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DIFFERENTIAL MULTIPLICATIVE HYPERRINGS

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Abstract. In a multiplicative hyperring, the multiplication is a hyperoperation, while the addition is a binary operation. In this paper, the notion of derivation on multiplicative hyperrings is introduced and some related properties are investigated.

1. Introduction

2. Derivation on multiplicative hyperrings

Let H be a non-empty set, $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a map $\star : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \star)$ is called a hypergrupoid (or hyperstructure). If A and B are non-empty subsets of H, then we denote $A \star B = \bigcup_{a\in A, b\in B} a \star b$, and if $x \in H$, then we denote $A \star x = A \star \{x\}$ and $x \star B = \{x\} \star B$. A hypergrupoid $(H, \star)$ is called a semihypergroup if for all x, y, z of H we have $(x \star y) \star z = x \star (y \star z)$. That is, $\bigcup_{u\in x\star y} u \star z = \bigcup_{v\in y\star z} x \star v$. A hypergrupoid $(H, \star)$ is called a quasihypergroup if for all $x \in H$, we have $x \star H = H \star x = H$. A hypergrupoid is called a hypergroup if it is both a semihypergroup and a quasihypergroup. A polygroup is a system $(P, \cdot, e, ^{-1})$, where $e \in P$, "$^{-1}$" is a unitary operation on P, "$\cdot$" maps $P \times P$ in to the nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$: (1) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; (2) $e \cdot x = x \cdot e = x$; (3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$. In every polygroup, we have $e \in x \cdot x^{-1} \cap x^{-1} \cap x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$, where $A^{-1} = \{a^{-1} | a \in A\}$. We can consider

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several definitions for a hyperring, by replacing at least one of the two operations by hyperoperations, for example see [2, 6, 8, 9, 12, 13]. The notion of multiplicative hyperring was introduced by R. Rota [19] in 1982. The multiplication is a hyperoperation, while the addition is a binary operation, that is why she called it a multiplicative hyperring. At the first, we recall the definition of a multiplicative hyperring. For more details and properties, we refer the readers to [5, 7, 11, 16, 17, 18]. A triple $(R, +, \cdot)$ is called a *multiplicative hyperring* if (1) $(R, +)$ is an abelian group; (2) $(R, \cdot)$ is a semihypergroup; (3) $x \cdot (y+z) \subseteq x \cdot y + x \cdot z$ and $(y+z) \cdot x \subseteq y \cdot x + z \cdot x$, for all $x, y, z \in R$; (4) $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$, for all $x, y, z \in R$. If in (3) we have equalities instead of inclusions, then we say that the multiplicative hyperring is *strongly distributive*. An element $e \in R$ is called a *weak identity* (*identity*, respectively) if $x \in e \cdot x \cap x \cdot e$ ($e \cdot x = x \cdot e = x$, respectively), for all $x \in R$. Throughout this paper, by a hyperring we mean a multiplicative hyperring. A nonempty subset H of a hyperring $(R, +, \cdot)$ is called *subhyperring* of R, if $(H, +, \cdot)$ is itself a hyperring. In other words, H is a subhyperring of $(R, +, \cdot)$ if $H - H \subseteq H$ and $x.y \subseteq H$, for all $x, y \in H$. A hyperring R is called an *integral hyperdomain*, if for all $x, y \in R$, $0 \in x \cdot y$ implies that $x = 0$ or $y = 0$. In this paper, the meaning of a *hyperfield* is a hyperring $(F, +, \cdot)$ such that $(F - \{0\}, \cdot)$ is a polygroup and " $\cdot$ " is strongly distributive with respect to " $+$ ". Hyperring $(R, +, \cdot)$ is called *commutative* (*weak commutative*, respectively), when $x \cdot y = y \cdot x$ ($x \cdot y \cap y \cdot x \neq \emptyset$, respectively), for all $x, y \in R$. The meaning of *center* of R is $Z(R) = \{x \in R | x \cdot y = y \cdot x, \text{for all } y \in R\}$. A nonempty subset I of a hyperring R is a *hyperideal* if $I - I \subseteq I$ and $x \cdot r \cup r \cdot x \subseteq I$, for all $x \in I$ and $r \in R$.

Example 2.1. Let $(R, +, \cdot)$ be a ring, I be an ideal of R and $\circ$ be the hyperoperation defined on R by $x \circ y = x \cdot y + I$, for all $x, y \in R$. Then, $(R, +, \circ)$ is a strongly distributive hyperring. For convenience, the multiplicative hyperring $(R, +, \circ)$ will be denoted by $(R, +, I)$. The ideal I is a hyperideal of hyperring $(R, +, I)$, since I is an additive subgroup of $(R, +)$ and for all $x \in I$ and $r \in R$, $x \circ r \cup r \circ x = (x \cdot r + I) \cup (r \cdot x + I) \subseteq I$.

A *homomorphism* (*good homomorphism*, respectively) between two hyperrings
$(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ is a map $f : R_1 \longrightarrow R_2$ such that for all $x, y \in R_1$, we have $f(x +_1 y) = f(x) +_2 f(y)$ and $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$ ($f(x \circ_1 y) = f(x) \circ_2 f(y)$, respectively). Let $f : R_1 \longrightarrow R_2$ be a good homomorphism. The kernel of f is the inverse image of $< 0 >$ (the hyperideal generated by the zero in R_2). It is denoted by $ker f$.

The concept of derivation on rings has been introduced by Posner [15], also see [3, 20]. In [1], Asokkumar introduced the notion of derivation on Krasner hyperrings. Now, we define the notion of derivation on multiplicative hyperrings.

Definition 2.2. Let $(R, +, \cdot)$ be a hyperring. The function $d : R \longrightarrow R$ is called *derivation* if for all $x, y \in R$,

(1) $d(x + y) = d(x) + d(y)$;
(2) $d(x \cdot y) = d(x) \cdot y + x \cdot d(y)$.

The function $d : R \longrightarrow R$ is called *weak derivation* if for all $x, y \in R$, it satisfies (1) and

(3) $d(x \cdot y) \subseteq d(x) \cdot y + x \cdot d(y)$.

It is clear that every derivation is a weak derivation. By the first condition of above definition for every (weak) derivation d of hyperring R, we have $d(0) = 0$ and $d(-x) = -d(x)$, for all $x \in R$.

We consider some examples.

Example 2.3. Let $(R, +, \cdot)$ be a hyperring and $0 \in r.0 \cap 0.r$, for all $r \in R$. Then, the function $d(x) = 0$, for all $x \in R$, is a weak derivation. It is called *trivial weak derivation*.

Example 2.4. Consider the ring $(\mathbb{Z}_m, +, \cdot)$. Let $p \in \mathbb{Z}_m$ and $p \neq 1$. We define hyperoperation $\circ$ on R by $x \circ y = \{x \cdot y,\ p \cdot x \cdot y\}$, for all $x, y \in \mathbb{Z}_m$. Then, $(\mathbb{Z}_m, +, \circ)$ is a hyperring. The function $d : \mathbb{Z}_m \longrightarrow Z_m$ defined by $d(x) = 0$, for all $x \in \mathbb{Z}_m$ is derivation, since $d(x) \circ y + x \circ d(y) = 0 \circ y + x \circ 0 = \{0\} = d(x \circ y)$, for all $x, y \in \mathbb{Z}_m$.

Example 2.5. Let $(R, +)$ be an abelian group and $\circ$ be the hyperoperation on R defined by $x \circ y = \langle x, y \rangle = \mathbb{Z}x + \mathbb{Z}y$, (the subgroup of $(R, +)$ generated by x and y), for all $x, y \in R$. Then, $(R, +, \circ)$ is a hyperring which is not generally strongly distributive. The functions $d_1, d_2 : R \longrightarrow R$ defined by $d_1(x) = x$ and $d_2(x) = -x$, for all $x \in R$, are derivations.

Example 2.6. Let R be an abelian group and S be a subgroup of R. For all $x, y \in R$, we define $x \circ y = S$. Then, $(R, +, \circ)$ is a hyperring. The functions $d_1, d_2 : R \longrightarrow R$ defined by $d_1(x) = x$ and $d_2(x) = -x$, for all $x \in R$, are derivations.

Example 2.7. Let $(R, +, \cdot)$ be a ring, P be a nonempty subset of R and $\circ$ be the hyperoperation defined on R by $x \circ y = x.P.y$, for all $x, y \in R$. Then, $(R, +, \circ)$ is a hyperring. For convenience, the hyperring $(R, +, \circ)$ will be denoted by $[R, +, P]$. Set $M = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | x, y \in [R, +, P] \right\}$ and define the hyperoperation $*$ on M as

$$\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a \in x_1 \circ x_2,\ b \in x_1 \circ y_2 \right\},$$

where $x_1, x_2, y_1, y_2 \in [R, +, P]$. Then, M with the usual addition of matrices and the hyperoperation $*$ is a hyperring. M may not be strongly distributive because $[R, +, P]$ may not be strongly distributive. It is easily to check that the function $d : M \longrightarrow M$ defined by $d\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ is a derivation.

A hyperring R is said to be of *characteristic* n, if n is the smallest positive integer such that $nx = 0$, for all $x \in R$. If no such of n exists, R is said to be of characteristic 0.

Lemma 2.8. *Let $(R, +, \cdot)$ be a hyperring and d be a weak derivation. Then, for all $n \in \mathbb{N}$ and $x, y \in R$,*

(1) *If R is commutative, then $d(x^n) \subseteq nx^{n-1}.d(x)$. The equality holds when R is strongly distributive and d is a derivation.*
(2) *$d^{(n)}(x.y) \subseteq \sum_{i=0}^{n} \binom{n}{i} d^{(n-i)}(x).d^{(i)}(y)$, where $d^{(n)}$ shows derivation of order n. The equality holds when d is a derivation.*

Proof. The proof follows easily by induction. □

Let a commutative hyperring R be strongly distributive and d be a derivation of R. If R is of characteristic n, then by the above Lemma, $0 \in d(x^n)$, for all $x \in R$.

Theorem 2.9. *Let $(R, +, \cdot)$ be a hyperring and the notation $[x, y]$ denotes the set $x \cdot y - y \cdot x$, for all $x, y \in R$. Then, for all $x, y, z \in R$,*

(1) *$[x + y, z] \subseteq [x, z] + [y, z]$, the equality holds when R is strongly distributive;*
(2) *If R is a strongly distributive, we have $[x \cdot y, z] \subseteq x \cdot [y, z] + [x, z] \cdot y$;*
(3) *If d is a weak derivation of R, then $d[x, y] \subseteq [d(x), y] + [x, d(y)]$; we have equality when d is a derivation.*

Proof. The proof is obvious. □

Definition 2.10. A hyperring R is called *prime* if $0 \in x \cdot r \cdot y$, for all $r \in R$, implies that either $x = 0$ or $y = 0$. R is called *semiprime* if $0 \in x \cdot r \cdot x$, for all $r \in R$, implies that $x = 0$. Obviously, every prime hyperring is a semiprime hyperring but the converse is not always true.

Example 2.11. Let $R = \{e, a, b\}$. Consider the following tables:

$+$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

and

$.$	e	a	b
e	e	e	e
a	e	$\{a,b\}$	$\{a,b\}$
b	e	$\{a,b\}$	$\{a,b\}$

It is easily to check $(R,+,\cdot)$ is prime.

Lemma 2.12. *Let I be a nonzero hyperideal on a prime hyperring R and $x,y \in R$, then*

1. *If $I \cdot x = 0$ or $x.I = 0$, then $x = 0$;*
2. *If $0 = x \cdot I \cdot y$, then $x = 0$ or $y = 0$;*
3. *If $0 \in r \cdot 0 \cap 0 \cdot r$, for all $r \in R$, $x \in Z$ and $0 \in x \cdot y$, then $x = 0$ or $y = 0$;*
4. *If R is strongly distributive, $x \in Z(R)$ and $x \cdot y \subseteq Z$, for all $y \in Z$, then $x = 0$ or R is weak commutative.*

Proof. (1) Suppose that $I \cdot x = 0$. Then, $u \cdot r \cdot x \subseteq I \cdot x = \{0\}$, for all $r \in R$ and $u \in I$. So, $x = 0$, since R is prime and $I \neq 0$. In the case $x \cdot I = 0$, the proof is similar.

(2) Suppose that $x \cdot I \cdot y = 0$. Then, $x \cdot I \cdot r \cdot y \subseteq x \cdot I \cdot y = \{0\}$, for all $r \in R$. Therefore, $x \cdot I \cdot r \cdot y = 0$, for all $r \in R$. Hence, $x \cdot I = 0$ or $y = 0$, since R is prime. So, by (1), $x = 0$ or $y = 0$.

(3) Suppose that $x \in Z$ and $0 \in x \cdot y$. Then, for all $r \in R$, $0 \in r \cdot 0 = r \cdot x \cdot y = x \cdot r \cdot y$. Therefore, $x = 0$ or $y = 0$, since R is prime.

(4) Suppose that $x \cdot y \subseteq Z$, for all $y \in R$. Then, $0 \in x \cdot y \cdot r - x \cdot y \cdot r = x \cdot y \cdot r - r \cdot x \cdot y = x \cdot y \cdot r - x \cdot r \cdot y = x \cdot (y \cdot r - r \cdot y) = x \cdot [y,r]$, for all $r \in R$. So, $0 \in t \cdot 0 \in t \cdot x \cdot [y,r] = x \cdot t \cdot [y,r]$, for all $t \in R$. Hence, $x = 0$ or $0 \in [y,r]$, since R is prime. This means that $x = 0$ or $y \cdot r \cap r \cdot y \neq \emptyset$, for all $r \in R$. □

Lemma 2.13. *Let d be a derivation on a prime hyperring $(R,+,\cdot)$ and I be a nonzero hyperideal of R. Also, let $0 \in 0 \cdot r \cap r \cdot 0$, for all $r \in R$, then for all $x \in R$,*

1. *If $d(I) = 0$, then $d = 0$;*
2. *If $d(I) \cdot x = 0$ or $x \cdot d(I) = 0$, then $x = 0$ or $d = 0$;*
3. *If $d(R) \cdot x = 0$ or $x \cdot d(R) = 0$, then $x = 0$ or $d = 0$.*

Proof. (1) For all $u \in I$ and $x \in R$, we have $0 = d(u \cdot x) = d(u) \cdot x + u \cdot d(x) \supseteq 0 + u \cdot d(x) = u \cdot d(x)$. Therefore, $u \cdot d(x) = 0$, for all $u \in I$. So, $I \cdot d(x) = 0$, which implies that $d = 0$, by Lemma 2.12 (1).

(2) Suppose that $d(I) \cdot x = 0$. Then, $0 = d(y \cdot u) \cdot x = d(y) \cdot u \cdot x + y \cdot d(u) \cdot x \supseteq d(y) \cdot u \cdot x$, for all $u \in I$ and $y \in R$. So, $d(y) \cdot u \cdot x = 0$, for all $u \in I$. Therefore, $d(y) \cdot I \cdot x = 0$, which implies that $d = 0$ or $x = 0$, by Lemma 2.12 (2). In the case $x \cdot d(I) = 0$, the proof is similar.

(3) In (2), put R instead of I. □

Definition 2.14. Let R be a hyperring and d be a derivation on R. Then, $x \in R$ is called a *constant element* if $d(x) = 0$. We denote by $C_d(R)$, the set of all of constant elements of R associated to derivation d.

Theorem 2.15. *Let d be a derivation on a prime strongly distributive hyperring R such that $d(R) \subseteq Z$. Also, let there is $c \in C_d(R)$ such that $0 \notin [c, x_0]$, for some $x_0 \in R$. Then, $d = 0$.*

Proof. We have $d(x \cdot c) = d(x) \cdot c + x \cdot d(c) \supseteq d(x) \cdot c$, for all $x \in R$. So, $d(x) \cdot c \subseteq d(x \cdot c) \subseteq Z$. Therefore, $d(x) \cdot c \cdot x_0 = x_0 \cdot d(x) \cdot c = d(x) \cdot x_0 \cdot c$. This means that $0 \in d(x) \cdot [c, x_0]$, since R is strongly distributive. Then, there is $t \in [c, x_0]$ such that $0 \in d(x) \cdot t$. So, $d(x) = 0$ or $t = 0$, by Lemma 2.12 (3). If $t = 0$, then $0 \in [c, x_0]$, this is a contradiction. Therefore, $d(x) = 0$, for all $x \in R$. □

Definition 2.16. A hyperring R is called *n-torsion free* if $nx = 0$, $x \in R$, implies that $x = 0$, where n is an integer number.

Theorem 2.17. *Let I be a nonzero hyperideal of a 2-torsion free prime hyperring $(R, +, \cdot)$ and $0 \in r \cdot 0 \cap 0 \cdot r$, for all $r \in R$.*

(1) *If d is a derivation of R such that $d^{(2)}(I) = 0$, then $d = 0$;*
(2) *If d_1 and d_2 are derivations of R such that $d_1 d_2(I) = 0$, then $d_1 = 0$ or $d_2 = 0$.*

Proof. (1) By Lemma 2.8, we have for all $u, v \in I$,

$$0 = d^{(2)}(u \cdot v) = d^{(2)}(u) \cdot v + 2d(u) \cdot d(v) + u \cdot d^{(2)}(v) \supseteq 2d(u) \cdot d(v).$$

So, $d(u) \cdot d(v) = 0$, since R is a $2-$torsion free hyperring. Therefore, $d = 0$, by Lemma 2.13 (1) and (2).

(2) We have for all $u, v \in I$,

$$\begin{aligned} 0 = d_1 d_2(u \cdot v) &= d_1(d_2(u) \cdot v + u \cdot d_2(v)) \\ &= d_1 d_2(u) \cdot v + d_2(u) \cdot d_1(v) + d_1(u) \cdot d_2(v) + u \cdot d_1 d_2(v) \\ &\supseteq d_2(u) \cdot d_1(v) + d_1(u) \cdot d_2(v). \end{aligned}$$

So, $d_2(u) \cdot d_1(v) + d_1(u) \cdot d_2(v) = 0$. By replaceing u by $d_2(u)$ in the above equation, we get $d_2^{(2)}(u) \cdot d_1(v) \subseteq d_2^{(2)}(u) \cdot d_1(v) + d_1 d_2(u) \cdot d_2(v) = 0$, that is $d_2^{(2)}(u) \cdot d_1(v) = 0$. Thus, $d_1 = 0$ or $d_2^{(2)}(I) = 0$, by Lemma 2.13 (1) and (2). Therefore, $d_1 = 0$ or $d_2 = 0$, by (1). □

3. Differential multiplicative hyperring

We denote by $\Delta(R, +, \cdot)$ ($D(R, +, \cdot)$, respectively), the set of all derivations (weak derivations, respectively) of hyperring $(R, +, \cdot)$. Note that

$$\Delta(R, +, \cdot) \subseteq D(R, +, \cdot) \subseteq Hom(R, +).$$

A hyperring with Δ (D, respectively) is called the *differential hyperring* (*weak differential hyperring*, respectively).

A hyperfield R is called *(weak) differential hyperfield* if R is (weak) differential hyperring. An integral hyperdomain R is called *(weak) differential integral hyperdomain* if R is (weak) differential hyperring. A subhyperring H of (weak) differential hyperring R is said *(weak) differential subhyperring* if for all (weak) derivation d of R, we have $d(h) \in H$, for all $h \in H$. A hyperideal I of (weak) differential hyperring R is called *(weak) differential hyperideal* if for all (weak) derivation d of R, we have $d(u) \in I$, for all $u \in I$.

Example 3.1. For every (weak) differential hyperring R, $\langle 0 \rangle_R$ is a (weak) differential hyperideal.

We usually use the perfix Δ (D, respectively) instead of we say that R is differential (weak differential, respectively) and Δ (D, respectively) is the set of all derivations (weak derivations, respectively) on R. Also, If R is a differential hyperring (weak differential hyperring, respectively) i.e. R is a Δ-hyperring (D-hyperring, respectively), then we usually use the notion Δ-hyperideal (D-hyperideal, respectively) instead of we say that I is a differential hyperideal (weak differential hyperideal, respectively) of R.

Example 3.2. Let $(R, +, \cdot)$ be a hyperring and $0 \in r.0 \cap 0.r$, for all $r \in R$. Then, by Example 2.3, the function $d : R \longrightarrow R$ defined as $d = 0$ is a weak derivation. So, $d \in D(R, +, \cdot)$ and this means that $D(R, +, \cdot) \neq \emptyset$.

Example 3.3. Let $(R, +, \circ)$ be the hyperring defined in Example 2.5. For all $f \in Hom(R, +)$, we have $f(x \circ y) = \mathbb{Z}f(x) + \mathbb{Z}f(y) \subseteq \mathbb{Z}f(x) + \mathbb{Z}y + \mathbb{Z}x + \mathbb{Z}f(y) = f(x) \circ y + x \circ f(y)$, for all $x, y \in R$. This implies that $f \in D(R, +, \circ)$ and so $Hom(R, +) \subseteq D(R, +, \circ)$. Also, we know that $D(R, +, \circ) \subseteq Hom(R, +)$. Therefore, $D(R, +, \circ) = Hom(R, +)$.

Example 3.4. In Example 2.1, if $I = R$, then every additive function $f : R \longrightarrow R$ is a weak derivation. For all $x, y \in R$, we have $d(x \circ y) = d(x \cdot y + R) = d(R) \subseteq R = d(x) \cdot y + R + x \cdot d(y) + R = d(x) \circ y + x \circ d(y)$. So, $D(R, +, R) = Hom(R, +)$. Also, in Example 2.1, if $(R, +, \cdot)$ and I are Δ-hyperring and Δ-hyperideal, then we have $\Delta(R, +, \cdot) \subseteq D(R, +, I)$. Because, for all $d \in \Delta(R, +, \cdot)$ and $x, y \in R$, we have $d(x \circ y) = d(x \cdot y + I) \subseteq d(x \cdot y) + I = d(x) \cdot y + x \cdot d(y) + I = d(x) \circ y + x \circ d(y)$.

Now, we analyze hyperring $(\mathbb{Z}, +, m\mathbb{Z})$, where m is a positive integer. We have $D(\mathbb{Z}, +, m\mathbb{Z}) \subseteq Hom(\mathbb{Z}, +) = \{g_a | a \in \mathbb{Z}\}$, where $g_a(x) = ax$, for all $x \in \mathbb{Z}$.

Theorem 3.5. *The following statements are valid:*

1. *For all $a \in \mathbb{Z}$, $g_a \in D(\mathbb{Z}, +, m\mathbb{Z})$ if and only if $m|a$.*
2. *$\{g_a | a \in m\mathbb{Z}\} = D(\mathbb{Z}, +, m\mathbb{Z})$, so $D(\mathbb{Z}, +, m\mathbb{Z})$ is infinite and only in the case $m = 1$, we have $D(\mathbb{Z}, +, m\mathbb{Z}) = Hom(\mathbb{Z}, +)$.*
3. *If $m > 1$, then $\{g_a | a \in m\mathbb{Z} + 1\} \subseteq Hom(\mathbb{Z}, +) \setminus D(\mathbb{Z}, +, m\mathbb{Z})$ and so $Hom(\mathbb{Z}, +) \setminus D(\mathbb{Z}, +, m\mathbb{Z})$ is infinite.*
4. *$m\mathbb{Z}$ is a D-hyperideal of $(\mathbb{Z}, +, m\mathbb{Z})$.*

Proof. (1) Suppose that $g_a \in D(\mathbb{Z}, +, m\mathbb{Z})$. Then, $a + am\mathbb{Z} = g_a(1 \circ 1) \subseteq g_a(1) \circ 1 + 1 \circ g_a(1) = a \circ 1 + 1 \circ a = 2a + m\mathbb{Z}$. So, $a \in m\mathbb{Z}$.

Conversely, suppose that $m|a$. Then, for all $x, y \in \mathbb{Z}$, we have $-axy \in m\mathbb{Z}$. Thus, $-axy + am\mathbb{Z} \subseteq am\mathbb{Z} + m\mathbb{Z} = m\mathbb{Z}$. Therefore, $g_a(x \circ y) = axy + am\mathbb{Z} \subseteq 2axy + m\mathbb{Z} = g_a(x) \circ y + x \circ g_a(y)$. Hence, $g_a \in D(\mathbb{Z}, +, m\mathbb{Z})$.

The rest parts follow by part (1). □

Consider the hyperring $(\mathbb{Z}_n, +, m\mathbb{Z}_n)$, where m and n are positive integers. We have $D(\mathbb{Z}_n, +, m\mathbb{Z}_n) \subseteq Hom(\mathbb{Z}_n, +) = \{h_{\bar{a}} | a \in \mathbb{Z}\}$, where $h_{\bar{a}}(\bar{x}) = \overline{ax}$, for all $\bar{x} \in \mathbb{Z}_n$.

Theorem 3.6. *The following statements are valid:*

1. *For all $a \in \mathbb{Z}$, $h_{\bar{a}} \in D(\mathbb{Z}_n, +, m\mathbb{Z}_n)$ if and only if $(m, n)|a$.*
2. *$\{h_{\bar{a}} | a \in (m, n)\mathbb{Z}\} = D(\mathbb{Z}_n, +, m\mathbb{Z}_n)$ and thus $|D(\mathbb{Z}_n, +, m\mathbb{Z}_n)| = \frac{n}{(m,n)}$. Also, only for $m = 1$, we have $D(\mathbb{Z}_n, +, m\mathbb{Z}_n) = Hom(\mathbb{Z}_n, +)$.*
3. *If $(m, n) > 1$, then $\{h_{\bar{a}} | a \in (m, n)\mathbb{Z} + 1\} \subseteq Hom(\mathbb{Z}_n, +) \setminus D(\mathbb{Z}_n, +, m\mathbb{Z}_n)$ and so $|Hom(\mathbb{Z}_n, +) \setminus D(\mathbb{Z}_n, +, m\mathbb{Z}_n)| \geq \frac{n}{(m,n)}$.*
4. *$m\mathbb{Z}_n$ is a D-hyperideal of $(\mathbb{Z}_n, +, m\mathbb{Z}_n)$.*

Proof. (1) Suppose that $h_{\bar{a}} \in D(\mathbb{Z}_n, +, m\mathbb{Z}_n)$, then $\bar{a} + \bar{a}m\mathbb{Z}_n = h_{\bar{a}}(\bar{1} \circ \bar{1}) \subseteq h_{\bar{a}}(\bar{1}) \circ \bar{1} + \bar{1} \circ h_{\bar{a}}(\bar{1}) = \bar{a} \circ \bar{1} + \bar{1} \circ \bar{a} = 2\bar{a} + m\mathbb{Z}_n$. Thus, $\bar{a} \in m\mathbb{Z}_n = (m, n)\mathbb{Z}_n$. Thus, $a = (m, n)s + nt$, for some $s, t \in \mathbb{Z}$. Since $(m, n)|(m, n)s + nt$, then $(m, n)|a$.

Conversely, suppose that $(m, n)|a$. Then, $a = (m, n)s$, for some $s \in \mathbb{Z}$. So, for all $x, y \in \mathbb{Z}$, we have $-\overline{axy} = -\overline{(m, n)sxy} \subseteq (m, n)\mathbb{Z}_n = m\mathbb{Z}_n$. Thus, $-\overline{axy} + am\mathbb{Z}_n \subseteq am\mathbb{Z}_n + m\mathbb{Z}_n = m\mathbb{Z}_n$. Therefore, $h_{\bar{a}}(\bar{x} \circ \bar{y}) = \bar{a}\bar{x}\bar{y} + \bar{a}m\mathbb{Z}_n = \overline{axy} + am\mathbb{Z}_n \subseteq 2\overline{axy} + m\mathbb{Z}_n = 2\bar{a}\bar{x}\bar{y} + m\mathbb{Z}_n = h_{\bar{a}}(\bar{x}) \circ \bar{y} + \bar{x} \circ h_{\bar{a}}(\bar{y})$. Hence, $h_{\bar{a}} \in D(\mathbb{Z}_n, +, m\mathbb{Z}_n)$.

The rest parts follow by part (1). □

For example, by the above theorems, we have

$D(\mathbb{Z}, +, 4\mathbb{Z}) = \{g_a \mid a \in 4\mathbb{Z}\}$ and $D(\mathbb{Z}_{20}, +, 4\mathbb{Z}_{20}) = \{h_{\bar{0}}, h_{\bar{4}}, h_{\bar{8}}, h_{\overline{12}}, h_{\overline{16}}\}$.

Now, consider the hyperring $[R,+,P]$ defined in Example 2.7. If we set $P=R$, then every additive function $f:R\longrightarrow R$ is a weak derivation. So, $D[R,+,R]=Hom(R,+)$.

Theorem 3.7. (1) *For all $a\in\mathbb{Z}$ and $\emptyset\neq P\subseteq\mathbb{Z}$, we have $g_a\in D[\mathbb{Z},+,P]$ if and only if $a\cdot P\subseteq 2a\cdot P$.*
(2) *For all $a\in\mathbb{Z}$ and $\emptyset\neq P\subseteq\mathbb{Z}_n$, we have $h_{\bar{a}}\in D[\mathbb{Z},+,P]$ if and only if $a\cdot P\subseteq 2a\cdot P$.*

Proof. (1) Suppose that $g_a\in D[\mathbb{Z},+,P]\subseteq Hom(\mathbb{Z},+)$. Then, $a\cdot P=g_a(P)=g_a(1\cdot P\cdot 1)=g_a(1\circ 1)\subseteq g_a(1)\circ 1+1\circ g_a(1)=a\circ 1+1\circ a=2a\cdot p$.

Conversely, suppose that $a\cdot P\subseteq 2a\cdot P$. Then, for all $x,y\in\mathbb{Z}$, we have $g_a(x\circ y)=g_a(x\cdot P\cdot y)=a\cdot x\cdot P\cdot y\subseteq 2a\cdot x\cdot P\cdot y=a\cdot x\cdot P\cdot y+x\cdot P\cdot a\cdot y=(a\cdot x)\circ y+x\circ(a\cdot y)=g_a(x)\circ y+x\circ g_a(y)$. Therefore, $g_a\in D[\mathbb{Z},+,P]$.

(2) The proof is similar to (1). □

Corollary 3.8. *If $0\in P$, then*

$$D[\mathbb{Z},+,P]=Hom(\mathbb{Z},+)\text{ and }D[\mathbb{Z}_n,+,P]=Hom(\mathbb{Z}_n,+).$$

Proof. By Theorem 3.7, the proof is obvious. □

Therefore, we have $D[\mathbb{Z},+,m\mathbb{Z}]=\Delta[\mathbb{Z},+,m\mathbb{Z}]=Hom(\mathbb{Z},+)$ and $D[\mathbb{Z}_n,+,m\mathbb{Z}_n]=\Delta[\mathbb{Z}_n,+,m\mathbb{Z}_n]=Hom(\mathbb{Z}_n,+)$.

Consider the hyperring $[\mathbb{Q},+,m\mathbb{Z}]$. Notice that $d(x)=xd(1)$, for all $x\in\mathbb{Q}$ and for all (weak) derivation d on $[\mathbb{Q},+,m\mathbb{Z}]$. Similar to Corollary 3.8, we have $D[\mathbb{Q},+,m\mathbb{Z}]=\Delta[\mathbb{Q},+,m\mathbb{Z}]=Hom(\mathbb{Q},+)=\{q_a|a\in\mathbb{Q}\}$, where $q_a(x)=ax$, for all $x\in\mathbb{Q}$.

Definition 3.9. Let R and S be Δ_1 and Δ_2-hyperrings, respectively. By a *differential (good) homomorphism* of R into S, we mean a (good) homomorphism φ such that $d_2\varphi(x)=\varphi d_1(x)$, for all $x\in R$, $d_1\in\Delta_1$ and $d_2\in\Delta_2$.

In the hyperrings

$(\mathbb{Z},+,m\mathbb{Z})$ and $[\mathbb{Z},+,m\mathbb{Z}]$ ($(\mathbb{Z}_n,+,m\mathbb{Z}_n)$ and $[\mathbb{Z}_n,+,m\mathbb{Z}_n]$, respectively)

we have $g_ag_b=g_bg_a$ ($h_{\bar{a}}h_{\bar{b}}=h_{\bar{b}}h_{\bar{a}}$, respectively), for all $a,b\in Z$. So, every homomorphism on them is a differential homomorphism. Therefore, all the results about homomorphisms on these hyperrings are valid about differential homomorphisms on them. For more details, refer to [10] and [14].

Let $(R,+,\circ)$ be Δ-hyperring. A Δ-hyperideal $I(\neq R)$ of a Δ-hyperring R is called *prime* of R, if for all $a,b\in R$, $a\circ b\subseteq I$ implies that $a\in I$ or $b\in I$. The intersection of all Δ-prime hyperideals of R

that contain Δ-hyperideal I is called *radical* I and denote by $Rad(I)$ or $\sqrt{I}$. If the Δ-hyperring R does not have any prime Δ-hyperideal containing I, we define $\sqrt{I} = R$. Δ-hyperideal I is called *differential radical hyperideal* if $\sqrt{I} = I$. Let $\mathcal{C}$ be the class of all finite products of elements of R i.e. $\mathcal{C} = \{r_1 \circ r_2 \circ \cdots \circ r_n | r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A Δ-hyperideal I of R is called Δ-$\mathcal{C}$-*ideal* of R, if for all $A \in \mathcal{C}$, $A \cap I \neq \emptyset$ implies that $A \subseteq I$. By a *maximal* Δ-*hyperideal* of R, we mean a Δ-hyperideal of R that is maximal among the proper Δ-hyperideals of R. Note that a maximal Δ-hyperideal need not to be a maximal hyperideal.

Theorem 3.10. *Let I be a Δ-hyperideal of a commutative Δ-hyperring R. Then, $N(I) \subseteq \sqrt{I}$, where $N(I) = \{r \in R | r^n \subseteq I, n \in \mathbb{N}\}$. The equality holds when I is a Δ-$\mathcal{C}$-ideal of R.*

Proof. The proof is similar to the proof of Proposition 3.2 of [4]. □

Theorem 3.11. *Let R and S be Δ_1 and Δ_2-hyperrings, respectively. Also, let $\varphi : R \longrightarrow S$ be a differential good homomorphism. Then,*

(1) *$ker\varphi$ is a Δ_1-hyperideal;*
(2) *If I is a Δ_2-hyperideal of S, then $\varphi^{-1}(I)$ is a Δ_1-hyperideal of R.*

Proof. According to [5] (p. 145), the inverse images of hyperideals are hyperideals. So, $ker\varphi$ is a hyperideal. For all $d_1 \in \Delta_1$, $d_2 \in \Delta_2$ and $x \in ker\varphi$, we have $\varphi d_1(x) = d_2\varphi(x) = d_2(0) = 0$. So, $d_1(x) \in ker\varphi$.

The proof of the part (2) is similar. □

Theorem 3.12. *Let $(R, +, \cdot)$ be a Δ-hyperring.*

(1) *If I and J are Δ-hyperideals of R, then $I \cdot J$ is also a Δ-hyperideal of R;*
(2) *If R is a Δ-hyperfield and I is a Δ-$\mathcal{C}$-hyperideal of R, then $\sqrt{I}$ is also a Δ-hyperideal;*
(3) *If R is a commutative strongly distributive Δ-hyperring and I is a Δ-hyperideal such that for all $\emptyset \neq A \subseteq R$, $nA \subseteq I$ implies $A \subseteq I$, where $n \in N$, then $\sqrt{I}$ is also a Δ-hyperideal;*
(4) *If R is commutative and I is a Δ-radical hyperideal, then $(I : r) = \{x \in R | x \cdot r \subseteq I\}$, for all $r \in R$, is also a Δ-radical hyperideal.*

Proof. (1) It is proved that $I \cdot J$ is a hyperideal [4]. If $x \in I \cdot J$, then $x \in \sum_{i=1}^{n} a_i \cdot b_i$, for some $a_i \in I$, $b_i \in J$ and $n \in \mathbb{N}$. So, for all $d \in \Delta$, we have $d(x) \in d(\sum_{i=1}^{n} a_i \cdot b_i) = \sum_{i=1}^{n} d(a_i \cdot b_i) = \sum_{i=1}^{n} d(a_i) \cdot b_i + a_i \cdot d(b_i) \subseteq I \cdot J$.

(2) It is clear that $\sqrt{I}$ is a hyperideal. Suppose that $x \in \sqrt{I}$, then $x^n \subseteq I$, for some $n \in \mathbb{N}$. So, $x^n \cdot d(x) \subseteq I$. Thus, $d(x) \in x^{-n} \cdot x^n \cdot d(x) \subseteq x^{-n} \cdot I \subseteq I \subseteq \sqrt{I}$. Therefore, $\sqrt{I}$ is a Δ-hyperideal.

(3) It is clear that $\sqrt{I}$ is a hyperideal. Let $x \in \sqrt{I}$, then $x^n \subseteq I$, for some $n \in \mathbb{N}$. Now, by induction we prove that for all $d \in \Delta$ and $k = 0, 1, \cdots, n$, $x^{n-k} \cdot d(x)^{2k} \subseteq I$. Let the statement is valid for k, i.e., $x^{n-k} \cdot d(x)^{2k} \subseteq I$. By Lemma 2.8, we get $(n-k)x^{n-k-1} \cdot d(x)^{2k+1} + 2kx^{n-k} \cdot d(x)^{2k-1} \cdot d^{(2)}(x) \subseteq I$. Multiply by $d(x)$ and use the hypothesis of induction, we have $(n-k)x^{n-k-1} \cdot d(x)^{2k+2} \subseteq I$. By hypothesis, we get $x^{n-(k+1)} \cdot d(x)^{2(k+1)} \subseteq I$. So, the statement is valid for $k+1$, which completes the induction. Now, set $k = n$, we have $d(x)^{2n} \subseteq I$. So, $d(x) \in \sqrt{I}$.

(4) Let $x, y \in (I : r)$. Then, $(x - y) \cdot r \subseteq x \cdot r - y \cdot r \subseteq I$. So, $x - y \in (I : r)$. Now, suppose that $x \in (I : r)$ and $t \in R$. Then, $x \cdot t \cdot r = x \cdot r \cdot t \subseteq I \cdot t \subseteq I$ and so $x \cdot t \subseteq (I : r)$. It shows that $(I : r)$ is a hyperideal. Let $x \in (I : r)$ and $d \in \Delta$. Then, $d(x) \cdot r \cdot d(x \cdot r) = (d(x) \cdot r)^2 + d(x) \cdot r \cdot x \cdot d(r)$. So, for all $t \in (d(x) \cdot r)^2$ and $s \in d(x) \cdot r \cdot d(x \cdot r) \subseteq I$, there is $z \in d(x) \cdot r \cdot x \cdot d(r) \subseteq I$ such that $s = t + z$. Thus, $t = s - z \subseteq I$. Then, $(d(x) \cdot r)^2 \subseteq I$. Therefore, $d(x) \cdot r \subseteq Rad(I) = I$, which means that $d(x) \in (I : r)$. So, I is a Δ-hyperideal. Obviously, $(I : r) \subseteq Rad((I : r))$. Let $x \in Rad((I : r))$. Then, there is $n \in \mathbb{N}$ such that $x^n \subseteq (I : r)$. Therefore, $x^n \cdot r \subseteq I$. So, we have $(x \cdot r)^n = x^n \cdot r^n = r^{n-1} \cdot (x^n \cdot r) \subseteq r^{n-1} \cdot I \subseteq I$, since R is commutative. Hence, $x \cdot r \subseteq Rad(I) = I$, which means that $x \in (I : r)$. So, $(I : r)$ is a Δ-radical hyperideal. □

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be Δ_1 and Δ_2- homomorphisms, respectively. Then, $(R_1 \times R_2, +, \circ)$ is a hyperring, where for all (a, b), $(c, d) \in R_1 \times R_2$ operation $+$ and hyperoperation $\circ$ are defined as $(a, b) + (c, d) = (a +_1 c, b +_2 d)$ and $(a, b) \circ (c, d) = \{(x, y) | x \in a \circ_1 c, y \in b \circ_2 d\}$. For all $d_1 \in \Delta_1$ and $d_2 \in \Delta_2$, we define the function $d_1 \times d_2 : R_1 \times R_2 \longrightarrow R_1 \times R_2$ as $(d_1 \times d_2)(x, y) = (d_1(x), d_2(y))$, for all $(x, y) \in R_1 \times R_2$. Then, $d_1 \times d_2$ is a derivation on $R_1 \times R_2$. If we set $\Delta = \{d_1 \times d_2 \mid d_1 \in \Delta_1, d_2 \in \Delta_2\}$, then $R_1 \times R_2$ is a Δ-hyperring.

Theorem 3.13. *Let I be a Δ-hyperideal of Δ-hyperring R. Then, R/I has a unique structure of differential hyperring so that the canonical mapping $\varphi : R \longrightarrow R/I$ is a differential homomorphism. So, there is a one to one correspondence between the set of differential hyperideals of R/I and the set of Δ-hyperideals of R which contain I.*

Proof. Suppose that $(R, +, \cdot)$ is a Δ-hyperring. It is proved in [5] that $(R/I, +, *)$ is a hyperring, where the hyperoperation $*$ is defined as

$(a+I)*(b+I) = \{c+I \mid c \in a \cdot b\}$, for all $a, b \in R$. We prove R/I is a differential hyperring. For all $d \in \Delta$, we define $D : R/I \longrightarrow R/I$ as $D(x+I) = d(x)+I$, for all $x \in R$. Let $x+I = y+I$, $x, y \in R$. Then,

$$\begin{aligned} x - y \in I \quad &\Rightarrow d(x) - d(y) \in d(I) \subseteq I \Rightarrow d(x) + I = d(y) + I \\ &\Rightarrow D(x+I) = D(y+I). \end{aligned}$$

So, D is well-defined.

Now, we show that D is a derivation of R/I. It is clear that D is an additive function. Also, for all $x, y \in R$, we have

$$\begin{aligned} D((x+I)*(y+I)) &= D(x \cdot y + I) = d(x \cdot y) + I \\ &= (d(x) \cdot y + I) + (x \cdot d(y) + I) \\ &= (d(x)+I)*(y+I) + (x+I)*(d(y)+I) \\ &= D(x+I)*(y+I) + (x+I)*D(y+I). \end{aligned}$$

Therefore, D is a derivation and R/I is a differential hyperring. The proof of the rest is easy. □

Corollary 3.14. *Let P be a Δ-$\mathcal{C}$-hyperideal of a commutative Δ-hyperring R. Then, P is a prime Δ-$\mathcal{C}$-hyperideal if and only if R/P is a Δ- integral hyperdomain.*

Proof. Suppose that P is a prime Δ-$\mathcal{C}$-hyperideal and $P \subseteq (a+P)*(b+P) = a \cdot b + P$, where $*$ is defined in the proof of Theorem 3.13. Then, for all $x \in P$ there are $z \in a \cdot b$ and $y \in P$ such that $x = z + y$. Thus, $z = x - y \in P$. Since P is a prime $\mathcal{C}$-hyperideal, then $a \in P$ or $b \in P$. Thus, $a + P = P$ or $b + P = P$. Therefore, by Theorem 3.13 R/P is a Δ- integral hyperdomain.

The proof of the converse is clear. □

Theorem 3.15. *(Fundamental differential isomorphism theorem) Let R and S be Δ_1 and Δ_2-hyperring, respectively. If $f : R \longrightarrow S$ is a differential epimorphism, then there exists a differential isomorphism such that $R/kerf \cong S/\langle 0\rangle$.*

Proof. Suppose that $f : R \longrightarrow S$ is a differential epimorphism. Denote $K = kerf$ and define $\varphi : R/K \longrightarrow S/\langle 0\rangle$ by $\varphi(r+K) = f(r) + \langle 0\rangle$, $r \in R$. It is easy to see that φ is a homomorphism. We show that φ is differential. For all $D_1 \in \Delta_{R/K}$ and $D_2 \in \Delta_{S/\langle 0\rangle}$, we have $D_1\varphi(r+K) = D_1(f(r) + \langle 0\rangle) = d_1(f(r)) + \langle 0\rangle = f(d_2(r)) + \langle 0\rangle = \varphi(d_2(r) + \langle 0\rangle) = \varphi D_2(r + \langle 0\rangle)$, where $d_1 \in \Delta_1$ and $d_2 \in \Delta_2$. □

The second and third isomorphism theorems are valid for Δ-hyperrings and Δ-hyperideals.

Let $(R, +, \cdot)$ be a hyperring. Then, set $\Omega = \langle \Delta \rangle$. Every element ω of Ω is as $\omega = d_1^{n_1} d_2^{n_2} \cdots d_m^{n_m}$, $n_1, n_2, \cdots n_m \in \mathbb{N}$. The unit of Ω is $1 = d_1^0 d_2^0 \cdots d_m^0$. We think of ω as an operator. If a is an element of a Δ-hyperring and $\omega = d_1^{n_1} d_2^{n_2} \cdots d_m^{n_m}$, then $\omega(a) = d_1^{n_1} d_2^{n_2} \cdots d_m^{n_m}(a)$. In this case, 1 is the identity operator, i.e., $1(a) = a$. For every $\omega = d_1^{n_1} d_2^{n_2} \cdots d_m^{n_m}$, we define $ord\omega = n_1 + n_2 + \cdots + n_m$.

Let S be a subset of R. Then, $[S]$ denotes the smallest Δ-hyperideal of R that contains S. Thus,

$$\begin{aligned}[S] &= (\{\omega_i(S) \mid \omega_i \in \Omega\}) + \{\sum_{i=1}^{n} x_i \cdot \omega_i(s_i) + \sum_{j=1}^{m} \omega_j(t_j) \cdot y_j \\ &+ \sum_{k=1}^{l} a_k \cdot \omega_k(r_k) \cdot b_k \mid \\ &x_i, y_j, a_k, b_k \in R; s_i, t_j, r_k \in S; n, m, l \in \mathbb{N}; \omega_i, \omega_j, \omega_k \in \Omega\},\end{aligned}$$

where $(\{\omega_i(S) | \omega_i \in \Omega\})$ is the subgroup of the group $(R, +)$, generated by the set $\{\omega_i(S) | \omega_i \in \Omega\}$.

Theorem 3.16. *Let $(R, +, \cdot)$ be a commutative strongly distributive Δ-hyperring, $a, b \in R$ and $\omega \in \Omega = \langle \Delta \rangle$. If $ord\omega = n$, then $a^{n+1} \cdot \omega(b) \subseteq [a \cdot b]$.*

Proof. We prove the statement by induction on n. If $n = 0$, then $\omega = 1$ and the result is obvious. Suppose that the statement is valid for $n = k$ (hypothesis of induction). Now, set $n = k + 1$. Then, there is $d \in \Delta$ such that $\omega = d\delta$, where $\delta \in \Omega$ and $ord\delta = k$. By hypothesis of induction we have $a^{k+1} \cdot \delta(b) \subseteq [a \cdot b]$. So, $a \cdot d(a^{k+1} \cdot \delta(b)) \subseteq [a \cdot b]$. Thus, by Lemma 2.8, $(k+1)a^{k+1} \cdot d(a) \cdot \delta(b) + a^{k+2} \cdot \omega(b) \subseteq [a \cdot b]$. Then, by the hypothesis of induction $(k + 1)a^{k+1} \cdot d(a) \cdot \delta(b) \subseteq [a \cdot b]$. Hence, $a^{k+2} \cdot \omega(b) \subseteq [a \cdot b]$, which completes the proof. □

Lemma 3.17. *Let S and T be subsets of a Δ-hyperring $(R, +, \cdot)$. Then,*

$$\sqrt{[S]} \cdot \sqrt{[T]} \subseteq \sqrt{[S]} \cap \sqrt{[T]} = \sqrt{[S \cdot T]}.$$

Proof. It is clear that $\sqrt{[S]} \cdot \sqrt{[T]} \subseteq \sqrt{[S]}, \sqrt{[T]}$. So, $\sqrt{[S]} \cdot \sqrt{[T]} \subseteq \sqrt{[S]} \cap \sqrt{[T]}$. Suppose that $a \in \sqrt{[S]} \cap \sqrt{[T]}$. Then, $a^s \subseteq [S]$ and $a^t \subseteq [T]$, for some $s, t \in \mathbb{N}$. So, $a^{s+t} \subseteq [S] \cdot [T] \subseteq \sqrt{[S \cdot T]}$. Therefore, $a \in \sqrt{[S \cdot T]}$.

Now, suppose that $a \in \sqrt{[S \cdot T]}$. Then, $a^n \subseteq [S \cdot T] \subseteq [S] \cap [T]$, for some $n \in \mathbb{N}$. Hence, $a \in \sqrt{[S]} \cap \sqrt{[T]}$. □

Definition 3.18. [4] A nonempty subset S of a hyoerring $(R, +, \cdot)$ is said to be a *multiplicative set* if $x, y \in S$ implies that $x \cdot y \cap S \neq \emptyset$.

Theorem 3.19. *Let $(R, +, \cdot)$ be a Δ-hyperring, Ω be a multiplicative set and M be a Δ-hyperideal that is maximal with respect to avoiding Ω. Then, M is prime.*

Proof. At the first, we prove $\sqrt{M} \cap \Omega = \emptyset$. Suppose that there is $t \in \sqrt{M} \cap \Omega$. So, $t^n \subseteq M$ and $t^n \cap \Omega \neq \emptyset$, for some $n \in \mathbb{N}$. Thus, $M \cap \Omega \neq \emptyset$, which is a contradiction. Then, $\sqrt{M} \cap \Omega = \emptyset$. So, by hypothesis $\sqrt{M} = M$.

Now, suppose that $a \cdot b \subseteq M$ and $a, b \notin M$. Then, $M \subsetneqq [a, M]$ and $[b, M]$. So, $M \subsetneqq \sqrt{[a, M]}$ and $\sqrt{[b, M]}$. Therefore, by hypothesis $\sqrt{[a, M]} \cap S \neq \emptyset$ and $\sqrt{[b, M]} \cap S \neq \emptyset$. Thus, there are $k \in \sqrt{[a, M]} \cap S$ and $t \in \sqrt{[b, M]} \cap S$. By Lemma 3.17, $k \cdot t \subseteq \sqrt{[a, M]}\sqrt{[b, M]} \subseteq \sqrt{[a \cdot b, M]} = \sqrt{M} = M$. So, there is $s \in k \cdot t$ such that $s \in M \cap S$, which is a contradiction. Therefore, M is a prime. □

Corollary 3.20. *Every maximal Δ-hyperideal is prime.*

Proof. In Theorem 3.19, set $\Omega = 1$. □

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