A CHARACTERIZATION OF BAER-IDEALS

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ABSTRACT. An ideal $I$ of a ring $R$ is called a right Baer-ideal if there exists an idempotent $e \in R$ such that $r(I) = eR$. We know that $R$ is quasi-Baer if every ideal of $R$ is a right Baer-ideal, $R$ is $n$-generalized right quasi-Baer if for each $I \subseteq R$ the ideal $I^n$ is a right Baer-ideal, and $R$ is right principaly quasi-Baer if every principal right ideal of $R$ is a right Baer-ideal. Therefore the concept of Baer ideal is important. In this paper we investigate some properties of Baer ideals and give a characterization of Baer ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings, semiprime ring and ring of continuous functions. Finally, we find equivalent conditions for which the 2-by-2 generalized triangular matrix ring be right $SA$.

1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity. Let $\emptyset \neq X \subseteq R$. Then $X \subseteq R$ denotes that $X$ is an ideal of $R$. For any subset $S$ of $R$, $l(S)$ and $r(S)$ denote the left annihilator and the right annihilator of $S$ in $R$. The ring of $n$-by-$n$ (upper triangular) matrices over $R$ is denoted by $M_n(R)$ ($T_n(R)$). An idempotent $e$ of a ring $R$ is called left (right) semicentral if $ae = eae$ ($ea = eae$) for all $a \in R$. It can be easily checked that an idempotent $e$ of $R$ is left (right) semicentral if and only if $eR$ ($Re$) is an ideal. Also note that an idempotent $e$ is left semicentral if and only if $1 - e$ is right semicentral. See [4] and [6], for a more detailed account of semicentral idempotents. Thus for a

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left (right) ideal $I$ of a ring $R$, if $l(I) = Re$ ($r(I) = eR$) with an idempotent $e$, then $e$ is right (left) semicentral, since $Re$ ($eR$) is an ideal, and we use $S_r(R)$ ($S_e(R)$) to denote the set of left (right) semicentral idempotents of $R$.

In [11], Clark defines $R$ to be a quasi-Baer ring if the left annihilator of every ideal of $R$ is generated, as a left ideal, by an idempotent. He uses the quasi-Baer concept to characterize when a finite-dimensional algebra with identity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The quasi-Baer condition are left-right symmetric. It is well known that $R$ is a quasi-Baer if and only if $M_n(R)$ is quasi-Baer if and only if $T_n(R)$ is a quasi-Baer ring (see [3], [7], [8] and [18]).

In [17], Moussavi, Javadi and Hashemi define a ring $R$ to be $n$-generalized right quasi-Baer if for each $I \trianglelefteq R$, the right annihilator of $I^n$ is generated (as a right ideal) by an idempotent. They proved in [17, Theorem 4.7] that $R$ is $n$-generalized quasi-Baer if and only if $M_n(R)$ is $n$-generalized. Moreover, they found equivalent conditions for which the 2-by-2 generalized triangular matrix ring be $n$-generalized quasi-Baer, see [17, Theorem 4.3].

In [9], Birkenmeier, Kim and Park introduced a principally quasi-Baer ring and used them to generalize many results on reduced (i.e., it has no nonzero nilpotent elements) p.p.-rings. A ring $R$ is called right principally quasi-Baer (or simply right p.q.-Baer) if the right annihilator of a principal right ideal is generated by an idempotent.

The above results are motivation for us to introduce Baer-ideal. An ideal $I$ of $R$ is called right Baer-ideal if $r(I) = eR$ for some idempotent $e \in R$, and if $l(I) = Rd$ for some idempotent $d \in R$, then we say $I$ is a left Baer-ideal. In section 2, we see an example of right Baer-ideals which are not left Baer-ideal. We also see that the set of Baer-ideals are closed under sum and direct product.

In section 3, we characterize Baer-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings. By these results we obtain new proofs for the well-known results about quasi-Baer and $n$-generalized quasi-Baer rings. Also, we find equivalent conditions for which the 2-by-2 generalized triangular matrix ring be right SA (i.e., for any two $I, J \trianglelefteq R$ there is a $K \trianglelefteq R$ such that $r(I) + r(J) = r(K)$).

In section 4, we prove that the product of two Baer ideals in a semiprime ring $R$ is a Baer-ideal. Also we show that an ideal $I$ of a semiprime ring $R$ is a Baer-ideal if and only if $intV(I)$ is a clopen subset of $Spec(R)$. Moreover, it is proved that an ideal $I$ of $C(X)$ is a Baer-ideal if and only if $int\bigcap_{f \in I} Z(f)$ is a clopen subset of space $X$. 
2. Preliminary results and examples

**Definition 2.1.** An ideal $I$ of $R$ is called **right Baer-ideal** if there exists an idempotent $e \in R$ such that $r(I) = eR$, similarly, we can define left Baer-ideal and we say $I$ is a Baer-ideal if $I$ is a right and left Baer-ideal.

**Example 2.2.** (i) The ideals $0$ and $R$ are Baer-ideals in any ring $R$.

(ii) For $e \in S_l(R)$ the ideal $ReR$ is a right Baer-ideal. Since, we have $r(ReR) = r(eR) = r(Re) = (1 - e)R$.

(iii) For $f \in S_l(R)$, the ideal $RfR$ is a left Baer-ideal. Since, $l(RfR) = l(Rf) = l(fR) = R(1 - f)$.

In the following, we provide an example of right Baer-ideals which are not left Baer-ideal. Also we see a non-quasi-Baer ring which has a Baer-ideal.

**Example 2.3.** Let $R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right) = \left\{ \begin{pmatrix} n & a \\ 0 & b \end{pmatrix} : n \in \mathbb{Z}, a, b \in \mathbb{Z}_2 \right\}$, where $\mathbb{Z}$ and $\mathbb{Z}_n$ are rings of integers and integers modulo $n$, respectively.

(i) For ideal $I = \left( \begin{array}{cc} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right)$, we have $l(I) = \left( \begin{array}{cc} 2\mathbb{Z} & 0 \\ 0 & 0 \end{array} \right)$, and is not containing any idempotent. Therefore $I$ is not a left Baer-ideal. On the other hand $r(I) = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) R$. Thus $I$ is a right Baer-ideal.

(ii) For ideal $J = \left( \begin{array}{cc} 2\mathbb{Z} & 0 \\ 0 & 0 \end{array} \right)$, we have $l(J) = \left( \begin{array}{cc} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right)$, and $r(J) = \left( \begin{array}{cc} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right) R$. Hence $J$ is a Baer-ideal.

**Lemma 2.4.** [20, Lemma 2.3]. Let $e_1$ and $e_2$ be two right semicentral idempotents.

(1) $e_1e_2$ is a right semicentral idempotent.

(2) $(e_1 + e_2 - e_1e_2)$ is a right semicentral idempotent.

(3) If $S \subseteq S_r(R)$ is finite, then there is a right semicentral idempotent $e$ such that $RSR = ReR = < e >$.

**Proposition 2.5.** The sum of two Baer-ideals in any ring $R$ is a Baer-ideal.

**Proof.** Let $I$ and $J$ be two Baer-ideals of $R$. Then there are idempotents $e, f \in S_l(R)$ such that $r(I) = eR = r(R(1 - e))$ and $r(J) = fR = r(R(1 - f))$. Therefore $r(I + J) = r(I) \cap r(J) = r(R(1 - e)) \cap r(R(1 - f)) = r(R(1 - e) + R(1 - f))$. Since $1 - e, 1 - f \in S_r(R)$, By Lemma 2.4, we have

$$h = ((1 - e) + (1 - f) - (1 - e)(1 - f)) \in S_r(R).$$
On the other hand, we can see that
\[ r(I + J) = r(R(1 - e) + R(1 - f)) = r(Rh) = (1 - h)R. \]
Hence \( I + J \) is a right Baer-ideal. Similarly, we can see that \( I + J \) is a left Baer-ideal. \( \square \)

**Proposition 2.6.** An ideal \( J \) of \( R = \prod_{x \in X} R_x \) a direct product of rings is a right Baer-ideal if and only if each \( \pi_x(J) = J_x \) is a right Baer-ideal of \( R_x \), where \( \pi_x : R \mapsto R_x \) denote the canonical projection homomorphism.

**Proof.** If \( J \) is a right Baer-ideal of \( R \), then there exists an idempotent \( e \in R \) such that \( r(J) = eR \). This implies that \( r(J_x) = \pi_x(e)R_x = e_xR_x \). Therefore each \( J_x \) is a right Baer-ideal of \( R_x \). Conversely, each \( J_x \) is a right Baer-ideal, hence for each \( x \in X \) there exists an idempotent \( e_x \in R_x \) such that \( r(J_x) = e_xR_x \). Thus \( r(J) = (e_x)_{x \in X}R \). Therefore \( J \) is a right Baer-ideal of \( R \). \( \square \)

**Corollary 2.7.** Let \( R = \prod_{x \in X} R_x \), a direct product of rings.

1. \( R \) is quasi-Baer if and only if each \( R_x \) is quasi-Baer.
2. \( R \) is \( n \)-generalized quasi-Baer if and only if each \( R_x \) is \( n \)-generalized quasi-Baer.

**Proof.** This is a consequence of Proposition 2.6. \( \square \)

### 3. Baer-ideals in extension rings

Throughout this section, \( T \) will denote a 2-by-2 generalized (or formal) triangular matrix ring \( \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \), where \( R \) and \( S \) are rings and \( M \) is an \((S, R)\)-bimodule. If \( N \) is an \((S, R)\)-submodule of \( M \) (briefly, \( sN_R \leq S M_R \)), then \( \text{Ann}_R N = \{ r \in R : Nr = 0 \} \) and \( \text{Ann}_S N = \{ s : sN = 0 \} \), see [16]. In this section we use a similar method as in Birkenmeier, Kim and Park in [10] and characterize Baer-ideals of 2-by-2 generalized triangular matrix rings. Also we characterize Baer-ideals in full and upper triangular matrix rings. By using of these results, we can prove the well-known results about quasi-Baer rings and generalized right quasi-Baer rings.

**Theorem 3.1.** An ideal \( J \) of \( M_n(R) \) is a right Baer-ideal if and only if \( J = M_n(I) \), for some right Baer-ideal \( I \) of \( R \).

**Proof.** Let \( J \) be a right Baer-ideal of \( M_n(R) \). By [15, Theorem 3.1], \( J = M_n(I) \), for some ideal \( I \) of \( R \). We claim that \( I \) is a right Baer-ideal. By hypothesis, there exists \( E \in S_l(M_n(R)) \) such that \( r(J) = EM_n(R) \). Hence \( e_{11}R \subseteq r(I) \), where \( e_{11} \) is the \((1, 1)\)-th entries in \( E \).
We show that \( r(I) \subseteq e_{11}R \). Suppose that \( x \in r(I) \). By [5, Lemma 3.1], \( r(J) = M_n(r(I)) \). Hence \( A \in r(J) \), where \( a_{11} = x \) and zero elsewhere. Therefore \( A \in EM_n(R) \). By [20, Theorem 3.3], in matrix \( E \), \( e_{ij} = e_{11}e_{ij} \). This implies that \( x \in e_{11}R \). Now let \( J = M_n(I) \) and \( I \) be a right Baer-ideal in \( R \). Then there exists an idempotent \( e \in R \) such that \( r(I) = eR \). By [5, Lemma 3.1], \( r(M_n(I)) = M_n(r(I)) = M_n(eR) = E M_n(R) \), where in matrix \( E \) for each \( 1 \leq i \leq n \), \( e_{ii} = e \) and \( e_{ij} = 0 \) for all \( i \neq j \). Thus \( J \) is a right Baer-ideal of \( M_n(R) \). □

**Theorem 3.2.** The following statements hold.

1. For every \( I \trianglelefteq T_n(R) \), there are ideals \( J_{ik} \) of \( R \), \( 1 \leq i, k \leq n \) such that

\[
I = \begin{pmatrix}
J_{11} & J_{12} & J_{13} & \ldots & J_{1n} \\
0 & J_{22} & J_{23} & \ldots & J_{2n} \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & 0 & J_{nn}
\end{pmatrix},
\]

and \( J_{i+1k} \subseteq J_{ik} \).

2. \( I \) is a right Baer-ideal of \( T_n(R) \) if and only if each \( J_{1k} \) is a right Baer-ideal of \( R \).

3. If \( K \) is a right Baer-ideal of \( R \), then \( T_n(K) \) is a right Baer-ideal of \( T_n(R) \).

**Proof.** (1) Let \( I \trianglelefteq T_n(R) \) and for each \( 1 \leq i \leq n \), \( K_i \) is the set consisting of all entries in the \( i \)th column of elements of \( I \). Then for each \( 1 \leq i \leq n \), \( K_i \subseteq R \). Put \( J_{ij} = K_i + \ldots + K_j \). Then \( J_{ik} \subseteq J_{ik+1} \) and \( J_{i+1k} \subseteq J_{ik} \). Always we have

\[
I \subseteq \begin{pmatrix}
K_1 & K_1 + K_2 & K_1 + K_2 + K_3 & \ldots & K_1 + \ldots + K_n \\
0 & K_2 & K_2 + K_3 & \ldots & K_2 + \ldots + K_n \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & 0 & K_n
\end{pmatrix}.
\]

On the other hand

\[
\begin{pmatrix}
K_1 & K_2 & K_3 & \ldots & K_n \\
0 & K_2 & K_3 & \ldots & K_n \\
& \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \ldots & 0 & K_n
\end{pmatrix} \subseteq I,
\]
and $I \trianglelefteq T_n(R)$, hence
\[
\begin{pmatrix}
K_1 & K_1 + K_2 & K_1 + K_2 + K_3 & \ldots & K_1 + \ldots + K_n \\
0 & K_2 & K_2 + K_3 & \ldots & K_2 + \ldots + K_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & K_n \\
\end{pmatrix}
\]
\[= \begin{pmatrix}
K_1 & K_2 & K_3 & \ldots & K_n \\
0 & K_2 & K_3 & \ldots & K_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & K_n \\
\end{pmatrix}
\]
\[+ \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 \\
0 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix}
\subseteq I.
\]

Therefore $I = \begin{pmatrix}
J_{11} & J_{12} & J_{13} & \ldots & J_{1n} \\
0 & J_{22} & J_{23} & \ldots & J_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & J_{nn} \\
\end{pmatrix}$.

(2) Assume that $I$ is a right Baer-ideal of $T_n(R)$. Then there exists an idempotent $E \in T_n(R)$ such that $r(I) = ET_n(R)$. On the other hand by (i), we can see that
\[
\begin{pmatrix}
J_{11} & J_{12} & J_{13} & \ldots & J_{1n} \\
0 & J_{22} & J_{23} & \ldots & J_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & J_{nn} \\
\end{pmatrix}
\]
\[
= \begin{pmatrix}
r_R(J_{11}) & r_R(J_{11}) & \ldots & r_R(J_{1n}) \\
0 & r_R(J_{12}) & \ldots & r_R(J_{12}) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & r_R(J_{1n}) \\
\end{pmatrix}.
\]

Thus for each $1 \leq k \leq n$, $r(J_{1k}) = e_{kk}R$, where $e_{kk}$ is the $(k,k)$-th entries in $E$. Conversely, let for each $1 \leq k \leq n$, $J_{1k}$ be a right Baer-ideal of $R$. Then there is an $e_{1k} \in S_i(R)$ such that $r(J_{1k}) = e_{1k}R$. Consider matrix $F$, where for each $1 \leq k \leq n$, $f_{kk} = e_{1k}$ and elsewhere is zero. Then we have $IF = 0$. If $A \in r(I)$, then for each $1 \leq j \leq n$, $a_{kj} \in r(J_{1k})$. Hence there exists $c_{kj} \in R$ such that $a_{kj} = e_{1k}c_{kj} =$
A CHARACTERIZATION OF BAER-IDEALS

$f_{kk}c_{kj}$, for all $1 \leq j \leq n$. Thus $A = FC \in FT_n(R)$, where $C = [c_{kj}]$. Therefore $r(I) = FT_n(R)$. Hence $I$ is a right Baer-ideal of $T_n(R)$.

(3) By (2), this is evident.

Corollary 3.3. The following statements hold.

1. [18, Proposition 2]. $R$ is quasi-Baer if and only if $M_n(R)$ is quasi-Baer.
2. [17, Theorem 4.7]. $R$ is $n$-generalized right quasi-Baer if and only if $M_n(R)$ is $n$-generalized right quasi-Baer.

Proof. (1) Let $R$ be quasi-Baer and $J \subseteq M_n(R)$. Then $J = M_n(I)$ for some $I \subseteq R$ and $I$ is a Baer-ideal. By Theorem 3.1, $J$ is a right Baer-ideal, hence $M_n(R)$ is a quasi-Baer ring. Now let $I \subseteq R$ and $M_n(R)$ be quasi-Baer. Then $M_n(I)$ is a right Baer-ideal of $M_n(R)$. Again by Theorem 3.1, $I$ is a right Baer-ideal in $R$, thus $R$ is a quasi-Baer-ring.

(2) Assume that $J \subseteq M_n(R)$ and $R$ is $n$-generalized right quasi-Baer. Then $J = M_n(I)$, where $I^n$ is a right Baer-ideal. By Theorem 3.1, $J^n = M_n(I^n)$ is a right Baer-ideal. This shows that $M_n(R)$ is $n$-generalized right quasi-Baer. The converse is evident.

Corollary 3.4. [18, Proposition 9]. $R$ is quasi-Baer if and only if $T_n(R)$ is quasi-Baer.

Proof. Let $J \subseteq T_n(R)$. By Theorem 3.2,

$$J = \begin{pmatrix}
J_{11} & J_{12} & J_{13} & \cdots & J_{1n} \\
0 & J_{22} & J_{23} & \cdots & J_{2n} \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots \\
0 & 0 & \cdots & 0 & J_{nn}
\end{pmatrix}.$$

By hypothesis, each $J_{ik}$ is a right Baer-ideal. Theorem 3.2, implies that $J$ is a right Baer-ideal. Thus $T_n(R)$ is quasi-Baer. The converse is evident.

Lemma 3.5. [10, Lemma 2.3]. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$ be an idempotent element of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

1. $e \in S_l(T)$ if and only if
   a. $e_1 \in S_l(S)$;
   b. $e_2 \in S_l(R)$;
   c. $e_1k = k$; and
(d) $e_1me_2 = me_2$, for all $m \in M$.

(2) $e_1k = k$ if and only if $eT \subseteq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T$.

(3) If $e_1me_2 = me_2$, for all $m \in M$, then $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T \subseteq eT$.

(4) If $e \in S_l(T)$, then $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T = eT$.

**Lemma 3.6.** [10, Lemma 3.1]. Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then $r(J) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \text{Ann}_R(N) \end{pmatrix}$ and $l(J) = \begin{pmatrix} l_S(I) \cap \text{Ann}_S(N) & l_M(L) \\ 0 & l_R(L) \end{pmatrix}$.

**Theorem 3.7.** Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

Then $J$ is a right Baer-ideal of $T$ if and only if

1. $I$ is a right Baer-ideal of $S$;
2. $r_M(I) = (r_S(I))M$; and
3. $r_R(L) \cap \text{Ann}_R(N) = aR$, for some $a^2 = a \in R$.

**Proof.** Let $J$ be a right Baer-ideal of $T$. Then there exists $e \in S_l(T)$ such that $r(J) = eT$. By Lemma 3.5, $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$, for some $e_1 \in S_l(S), e_2 \in S_l(R), k \in M$ and $kR = e_1kR$. Thus $e_1M = e_1M + kR$. By Lemma 3.5, $e_1S = r_S(I), r_M(I) = e_1M = e_1SM = (r_S(I))M$ and $r_R(L) \cap \text{Ann}_R(N) = e_2R$.

Conversely, by hypothesis, there are $e_1 \in S_l(S)$ and $a^2 = a \in R$ such that $r_S(I) = e_1S$ and $r_R(L) \cap \text{Ann}_R(N) = aR$. Since $\text{Ann}_R(N) \subseteq R$, then $a \in S_l(R)$. By (ii), $r_M(I) = (r_S(I))M = e_1M$. Now let $e = \begin{pmatrix} e_1 & 0 \\ 0 & a \end{pmatrix}$. Then $eT = \begin{pmatrix} e_1S & e_1M \\ 0 & aR \end{pmatrix} = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap \text{Ann}_R(N) \end{pmatrix}$. From Lemma 3.6, $eT = r(J)$. Therefore $J$ is a right Baer-ideal of $T$. \qed

**Corollary 3.8.** [10, Theorem 3.2]. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent.

1. $T$ is quasi-Baer.
2. (i) $R$ and $S$ are quasi-Baer;
   (ii) $r_M(I) = (r_S(I))M$ for all $I \subseteq S$; and
A CHARACTERIZATION OF BAER-IDEALS

(iii) If \( sN_R \leq_S M_R \), then we have \( \text{Ann}_R(N) = aR \) for some \( a^2 = a \in R \).

Proof. \( 1 \Rightarrow 2 \). Let \( I \leq S \), \( N \) be a \((S, R)\) submodule of \( M \) and \( J \leq R \). Then \( \begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix} \) are Baer-ideals of \( T \). By Theorem 3.7, \( I \) and \( J \) are Baer-ideals, hence \( R, S \) are quasi-Baer and \( r_R(0) \cap \text{Ann}_R(N) = \text{Ann}_R(N) = aR \), for some \( a^2 = a \in R \).

\( 2 \Rightarrow 1 \). Let \( J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix} \leq T \). By hypothesis, there are \( a, e \in S_l(R) \) such that \( \text{Ann}_R(N) = aR \), \( r_R(L) = eR \) and \( I \) is a Baer-ideal. Hence \( r_R(L) \cap \text{Ann}_R(N) = r(R(1 - e)) \cap r(R(1 - a)) = eaR \). By Theorem 3.7, \( J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix} \) is a Baer-ideal, thus \( T \) is a quasi-Baer ring. \( \square \)

Corollary 3.9. \([17, \text{Theorem 4.3}]\). Let \( T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \). Then the following are equivalent.

1. \( T \) is \( n \)-generalized right (principally) quasi-Baer.
2. (i) \( S \) is \( n \)-generalized right quasi-Baer;
   (ii) \( r_M(I^n) = (r_S(I^n))M \) for all \( I \leq S \); and
   (iii) If \( \begin{pmatrix} 0 & N \\ 0 & J \end{pmatrix} \leq T \), then there is some \( e^2 = e \in R \) such that

\[ r_R(J^n) \cap \text{Ann}_R(I^{n-1}N) \cap \text{Ann}_R(I^{n-2}NJ) \cap \ldots \cap \text{Ann}_R(NJ^{n-1}) = eR. \]

Proof. \( 1 \Rightarrow 2 \). (i), (ii) Let \( I \leq S \). Then \( \begin{pmatrix} I^n & I^{n-1}M \\ 0 & M \end{pmatrix} \) is a Baer-ideal of \( T \). By Theorem 3.7, \( I^n \) is a Baer-ideal in \( S \), hence \( S \) is \( n \)-generalized right (principally) quasi-Baer and \( r_M(I^n) = (r_S(I^n))M \).

(iii) If \( \begin{pmatrix} 0 & N \\ 0 & J \end{pmatrix} \leq T \). Then \( \begin{pmatrix} I^n & I^{n-1}N + I^{n-2}NJ + \ldots + NJ^{n-1} \\ 0 & J^n \end{pmatrix} \) is a Baer-ideal in \( T \). By Theorem 3.7, there is some \( e^2 = e \in R \) such that

\[ r_R(J^n) \cap \text{Ann}_R(I^{n-1}N) \cap \text{Ann}_R(I^{n-2}NJ) \cap \ldots \cap \text{Ann}_R(NJ^{n-1}) = eR. \]

\( 2 \Rightarrow 1 \). Let \( K = \begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \leq T \). By hypothesis and Theorem 3.7, \( K^n = \begin{pmatrix} I^n & I^{n-1}N + I^{n-2}NJ + \ldots + NJ^{n-1} \\ 0 & J^n \end{pmatrix} \) is a Baer-ideal in \( T \). Hence \( T \) is \( n \)-generalized right (principally) quasi-Baer. \( \square \)

Recall that a ring \( R \) is a right \( SA \) if for each \( I, J \leq R \) there exists \( K \leq R \) such that \( r(I) + r(J) = r(K) \) (see \([5]\)).
Theorem 3.10. Let \( T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix} \). Then the following are equivalent.

(1) \( T \) is a right \( SA \)-ring.

(2) (i) For \( I_1, I_2 \subseteq S \), there exists \( I_3 \subseteq S \), such that \( r_M(I_1) + r_M(I_2) = r_M(I_3) \), \( r_S(I_1) + r_S(I_2) = r_S(I_3) \) (i.e., \( S \) is right \( SA \)); and

(ii) For each \( I, J \subseteq R \) and \( (S, R) \) submodules \( N_1, N_2 \), of \( M \), there are \( K \subseteq R \) and \( SN_R \leq_s M_R \), such that

\[
r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).
\]

Proof. \( 1 \Rightarrow 2 \).

(i) Let \( I_1, I_2 \subseteq S \). Then \( \begin{pmatrix} I_1 & M \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} I_2 & M \\ 0 & 0 \end{pmatrix} \) are ideals of \( T \). By hypothesis, there is \( \begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \subseteq T \) such that

\[
r\left(\begin{pmatrix} I_1 & M \\ 0 & 0 \end{pmatrix}\right) + r\left(\begin{pmatrix} I_2 & M \\ 0 & 0 \end{pmatrix}\right) = r\left(\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}\right).
\]

By Lemma 3.6, we have \( r_S(I_1) + r_S(I_2) = r_S(I) \) and \( r_M(I_1) + r_M(I_2) = r_M(I) \).

(ii) Let \( I, J \subseteq R \) and \( N_1, N_2 \) are \( (S, R) \) submodules of \( M \). Then

\[
\begin{pmatrix} 0 & N_1 \\ 0 & I \end{pmatrix}\) and \( \begin{pmatrix} 0 & N_2 \\ 0 & J \end{pmatrix}\) are ideals of \( T \). By hypothesis, there are \( K \subseteq R \), \( I \subseteq S \) and \( SN_R \leq_s M_R \), such that

\[
r\left(\begin{pmatrix} 0 & N_1 \\ 0 & I \end{pmatrix}\right) + r\left(\begin{pmatrix} 0 & N_2 \\ 0 & J \end{pmatrix}\right) = r\left(\begin{pmatrix} I & N \\ 0 & K \end{pmatrix}\right).
\]

Now, Lemma 3.6, implies that

\[
r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).
\]

\( 2 \Rightarrow 1 \). Suppose that \( K_1 = \begin{pmatrix} I_1 & N_1 \\ 0 & J_1 \end{pmatrix} \) and \( K_2 = \begin{pmatrix} I_2 & N_2 \\ 0 & J_2 \end{pmatrix} \) are two ideals of \( T \). By Lemma 3.6, we have

\[
r(K_1) + r(K_2) = \begin{pmatrix} r_S(I_1) + r_S(I_2) & r_M(I_1) + r_M(I_2) \\ 0 & r_R(J_1) \cap Ann_R(N_1) + r_R(J_2) \cap Ann_R(N_2) \end{pmatrix}.
\]

By hypothesis, there are \( I_3 \subseteq S \), \( K \subseteq R \) and \( SN_R \leq_s M_R \), such that

\[
r_S(I_1) + r_S(I_2) = r_S(I_3), r_M(I_1) + r_M(I_2) = r_M(I_3),
\]

and

\[
r_R(J_1) \cap Ann_R(N_1) + r_R(J_2) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).
\]

Therefore, by Lemma 3.6, \( r(K_1) + r(K_2) = r\left(\begin{pmatrix} I_3 & N \\ 0 & K \end{pmatrix}\right) \). \( \square \)
The following statements hold.

1. Let $R = S$ and for every $I \subseteq S$, $r_M(I) = (r_S(I))M$. Then $T$ is right $SA$ if and only if $R$ is right $SA$.
2. Let $R = S$ and $M \subseteq R$, then $T$ is right $SA$ if and only if $R$ is right $SA$.
3. Let $S = M$. Then $T$ is right $SA$ if and only if $S$ is a right $SA$ and for each $I, J \subseteq R$ and $N_1, N_2 \subseteq S$, there are $K \subseteq R$ and $N \subseteq S$, such that $r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N)$.

Proof. This is a consequence of Theorem 3.10. □

4. BAER-IDEALS IN SEMIPRIME RING AND RING OF CONTINUOUS FUNCTIONS

In this section, first, we show that an ideal $I$ of $C(X)$ is a Baer-ideal if and only if $\text{int}\bigcap_{f \in I} Z(f)$ is a clopen subset of $X$. Then we show that an ideal $I$ of semiprime ring $R$ is a Baer-ideal if and only if $\text{int}V(I)$ is a clopen subset of $\text{Spec}(R)$. Also we prove that the product of two Baer-ideals in a semiprime ring $R$ is a Baer-ideal.

A non-zero ideal $I$ of $R$ is an essential ideal if for any ideal $J$ of $R$, $I \cap J = 0$ implies that $J = 0$. Also an ideal $P$ of a commutative ring $R$ is called pseudoprime ideal if $ab = 0$, implies that $a \in P$ or $b \in P$ (see [13]).

We denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$. For any $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called a zero-set. We can see that a subset $A$ of $X$ is clopen if and only if $A = Z(f)$ for some idempotent $f \in C(X)$. For any subset $A$ of $X$ we denote by $\text{int}A$ the interior of $A$ (i.e., the largest open subset of $X$ contained in $A$). For terminology and notations, the reader is referred to [12] and [14].

Lemma 4.1. For $I, J \subseteq C(X)$, $r(I) = r(J)$ if and only if $\text{int}\bigcap_{f \in I} Z(f) = \text{int}\bigcap_{g \in J} Z(g)$.

Proof. ($\Rightarrow$) Let $x \in \text{int}\bigcap_{f \in I} Z(f)$. Then $x \notin X \setminus \text{int}\bigcap_{f \in I} Z(f)$. By completely regularity of $X$, there exists $h \in C(X)$ such that $x \in X \setminus Z(h) \subseteq \text{int}\bigcap_{f \in I} Z(f)$. Therefore $fh = 0$ for all $f \in I$. This implies that $h \in r(I) = r(J)$. Hence $gh = 0$ for each $g \in J$. Thus $x \in X \setminus \text{int}Z(h) \subseteq \text{int}\bigcap_{g \in J} Z(g)$. Similarly, we can prove that $\text{int}\bigcap_{g \in J} Z(g) \subseteq \text{int}\bigcap_{f \in I} Z(f)$.

($\Leftarrow$) Suppose that $h \in r(I)$. Then $X \setminus Z(h) \subseteq \text{int}\bigcap_{f \in I} Z(f)$, so $X \setminus Z(h) \subseteq Z(f)$ for all $f \in I$. Hence for each $f \in I$, $fh = 0$. This implies that $r(I) \subseteq r(J)$. Similarly, we can prove that $r(J) \subseteq r(I)$. □
Proposition 4.2. The following statements hold.

(1) An ideal \( I \) of \( C(X) \) is a Baer-ideal if and only if \( \text{int}\bigcap_{f\in I} Z(f) \) is a clopen subset of \( X \).

(2) Every pseudoprime ideal of \( C(X) \) is a Baer-ideal.

Proof. (1) Let \( I \) be a Baer-ideal of \( C(X) \). Then there exists an idempotent \( e \in C(X) \) such that \( r(I) = eC(X) = r(C(X))(1 - e) \). By Lemma 4.1, \( \text{int}\bigcap_{f\in I} Z(f) = \text{int}Z(1 - e) = Z(1 - e) \). This shows that \( \text{int}\bigcap_{f\in I} Z(f) \) is a clopen subset of \( X \). Now let \( \text{int}\bigcap_{f\in I} Z(f) \) be a clopen subset of \( X \). Then there exists an idempotent \( e \in C(X) \) such that \( \text{int}\bigcap_{f\in I} Z(f) = Z(e) = \text{int}Z(e) \). By Lemma 4.1, \( r(I) = r(e) = (1 - e)C(X) \). Hence \( I \) is a Baer-ideal.

(2) By [1, Corollary 3.3], every pseudoprime ideal in \( C(X) \) is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal. Now let \( P \) be a pseudoprime ideal in \( C(X) \). If \( P \) is essential, then by [1, Theorem 3.1], \( \text{int}\bigcap_{f\in P} Z(f) = \emptyset \), so (i), implies that \( P \) is a Baer-ideal. Otherwise \( P \) is a maximal ideal which is also a minimal prime ideal. Then there exists an isolated point \( x \in X \) such that \( P = M_x = \{ f \in C(X) : x \in Z(f) \} \). This shows that \( \text{int}\bigcap_{f\in P} Z(f) = \{ x \} \) is a clopen subset of \( X \), so \( P \) is a Baer-ideal. □

Recall that a topological space \( X \) is extremally disconnected if the interior of any closed subset is closed, see [14, 1.H]. The next result is proved in [2, Theorem 3.5] and [21, Theorem 2.12]. Now we give a new proof.

Corollary 4.3. \( C(X) \) is a Baer-ring if and only if \( X \) is an extremally disconnected space.

Proof. Let \( F \) be a closed subset of \( X \) and \( C(X) \) is a Baer-ring. By completely regularity of \( X \), there exists an ideal \( I \) of \( C(X) \) such that \( F = \bigcap_{f\in I} Z(f) \). By Proposition 4.2, \( \text{int}F \) is closed, hence \( X \) is extremally disconnected. Conversely, suppose that \( I \subseteq C(X) \). Then \( \text{int}\bigcap_{f\in I} Z(f) \) is closed. By Proposition 4.2, \( I \) is a Baer-ideal, thus \( C(X) \) is a Baer-ring. □

For any \( a \in R \), let \( \text{supp}(a) = \{ P \in \text{Spec}(R) : a \notin P \} \). Shin [19, Lemma 3.1] proved that for any \( R \), \( \{ \text{supp}(a) : a \in R \} \) forms a basis of open sets on \( \text{Spec}(R) \). This topology is called hull-kernel topology. We mean of \( V(I) \) is the set of \( P \in \text{Spec}(R) \), where \( I \subseteq P \). Note that \( V(I) = \bigcap_{a \in I} V(a) \).

Lemma 4.4. [5, Lemma 4.2]. The following statements hold.
(1) If $I$ and $J$ are two ideals of a semiprime ring $R$, then $r(I) = r(J)$ if and only if $intV(I) = intV(J)$.
(2) $A \subseteq Spec(R)$ is a clopen subset if and only if there exists an idempotent $e \in R$ such that $A = V(e)$.

**Proposition 4.5.** Let $R$ be a semiprime ring.

(1) An ideal $I$ of $R$ is a Baer-ideal if and only if $intV(I)$ is a clopen subset of $Spec(R)$.
(2) The product of two Baer-ideals is a Baer-ideal.
(3) If $R$ is a commutative ring, then any essential ideal of $R$ is a Baer-ideal.

**Proof.** (1) Let $I$ be a Baer ideal of $R$. Then there exists an idempotent $e \in R$ such that $r(I) = eR = r(R(1 - e))$. By Lemma 4.4, $intV(I) = intV(1 - e) = V(1 - e)$. Thus $intV(I)$ is closed. Conversely, let $I \subseteq R$. By hypothesis and Lemma 4.4, there exists an idempotent $e \in R$ such that $intV(I) = V(e)$. So, Lemma 4.4, implies that $r(I) = r(Re) = (1 - e)R$. Therefore, $I$ is a right Baer-ideal. By semiprime hypothesis, $I$ is a left Baer-ideal.

(2) Let $I, J$ be two Baer-ideals of $R$. Then there are idempotents $e, f \in R$ such that $r(I) = eR$ and $r(J) = fR$. We will prove $r(IJ) = fR + eR + feR$. By Lemma 2.4, there exists $h \in S_I(R)$ such that $r(IJ) = fR + eR + feR = hR$. Therefore $IJ$ is a right Baer-ideal.

By semiprime hypothesis, $IJ$ is a left Baer-ideal. Now let $x \in r(IJ)$. Then $Jx \subseteq r(I) = r(R(1 - e))$. So $R(1 - e)Jx = 0$. This implies that $(JxR(1 - e))^2 = 0$. Since $R$ is semiprime, we have $JxR(1 - e) = 0$. Thus $x(1 - e) \in r(J) = r(R(1 - f))$. Hence $(1 - f)x(1 - e) = 0$. This shows that $x = -fexe + fexe + fexe + exe + fxe \in feR + eR + fR$. On the other hand we have $(IJ)(feR + eR + fR) = 0$, so $feR + eR + fR \subseteq r(IJ)$.

(3) It is easily seen that an ideal $I$ of a commutative semiprime ring $R$ is essential if and only if $r(I) = 0 = r(R)$. Now, Lemma 4.4, implies that $I$ is a Baer-ideal. \hfill \Box

Now we apply the theory of Baer ideals to give the following well-known result.

**Corollary 4.6.** Let $R$ be a semiprime ring. Then $R$ is quasi-Baer if and only if $Spec(R)$ is extremally disconnected.

**Proof.** Let $A$ be a closed subset of $Spec(R)$ and $R$ is quasi-Baer. Since $\{V(a) : a \in R\}$ is a base for closed subsets in $Spec(R)$, there exists $S \subseteq R$ such that $A = \bigcap_{a \in S} V(a)$. Take $I = RSR$. Then $A = V(I)$. By Lemma 4.5, $intA$ is closed. Thus $Spec(R)$ is extremally disconnected.
Conversely, let $I \subseteq R$. We know that $V(I)$ is a closed subset of $\text{Spec}(R)$. By hypothesis and Lemma 4.5, $\text{int}V(I)$ is a clopen subset of $\text{Spec}(R)$, and hence $I$ is Baer-ideal. Thus $R$ is a quasi Baer-ring. □

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References


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