $\mathfrak{M}$ of R with* $dim_{\frac{R}{\mathfrak{M}}} \frac{\mathfrak{M}}{\mathfrak{M}^2} = 1$.

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