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# A CHARACTERIZATION OF BAER-IDEALS

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Abstract. An ideal $I$ of a ring $R$ is called a right Baer-ideal if there exists an idempotent $e \in R$ such that $r(I) = eR$. We know that $R$ is quasi-Baer if every ideal of $R$ is a right Baer-ideal, $R$ is $n$-generalized right quasi-Baer if for each $I \trianglelefteq R$ the ideal $I^n$ is a right Baer-ideal, and $R$ is right principaly quasi-Baer if every principal right ideal of $R$ is a right Baer-ideal. Therefore the concept of Baer ideal is important. In this paper we investigate some properties of Baer-ideals and give a characterization of Baer-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings, semiprime ring and ring of continuous functions. Finally, we find equivalent conditions for which the 2-by-2 generalized triangular matrix ring be right $SA$.

## 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity. Let $\emptyset \neq X \subseteq R$. Then $X \trianglelefteq R$ denotes that $X$ is an ideal of $R$. For any subset $S$ of $R$, $l(S)$ and $r(S)$ denote the left annihilator and the right annihilator of $S$ in $R$. The ring of $n$-by-$n$ (upper triangular) matrices over $R$ is denoted by $\mathbf{M_n(R)}$ ($\mathbf{T_n(R)}$). An idempotent $e$ of a ring $R$ is called *left (right) semicentral* if $ae = eae$ ($ea = eae$) for all $a \in R$. It can be easily checked that an idempotent $e$ of $R$ is left (right) semicentral if and only if $eR$ ($Re$) is an ideal. Also note that an idempotent $e$ is left semicentral if and only if $1 - e$ is right semicentral. See [4] and [6], for a more detailed account of semicentral idempotents. Thus for a

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left (right) ideal $I$ of a ring $R$, if $l(I) = Re$ $(r(I) = eR)$ with an idempotent $e$, then $e$ is right (left) semicentral, since $Re$ $(eR)$ is an ideal, and we use $S_l(R)$ $(S_r(R))$ to denote the set of left (right) semicentral idempotents of $R$.

In [11], Clark defines $R$ to be a *quasi-Baer ring* if the left annihilator of every ideal of $R$ is generated, as a left ideal, by an idempotent. He uses the quasi-Baer concept to characterize when a finite-dimensional algebra with identity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. The quasi-Baer condition are left-right symmetric. It is well known that $R$ is a quasi-Baer if and only if $\mathbf{M_n(R)}$ is quasi-Baer if and only if $\mathbf{T_n(R)}$ is a quasi-Baer ring (see [3], [7], [8] and [18]).

In [17], Moussavi, Javadi and Hashemi define a ring $R$ to be $n$-generalized right quasi-Baer if for each $I \trianglelefteq R$, the right annihilator of $I^n$ is generated (as a right ideal) by an idempotent. They proved in [17, Theorem 4.7] that $R$ is $n$-generalized quasi-Baer if and only if $\mathbf{M_n(R)}$ is $n$-generalized. Moreover, they found equivalent conditions for which the 2-by-2 generalized triangular matrix ring be $n$-generalized quasi-Baer, see [17, Theorem 4.3].

In [9], Birkenmeier, Kim and Park introduced a principally quasi-Baer ring and used them to generalize many results on *reduced* (i.e., it has no nonzero nilpotent elements) p.p.-rings. A ring $R$ is called *right principally quasi-Baer* (or simply right p.q.-Baer) if the right annihilator of a principal right ideal is generated by an idempotent.

The above results are motivation for us to introduce Baer-ideal. An ideal $I$ of $R$ is called *right Baer-ideal* if $r(I) = eR$ for some idempotent $e \in R$, and if $l(I) = Rf$, for some idempotent $f \in R$, then we say $I$ is a left Baer-ideal. In section 2, we see an example of right Baer-ideals which are not left Baer-ideal. We also see that the set of Baer-ideals are closed under sum and direct product.

In section 3, we characterize Baer-ideals in 2-by-2 generalized triangular matrix rings, full and upper triangular matrix rings. By these results we obtain new proofs for the well-known results about quasi-Baer and $n$-generalized quasi-Baer rings. Also, we find equivalent conditions for which the 2-by-2 generalized triangular matrix ring be right $SA$ (i.e., for any two $I, J \trianglelefteq R$ there is a $K \trianglelefteq R$ such that $r(I) + r(J) = r(K)$).

In section 4, we prove that the product of two Baer ideals in a semiprime ring $R$ is a Baer-ideal. Also we show that an ideal $I$ of a semiprime ring $R$ is a Baer-ideal if and only if $intV(I)$ is a clopen subset of $Spec(R)$. Moreover, it is proved that an ideal $I$ of $C(X)$ is a Baer-ideal if and only if $int \bigcap_{f \in I} Z(f)$ is a clopen subset of space $X$.

## 2. Preliminary results and examples

**Definition 2.1.** An ideal $I$ of $R$ is called *right Baer-ideal* if there exists an idempotent $e \in R$ such that $r(I) = eR$, similarly, we can define left Baer-ideal and we say $I$ is a Baer-ideal if $I$ is a right and left Baer-ideal.

**Example 2.2.** (i) The ideals 0 and $R$ are Baer-ideals in any ring $R$.

(ii) For $e \in S_r(R)$ the ideal $ReR$ is a right Baer-ideal. Since, we have $r(ReR) = r(eR) = r(Re) = (1-e)R$.

(iii) For $f \in S_l(R)$, the ideal $RfR$ is a left Baer-ideal. Since, $l(RfR) = l(Rf) = l(fR) = R(1-f)$.

In the following, we provide an example of right Baer-ideals which are not left Baer-ideal. Also we see a non-quasi-Baer ring which has a Baer-ideal.

**Example 2.3.** Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = \{\begin{pmatrix} n & a \\ 0 & b \end{pmatrix} : n \in \mathbb{Z}, a, b \in \mathbb{Z}_2\}$, where $\mathbb{Z}$ and $\mathbb{Z}_n$ are rings of integers and integers modulo $n$, respectively.

(i) For ideal $I = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$, we have $l(I) = \begin{pmatrix} 2\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$, and is not containing any idempotent. Therefore $I$ is not a left Baer-ideal. On the other hand $r(I) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R$. Thus $I$ is a right Baer-ideal.

(ii) For ideal $J = \begin{pmatrix} 2\mathbb{Z} & 0 \\ 0 & 0 \end{pmatrix}$, we have $l(J) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = R \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, and $r(J) = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} R$. Hence $J$ is a Baer-ideal.

**Lemma 2.4.** [20, Lemma 2.3]. Let $e_1$ and $e_2$ be two right semicentral idempotents.

(1) $e_1 e_2$ is a right semicentral idempotent.
(2) $(e_1 + e_2 - e_1 e_2)$ is a right semicentral idempotent.
(3) If $S \subseteq S_r(R)$ is finite, then there is a right semicentral idempotent $e$ such that $RSR = ReR =< e >$.

**Proposition 2.5.** The sum of two Baer-ideals in any ring $R$ is a Baer-ideal.

*Proof.* Let $I$ and $J$ be two Baer-ideals of $R$. Then there are idempotents $e, f \in S_l(R)$ such that $r(I) = eR = r(R(1-e))$ and $r(J) = fR = r(R(1-f))$. Therefore $r(I+J) = r(I) \cap r(J) = r(R(1-e)) \cap r(R(1-f)) = r(R(1-e) + R(1-f))$. Since $1-e, 1-f \in S_r(R)$ . By Lemma 2.4, we have

$$h = ((1-e) + (1-f) - (1-e)(1-f)) \in S_r(R).$$

On the other hand, we can see that

$$r(I+J) = r(R(1-e)+R(1-f)) = r(Rh) = (1-h)R.$$

Hence $I+J$ is a right Baer-ideal. Similarly, we can see that $I+J$ is a left Baer-ideal. □

**Proposition 2.6.** An ideal $J$ of $R=\prod_{x\in X} R_x$ a direct product of rings is a right Baer-ideal if and only if each $\pi_x(J)=J_x$ is a right Baer-ideal of $R_x$, where $\pi_x : R \mapsto R_x$ denote the canonical projection homomorphism.

*Proof.* If $J$ is a right Baer-ideal of $R$, then there exists an idempotent $e\in R$ such that $r(J)=eR$. This implies that $r(J_x)=\pi_x(e)R_x=e_xR_x$. Therefore each $J_x$ is a right Baer-ideal of $R_x$. Conversely, each $J_x$ is a right Baer-ideal, hence for each $x\in X$ there exists an idempotent $e_x\in R_x$ such that $r(J_x)=e_xR_x$. Thus $r(J)=(e_x)_{x\in X}R$. Therefore $J$ is a right Baer-ideal of $R$. □

**Corollary 2.7.** Let $R=\prod_{x\in X} R_x$, a direct product of rings.
(1) $R$ is quasi-Baer if and only if each $R_x$ is quasi-Baer.
(2) $R$ is $n$-generalized quasi-Baer if and only if each $R_x$ is $n$-generalized quasi-Baer.

*Proof.* This is a consequence of Proposition 2.6. □

## 3. Baer-ideals in extension rings

Throughout this section, $T$ will denote a 2-by-2 generalized (or formal) triangular matrix ring $\begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where $R$ and $S$ are rings and $M$ is an $(S,R)$-bimodule. If $N$ is an $(S,R)$-submodule of M (briefly, ${}_SN_R \leq_S M_R$) , then $Ann_RN=\{r\in R : Nr=0\}$ and $Ann_SN=\{s : sN=0\}$, see [16] . In this section we use a similar method as in Birkenmeier, Kim and Park in [10] and characterize Bear-ideals of 2-by-2 generalized triangular matrix rings. Also we characterize Baer-ideals in full and upper triangular matrix rings. By using of these results , we can prove the well-known results about quasi-Baer rings and generalized right quasi-Baer rings.

**Theorem 3.1.** An ideal $J$ of $\mathbf{M_n(R)}$ is a right Baer-ideal if and only if $J=\mathbf{M_n(I)}$, for some right Baer-ideal $I$ of $R$.

*Proof.* Let $J$ be a right Baer-ideal of $\mathbf{M_n(R)}$. By [15, Theorem 3.1], $J=\mathbf{M_n(I)}$, for some ideal $I$ of $R$. We claim That $I$ is a right Baer-ideal. By hypothesis, there exists $E\in S_l(\mathbf{M_n(R)})$ such that $r(J)=E\mathbf{M_n(R)}$. Hence $e_{11}R\subseteq r(I)$, where $e_{11}$ is the $(1,1)$-th entries in $E$.

We show that $r(I) \subseteq e_{11}R$. Suppose that $x \in r(I)$. By [5, Lemma 3.1], $r(J) = \mathbf{M_n}(\mathbf{r}(\mathbf{I}))$. Hence $A \in r(J)$, where $a_{11} = x$ and zero elsewhere. Therefore $A \in E\mathbf{M_n}(\mathbf{R})$. By [20, Theorem 3.3], in matrix $E$, $e_{ij} = e_{11}e_{ij}$. This implies that $x \in e_{11}R$. Now let $J = \mathbf{M_n}(\mathbf{I})$ and $I$ be a right Baer-ideal in $R$. Then there exists an idempotent $e \in R$ such that $r(I) = eR$. By [5, Lemma 3.1], $r(\mathbf{M_n}(\mathbf{I})) = \mathbf{M_n}(\mathbf{r}(\mathbf{I})) = \mathbf{M_n}(\mathbf{eR}) = \mathbf{EM_n}(\mathbf{R})$, where in matrix $E$ for each $1 \leq i \leq n$, $e_{ii} = e$ and $e_{ij} = 0$ for all $i \neq j$. Thus $J$ is a right Baer-ideal of $\mathbf{M_n}(\mathbf{R})$. □

**Theorem 3.2.** The following statements hold.

(1) For every $I \trianglelefteq \mathbf{T_n}(\mathbf{R})$, there are ideals $J_{ik}$ of $R$, $1 \leq i, k \leq n$ such that

$$I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}, J_{ik} \subseteq J_{ik+1}$$

and $J_{i+1k} \subseteq J_{ik}$.

(2) $I$ is a right Baer-ideal of $\mathbf{T_n}(\mathbf{R})$ if and only if each $J_{1k}$ is a right Baer-ideal of $R$.

(3) If $K$ is a right Baer-ideal of $R$, then $\mathbf{T_n}(\mathbf{K})$ is a right Baer-ideal of $\mathbf{T_n}(\mathbf{R})$.

*Proof.* (1) Let $I \trianglelefteq \mathbf{T_n}(\mathbf{R})$ and for each $1 \leq i \leq n$, $K_i$ is the set consisting of all entries in the $i$'th column of elements of $I$. Then for each $1 \leq i \leq n$, $K_i \trianglelefteq R$. Put $J_{ij} = K_i + ... + K_j$. Then $J_{ik} \subseteq J_{ik+1}$ and $J_{i+1k} \subseteq J_{ik}$. Always we have

$$I \subseteq \begin{pmatrix} K_1 & K_1 + K_2 & K_1 + K_2 + K_3 & . & . & . & K_1 + ... + K_n \\ 0 & K_2 & K_2 + K_3 & . & . & . & K_2 + ... + K_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & K_n \end{pmatrix}.$$

On the other hand

$$\begin{pmatrix} K_1 & K_2 & K_3 & . & . & . & K_n \\ 0 & K_2 & K_3 & . & . & . & K_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & K_n \end{pmatrix} \subseteq I,$$

and $I \trianglelefteq \mathbf{T_n}(\mathbf{R})$, hence

$$\begin{pmatrix} K_1 & K_1+K_2 & K_1+K_2+K_3 & . & . & . & K_1+...+K_n \\ 0 & K_2 & K_2+K_3 & . & . & . & K_2+...+K_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & K_n \end{pmatrix} =$$

$$\begin{pmatrix} K_1 & K_2 & K_3 & . & . & . & K_n \\ 0 & K_2 & K_3 & . & . & . & K_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & K_n \end{pmatrix} +$$

$$\begin{pmatrix} K_1 & K_2 & K_3 & . & . & . & K_n \\ 0 & K_2 & K_3 & . & . & . & K_n \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & K_n \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & . & . & . & 1 \\ 0 & 0 & 1 & . & . & . & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ 0 & 0 & . & . & . & 0 & 0 \end{pmatrix} \subseteq I.$$

Therefore $I = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}$.

(2) Assume that $I$ is a right Baer-ideal of $\mathbf{T_n}(\mathbf{R})$. Then there exists an idempotent $E \in \mathbf{T_n}(\mathbf{R})$ such that $r(I) = E\mathbf{T_n}(\mathbf{R})$. On the other hand by (i), we can see that

$$r_{T_n(R)}(I) = \begin{pmatrix} r_R(J_{11}) & r_R(J_{11}) & . & . & . & r_R(J_{11}) \\ 0 & r_R(J_{12}) & . & . & . & r_R(J_{12}) \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & r_R(J_{1n}) \end{pmatrix}.$$

Thus for each $1 \leq k \leq n$, $r(J_{1k}) = e_{kk}R$, where $e_{kk}$ is the $(k,k)$-th entries in $E$. Conversely, let for each $1 \leq k \leq n$, $J_{1k}$ be a right Baer-ideal of $R$. Then there is an $e_{1k} \in S_l(R)$ such that $r(J_{1k}) = e_{1k}R$. Consider matrix $F$, where for each $1 \leq k \leq n$, $f_{kk} = e_{1k}$ and elsewhere is zero. Then we have $IF = 0$. If $A \in r(I)$, then for each $1 \leq j \leq n$, $a_{kj} \in r(J_{1k})$. Hence there exists $c_{kj} \in R$ such that $a_{kj} = e_{1k}c_{kj} =$

$f_{kk}c_{kj}$, for all $1 \leq j \leq n$. Thus $A = FC \in F\mathbf{T_n}(\mathbf{R})$, where $C = [c_{kj}]$. Therefore $r(I) = F\mathbf{T_n}(\mathbf{R})$. Hence $I$ is a right Baer-ideal of $\mathbf{T_n}(\mathbf{R})$.

(3) By (2), this is evident. □

**Corollary 3.3.** The following statements hold.

(1) [18, Proposition 2]. $R$ is quasi-Baer if and only if $\mathbf{M_n}(\mathbf{R})$ is quasi-Baer.

(2) [17, Theorem 4.7]. $R$ is $n$-generalized right quasi-Baer if and only if $\mathbf{M_n}(\mathbf{R})$ is $n$-generalized right quasi-Baer.

*Proof.* (1) Let $R$ be quasi-Baer and $J \trianglelefteq \mathbf{M_n}(\mathbf{R})$. Then $J = \mathbf{M_n}(\mathbf{I})$ for some $I \trianglelefteq R$ and $I$ is a Baer-ideal. By Theorem 3.1, $J$ is a right Baer-ideal, hence $\mathbf{M_n}(\mathbf{R})$ is a quasi-Baer ring. Now let $I \trianglelefteq R$ and $\mathbf{M_n}(\mathbf{R})$ be quasi-Baer. Then $\mathbf{M_n}(\mathbf{I})$ is a right Baer-ideal of $\mathbf{M_n}(\mathbf{R})$. Again by Theorem 3.1, $I$ is a right Baer-ideal in $R$, thus $R$ is a quasi-Baer-ring.

(2) Assume that $J \trianglelefteq \mathbf{M_n}(\mathbf{R})$ and $R$ is $n$-generalized right quasi-Baer. Then $J = \mathbf{M_n}(\mathbf{I})$, where $I^n$ is a right Baer-ideal. By Theorem 3.1, $J^n = \mathbf{M_n}(\mathbf{I^n})$ is a right Baer-ideal. This shows that $\mathbf{M_n}(\mathbf{R})$ is $n$-generalized right quasi-Baer. The converse is evident. □

**Corollary 3.4.** [18, Proposition 9]. $R$ is quasi-Baer if and only if $\mathbf{T_n}(\mathbf{R})$ is quasi-Baer.

*Proof.* Let $J \trianglelefteq T_n(R)$. By Theorem 3.2,

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} & . & . & . & J_{1n} \\ 0 & J_{22} & J_{23} & . & . & . & J_{2n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & 0 & J_{nn} \end{pmatrix}.$$

By hypothesis, each $J_{ik}$ is a right Baer-ideal. Theorem 3.2, implies that $J$ is a right Baer-ideal. Thus $T_n(R)$ is quasi-Baer. The converse is evident. □

**Lemma 3.5.** [10, Lemma 2.3]. Let $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$ be an idempotent element of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

(1) $e \in S_l(T)$ if and only if

(a) $e_1 \in S_l(S)$;

(b) $e_2 \in S_l(R)$;

(c) $e_1 k = k$; and

(d) $e_1me_2 = me_2$, for all $m \in M$.

(2) $e_1k = k$ if and only if $eT \subseteq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T$.

(3) If $e_1me_2 = me_2$, for all $m \in M$, then $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T \subseteq eT$.

(4) If $e \in S_l(T)$, then $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} T = eT$.

**Lemma 3.6.** [10, Lemma 3.1]. Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then $r(J) = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}$ and $l(J) = \begin{pmatrix} l_S(I) \cap Ann_S(N) & l_M(L) \\ 0 & l_R(L) \end{pmatrix}$.

**Theorem 3.7.** Let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ be an ideal of $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$.

Then $J$ is a right Baer-ideal of $T$ if and only if

(1) $I$ is a right Baer-ideal of $S$;
(2) $r_M(I) = (r_S(I))M$; and
(3) $r_R(L) \cap Ann_R(N) = aR$, for some $a^2 = a \in R$.

*Proof.* Let $J$ be a right Baer-ideal of $T$. Then there exists $e \in S_l(T)$ such that $r(J) = eT$. By Lemma 3.5, $e = \begin{pmatrix} e_1 & k \\ 0 & e_2 \end{pmatrix}$, for some $e_1 \in S_l(S), e_2 \in S_l(R), k \in M$ and $kR = e_1kR$. Thus $e_1M = e_1M + kR$. By Lemma 3.5, $e_1S = r_S(I)$, $r_M(I) = e_1M = e_1SM = (r_S(I))M$ and $r_R(L) \cap Ann_R(N) = e_2R$.

Conversely, by hypothesis, there are $e_1 \in S_l(S)$ and $a^2 = a \in R$ such that $r_S(I) = e_1S$ and $r_R(L) \cap Ann_R(N) = aR$. Since $Ann_R(N) \trianglelefteq R$, then $a \in S_l(R)$. By (ii), $r_M(I) = (r_S(I))M = e_1M$. Now let $e = \begin{pmatrix} e_1 & 0 \\ 0 & a \end{pmatrix}$. Then $eT = \begin{pmatrix} e_1S & e_1M \\ 0 & aR \end{pmatrix} = \begin{pmatrix} r_S(I) & r_M(I) \\ 0 & r_R(L) \cap Ann_R(N) \end{pmatrix}$. From Lemma 3.6, $eT = r(J)$. Therefore $J$ is a right Baer-ideal of $T$. □

**Corollary 3.8.** [10, Theorem 3.2]. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent.

(1) $T$ is quasi-Baer.
(2) (i) $R$ and $S$ are quasi-Baer;
(ii) $r_M(I) = (r_S(I))M$ for all $I \trianglelefteq S$; and

(iii) If ${}_SN_R \leq_S M_R$, then we have $Ann_R(N) = aR$ for some $a^2 = a \in R$.

*Proof.* 1 $\Rightarrow$ 2. Let $I \trianglelefteq S$, $N$ be a $(S, R)$ submodule of $M$ and $J \trianglelefteq R$. Then $\begin{pmatrix} I & M \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$ are Baer-ideals of $T$. By Theorem 3.7, $I$ and $J$ are Baer-ideals, hence $R, S$ are quasi-Baer and $r_R(0) \cap Ann_R(N) = Ann_R(N) = aR$, for some $a^2 = a \in R$.

2 $\Rightarrow$ 1. let $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix} \trianglelefteq T$. By hypothesis, there are $a, e \in S_l(R)$ such that $Ann_R(N) = aR$, $r_R(L) = eR$ and $I$ is a Baer-ideal. Hence $r_R(L) \cap Ann_R(N) = r(R(1-e)) \cap r(R(1-a)) = eaR$. By Theorem 3.7, $J = \begin{pmatrix} I & N \\ 0 & L \end{pmatrix}$ is a Baer-ideal, thus $T$ is a quasi-Baer ring. □

**Corollary 3.9.** [17, Theorem 4.3]. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent.

(1) $T$ is $n$-generalized right (principally) quasi-Baer.
(2) (i) $S$ is $n$-generalized right quasi-Baer;
(ii) $r_M(I^n) = (r_S(I^n))M$ for all $I \trianglelefteq S$; and
(iii) If $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \trianglelefteq T$, then there is some $e^2 = e \in R$ such that

$$r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(i^{n-2}NJ) \cap ... \cap Ann_R(NJ^{n-1}) = eR.$$

*Proof.* 1 $\Rightarrow$ 2. (i), (ii) Let $I \trianglelefteq S$. Then $\begin{pmatrix} I^n & I^{n-1}M \\ o & o \end{pmatrix}$ is a Baer-ideal of $T$. By Theorem 3.7, $I^n$ is a Baer-ideal in $S$, hence $S$ is $n$-generalized right (principally) quasi-Baer and $r_M(I^n) = (r_S(I^n))M$.

(iii) If $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \trianglelefteq T$. Then $\begin{pmatrix} I^n & I^{n-1}N + I^{n-2}NJ + ... + NJ^{n-1} \\ 0 & J^n \end{pmatrix}$ is a Baer-ideal in $T$. By Theorem 3.7, there is some $e^2 = e \in R$ such that

$$r_R(J^n) \cap Ann_R(I^{n-1}N) \cap Ann_R(I^{n-2}NJ) \cap ... \cap Ann_R(NJ^{n-1}) = eR.$$

2 $\Rightarrow$ 1. Let $K = \begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \trianglelefteq T$. By hypothesis and Theorem 3.7, $K^n = \begin{pmatrix} I^n & I^{n-1}N + I^{n-2}NJ + ... + NJ^{n-1} \\ 0 & J^n \end{pmatrix}$ is a Baer-ideal in $T$. Hence $T$ is $n$-generalized right (principally) quasi-Baer. □

Recall that a ring $R$ is a *right SA* if for each $I, J \trianglelefteq R$ there exists $K \trianglelefteq R$ such that $r(I) + r(J) = r(K)$ (see [5]).

**Theorem 3.10.** Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following are equivalent.

(1) $T$ is a right $SA$-ring.

(2) (i) For $I_1, I_2 \trianglelefteq S$, there exists $I_3 \trianglelefteq S$, such that $r_M(I_1)+r_M(I_2) = r_M(I_3)$, $r_S(I_1) + r_S(I_2) = r_S(I_3)$ (i.e., $S$ is right $SA$); and
(ii) For each $I, J \trianglelefteq R$ and $(S, R)$ submodules $N_1, N_2$, of $M$, there are $K \trianglelefteq R$ and ${}_SN_R \leq_S M_R$, such that

$$r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).$$

*Proof.* 1 ⇒ 2. (i) Let $I_1, I_2 \trianglelefteq S$. Then $\begin{pmatrix} I_1 & M \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} I_2 & M \\ 0 & 0 \end{pmatrix}$ are ideals of $T$. By hypothesis, there is $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix} \trianglelefteq T$ such that

$$r(\begin{pmatrix} I_1 & M \\ 0 & 0 \end{pmatrix}) + r(\begin{pmatrix} I_2 & M \\ 0 & 0 \end{pmatrix}) = r(\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}).$$

By Lemma 3.6, we have $r_S(I_1) + r_S(I_2) = r_S(I)$ and $r_M(I_1) + r_M(I_2) = r_M(I)$.

(ii) Let $I, J \trianglelefteq R$ and $N_1, N_2$ are $(S, R)$ submodules of $M$. Then $\begin{pmatrix} 0 & N_1 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} 0 & N_2 \\ 0 & J \end{pmatrix}$ are ideals of $T$. By hypothesis, there are $K \trianglelefteq R$, $I \trianglelefteq S$ and ${}_SN_R \leq_S M_R$, such that

$$r(\begin{pmatrix} 0 & N_1 \\ 0 & I \end{pmatrix}) + r(\begin{pmatrix} 0 & N_2 \\ 0 & J \end{pmatrix}) = r(\begin{pmatrix} I & N \\ 0 & K \end{pmatrix}).$$

Now, Lemma 3.6, implies that

$$r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).$$

2 ⇒ 1. Suppose that $K_1 = \begin{pmatrix} I_1 & N_1 \\ 0 & J_1 \end{pmatrix}$ and $K_2 = \begin{pmatrix} I_2 & N_2 \\ 0 & J_2 \end{pmatrix}$ are two ideals of $T$. By Lemma 3.6, we have $r(K_1) + r(K_2) =$

$$\begin{pmatrix} r_S(I_1) + r_S(I_2) & r_M(I_1) + r_M(I_2) \\ 0 & r_R(J_1) \cap Ann_R(N_1) + r_R(J_2) \cap Ann_R(N_2) \end{pmatrix}.$$

By hypothesis, there are $I_3 \trianglelefteq S$, $K \trianglelefteq R$ and ${}_SN_R \leq_S M_R$, such that

$$r_S(I_1) + r_S(I_2) = r_S(I_3), r_M(I_1) + r_M(I_2) = r_M(I_3),$$

and

$$r_R(J_1) \cap Ann_R(N_1) + r_R(J_2) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N).$$

Therefore, by Lemma 3.6, $r(K_1) + r(K_2) = r(\begin{pmatrix} I_3 & N \\ 0 & K \end{pmatrix}$. □

**Corollary 3.11.** The following statements hold.

(1) Let $R = S$ and for every $I \trianglelefteq S$, $r_M(I) = (r_S(I))M$. Then $T$ is right $SA$ if and only if $R$ is right $SA$.
(2) Let $R = S$ and $M \trianglelefteq R$, then $T$ is right $SA$ if and only if $R$ is right $SA$.
(3) Let $S = M$. Then $T$ is right $SA$ if and only if $S$ is a right $SA$ and for each $I, J \trianglelefteq R$ and $N_1, N_2 \trianglelefteq S$, there are $K \trianglelefteq R$ and $N \trianglelefteq S$, such that $r_R(I) \cap Ann_R(N_1) + r_R(J) \cap Ann_R(N_2) = r_R(K) \cap Ann_R(N)$.

*Proof.* This is a consequence of Theorem 3.10. □

## 4. Baer-ideals in semiprime ring and ring of continuous functions

In this section, first, we show that an ideal $I$ of $C(X)$ is a Baer-ideal if and only if $int \bigcap_{f\in I} Z(f)$ is a clopen subset of $X$. Then we show that an ideal $I$ of semiprime ring $R$ is a Baer-ideal if and only if $intV(I)$ is a clopen subset of $Spec(R)$. Also we prove that the product of two Baer-ideals in a semiprime ring $R$ is a Baer-ideal.

A non-zero ideal $I$ of $R$ is an *essential ideal* if for any ideal $J$ of $R$, $I \cap J = 0$ implies that $J = 0$. Also an ideal $P$ of a commutative ring $R$ is called *pseuodoprime ideal* if $ab = 0$, implies that $a \in P$ or $b \in P$ (see [13]).

We denote by $C(X)$, the ring of all real-valued continuous functions on a completely regular Hausdorff space $X$. For any $f \in C(X)$, $Z(f) = \{x \in X : f(x) = 0\}$ is called a zero-set. We can see that a subset $A$ of $X$ is clopen if and only if $A = Z(f)$ for some idempotent $f \in C(X)$. For any subset $A$ of $X$ we denote by $intA$ the interior of $A$ (i.e., the largest open subset of $X$ contained in $A$). For terminology and notations, the reader is referred to [12] and [14].

**Lemma 4.1.** For $I, J \trianglelefteq C(X)$, $r(I) = r(J)$ if and only if $int \bigcap_{f\in I} Z(f) = int \bigcap_{g\in J} Z(g)$.

*Proof.* ($\Rightarrow$) Let $x \in int \bigcap_{f\in I} Z(f)$. Then $x \notin X \setminus int \bigcap_{f\in I} Z(f)$. By completely regularity of $X$, there exists $h \in C(X)$ such that $x \in X \setminus Z(h) \subseteq int \bigcap_{f\in I} Z(f)$. Therefore $fh = 0$ for all $f \in I$. This implies that $h \in r(I) = r(J)$. Hence $gh = 0$ for each $g \in J$. Thus $x \in X \setminus intZ(h) \subseteq int \bigcap_{g\in J} Z(g)$. Similarly, we can prove that $int \bigcap_{g\in J} Z(g) \subseteq int \bigcap_{f\in I} Z(f)$.

($\Leftarrow$) Suppose that $h \in r(I)$. Then $X \setminus Z(h) \subseteq int \bigcap_{f\in I} Z(f)$, so $X \setminus Z(h) \subseteq Z(f)$ for all $f \in I$. Hence for each $f \in I$, $fh = 0$. This implies that $r(I) \subseteq r(J)$. Similarly, we can prove that $r(J) \subseteq r(I)$. □

**Proposition 4.2.** The following statements hold.

(1) An ideal $I$ of $C(X)$ is a Baer-ideal if and only if $int\bigcap_{f\in I} Z(f)$ is a clopen subset of $X$.
(2) Every pseudoprime ideal of $C(X)$ is a Baer-ideal.

*Proof.* (1) Let $I$ be a Baer-ideal of $C(X)$. Then there exists an idempotent $e \in C(X)$ such that $r(I) = eC(X) = r(C(X)(1-e))$. By Lemma 4.1, $int\bigcap_{f\in I} Z(f) = intZ(1-e) = Z(1-e)$. This shows that $int\bigcap_{f\in I} Z(f)$ is a clopen subset of $X$. Now let $int\bigcap_{f\in I} Z(f)$ is a clopen subset of $X$. Then there exists an idempotent $e \in C(X)$ such that $int\bigcap_{f\in I} Z(f) = Z(e) = intZ(e)$. By Lemma 4.1, $r(I) = r(e) = (1-e)C(X)$. Hence $I$ is a Baer-ideal.

(2) By [1, Corollary 3.3], every pseudoprime ideal in $C(X)$ is either an essential ideal or a maximal ideal which is at the same time a minimal prime ideal. Now let $P$ be a pseudoprime ideal in $C(X)$. If $P$ is essential, then by [1, Theorem 3.1], $int\bigcap_{f\in P} Z(f) = \emptyset$, so (i), implies that $P$ is a Baer-ideal. Otherwise $P$ is a maximal ideal which is also a minimal prime ideal. Then there exists an isolated point $x \in X$ such that $P = M_x = \{f \in C(X) : x \in Z(f)\}$. This shows that $int\bigcap_{f\in P} Z(f) = \{x\}$ is a clopen subset of $X$, so $P$ is a Baer-ideal. □

Recall that a topological space $X$ is *extremally disconnected* if the interior of any closed subset is closed, see [14, 1.H]. The next result is proved in [2, Theorem 3.5] and [21, Theorem 2.12]. Now we give a new proof.

**Corollary 4.3.** $C(X)$ is a Baer-ring if and only if $X$ is an extremally disconnected space.

*Proof.* Let $F$ be a closed subset of $X$ and $C(X)$ is a Baer-ring. By completely regularity of $X$, there exists an ideal $I$ of $C(X)$ such that $F = \bigcap_{f\in I} Z(f)$. By Proposition 4.2, $intF$ is closed, hence $X$ is extremally disconnected. Conversely, suppose that $I \trianglelefteq C(X)$. Then $int\bigcap_{f\in I} Z(f)$ is closed. By Proposition 4.2, $I$ is a Baer-ideal, thus $C(X)$ is a Baer-ring. □

For any $a \in R$, let $supp(a) = \{P \in Spec(R) : a \notin P\}$. Shin [19, Lemma 3.1] proved that for any $R$, $\{supp(a) : a \in R\}$ forms a basis of open sets on Spec(R). This topology is called *hull-kernel topology.* We mean of $V(I)$ is the set of $P \in Spec(R)$, where $I \subseteq P$. Note that $V(I) = \bigcap_{a\in I} V(a)$.

**Lemma 4.4.** [5, Lemma 4.2]. The following statements hold.

(1) If $I$ and $J$ are two ideals of a semiprime ring $R$, then $r(I) = r(J)$ if and only if $intV(I) = intV(J)$.
(2) $A \subseteq Spec(R)$ is a clopen subset if and only if there exists an idempotent $e \in R$ such that $A = V(e)$.

**Proposition 4.5.** Let $R$ be a semiprime ring.

(1) An ideal $I$ of $R$ is a Baer-ideal if and only if $intV(I)$ is a clopen subset of $Spec(R)$.
(2) The product of two Baer-ideals is a Baer-ideal.
(3) If $R$ is a commutative ring, then any essential ideal of $R$ is a Baer-ideal.

*Proof.* (1) Let $I$ be a Baer ideal of $R$. Then there exists an idempotent $e \in R$ such that $r(I) = eR = r(R(1-e))$. By Lemma 4.4, $intV(I) = intV(1-e) = V(1-e)$. Thus $intV(I)$ is closed. Conversely, let $I \trianglelefteq R$. By hypothesis and Lemma 4.4, there exists an idempotent $e \in R$ such that $intV(I) = V(e)$. So, Lemma 4.4, implies that $r(I) = r(Re) = (1-e)R$. Therefore, $I$ is a right Baer-ideal. By semiprime hypothesis, $I$ is a left Baer-ideal.

(2) Let $I, J$ be two Baer-ideals of $R$. Then there are idempotents $e, f \in R$ such that $r(I) = eR$ and $r(J) = fR$. We will prove $r(IJ) = fR + eR + feR$. By Lemma 2.4, there exists $h \in S_l(R)$ such that $r(IJ) = fR + eR + feR = hR$. Therefore $IJ$ is a right Baer-ideal. By semiprime hypothesis, $IJ$ is a left Baer-ideal. Now let $x \in r(IJ)$. Then $Jx \subseteq r(I) = r(R(1-e))$. So $R(1-e)Jx = 0$. This implies that $(JxR(1-e))^2 = 0$. Since $R$ is semiprime, we have $JxR(1-e) = 0$. Thus $x(1-e) \in r(J) = r(R(1-f))$. Hence $(1-f)x(1-e) = 0$. This shows that $x = -fxe + fx + xe = fexe + exe + fx \in feR + eR + fR$. On the other hand we have $(IJ)(feR + eR + fR) = 0$, so $feR + eR + fR \subseteq r(IJ)$.

(3) It is easily seen that an ideal $I$ of a commutative semiprime ring $R$ is essential if and only if $r(I) = 0 = r(R)$. Now, Lemma 4.4, implies that $I$ is a Baer-ideal. □

Now we apply the theory of Baer ideals to give the following well-known result.

**Corollary 4.6.** Let $R$ be a semiprime ring. Then $R$ is quasi-Baer if and only if $Spec(R)$ is extremally disconnected.

*Proof.* Let $A$ be a closed subset of $Spec(R)$ and $R$ is quasi-Baer. Since $\{V(a) : a \in R\}$ is a base for closed subsets in Spec(R), there exists $S \subseteq R$ such that $A = \bigcap_{a \in S} V(a)$. Take $I = RSR$. Then $A = V(I)$. By Lemma 4.5, $intA$ is closed. Thus $Spec(R)$ is extremally disconnected.

Conversely, let $I \trianglelefteq R$. We know that $V(I)$ is a closed subset of $Spec(R)$. By hypothesis and Lemma 4.5, $intV(I)$ is a clopen subset of $Spec(R)$, and hence $I$ is Baer-ideal. Thus $R$ is a quasi Baer-ring. □


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