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APPROXIMATE IDENTITY IN CLOSED CODIMENSION ONE IDEALS OF SEMIGROUP ALGEBRAS

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ABSTRACT. Let S be a foundation semigroup with identity and $M_a(S)$ be its semigroup algebra. In this paper, we give necessary and sufficient conditions for the existence of a bounded approximate identity in closed codimension one ideals of semigroup algebra $M_a(S)$ of a locally compact topological foundation semigroup with identity.

1. Introduction

Throughout this paper, S denotes a locally compact Hausdorff topological semigroup. The space of all bounded complex regular Borel measures on S is denoted by $M(S)$. This space with the convolution multiplication $*$ and the total variation norm defines a Banach algebra. The space of all measures $\mu \in M(S)$ for which the maps $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous is denoted by $M_a(S)$ (or $\widetilde{L}(S)$ as in [1]), where δ_x denotes the Dirac measure at x. It is well-known that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$ (see for example [1]). S is called *foundation semigroup* if S coincides with the closure of the set $\bigcup\{\text{supp}(\mu) : \mu \in M_a(S)\}$. This family of semigroups is quite extensive and it contains topological groups and discrete semigroups as elementary examples.

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Denote by $L^\infty(S, M_a(S))$ the set of all complex-valued bounded functions g on S that are μ-measurable for all $\mu \in M_a(S)$. We identify functions in $L^\infty(S, M_a(S))$ that agree μ-almost everywhere for all $\mu \in M_a(S)$. For every $g \in L^\infty(S; M_a(S))$, define $\|g\|_\infty = \sup\{\ \|g\|_{\infty,|\mu|} : \mu \in M_a(S)\ \}$, where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(S, M_a(S))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra. The duality

$$\tau(g)(\mu) := \mu(g) = \int_S g\ d\mu$$

defines a linear mapping τ from $L^\infty(S, M_a(S))$ into $M_a(S)^*$. It is well-known that if S is a foundation semigroup with identity, then τ is an isometric isomorphism of $L^\infty(S, M_a(S))$ onto $M_a(S)^*$; see Proposition 3.6 of Sleijpen [8].

Let S be a foundation semigroup with identity. A *semicharacter* ρ is a non-zero complex function on S satisfying $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in S$. We denote by $\hat{S}$ the set of all bonded and continuous semicharacters on S. For each bounded and continuous semicharacters $\rho \in \hat{S}$, denote by $I_{0,\rho}(M(S))$ the closed codimension one ideal $\{\mu \in M(S) : \phi_\rho(\mu) := \int_S \rho(x)\ d\mu(x) = 0\}$, and write

$$I_{0,\rho}(M_a(S)) := M_a(S) \cap I_{0,\rho}(M(S)).$$

From Lemma 2.2 of [7], we have there exists a bijective between $\hat{S}$ and the set of all closed codimension one ideal in semigroup algebras $M_a(S)$. Indeed; any closed codimension one ideal of $M_a(S)$ is the form $I_{0,\rho}(M_a(S))$ for some semicharacter $\rho \in \hat{S}$.

Remark 1.1. *Note that Lemma 2.2 of* [7] *is not in general valid if the hypothesis that S is foundation is dropped. For example, the set* $S = [0, 1]$ *with the operation* $xy = min\{x, y\}$ *and the usual topology of the real line is a non-foundation compact semigroup with identity such that* $\hat{S} = \{\chi_S\}$*, where* $\chi_S(x) = 1$ *for all* $x \in S$*, but* $M_a(S)$ *has not any codimension one ideal.*

Recall that a net $(\upsilon_\alpha) \subseteq I_{0,\rho}(M_a(S))$ is a bounded approximate identity for $I_{0,\rho}(M_a(S))$ if there is a constant $M > 0$ such that $||\upsilon_\alpha|| \leq M$ for all α and $||\nu * \upsilon_\alpha - \nu|| \to 0$ for all $\nu \in I_{0,\rho}(M_a(S))$.

In this paper, we give necessary and sufficient condition for the existence of a bounded approximate identity in closed codimension one ideal $I_{0,\rho}(M_a(S))$ in semigroup algebras $M_a(S)$ of a foundation semigroup S with identity.

2. The results

In proving our result we will make use of a modification of the following condition, in the group case known as Reiter's condition P_1. Recall that a locally compact semigroup S satisfy the condition P_1, if for each $\varepsilon > 0$ and every compact subset $C \subseteq S$ there exists some positive measure $\mu \in M_a(S)$ with the properties $\|\mu\| \leq 1$ and $||\delta_x * \mu - \mu|| < \varepsilon$ for all $x \in C$.

Definition 2.1. We say that the P_1-condition with bound M is satisfy in $\rho \in \hat{S}$ *(*$P_1(\alpha, M)$-condition for short*)* if for each $\varepsilon > 0$ and every compact subset C of S there exists some $\mu \in M_a(S)$ such that $\phi_\rho(\mu) = 1$, $||\mu|| \leq M$ and $||\delta_x * \mu - \rho(x)\mu|| < \varepsilon$ for all $x \in C$.

We also, say that the P_1^*-condition with bound M is satisfy *(*$P_1^*(\alpha, M)$ for short*)* if the above condition happens for a finite subset C of S.

Proposition 2.2. *The condition $P_1(\rho, M')$ follows from the condition $P_1^*(\rho, M)$, where M' depends only on M and ρ.*

Proof. Let $\mu \in M_a(S)$, $C \subseteq S$ be a compact subset and $\varepsilon > 0$. Since S is a foundation semigroup with identity, the map $x \mapsto \delta_x * |\mu|$ from S into $M_a(S)$ are norm continuous, and so we can choose finitely many open neighbourhoods $U_i = U(x_i)$, $i = 1, 2, ..., n$ are such that $C \subseteq \bigcup U_i$ and

$$|\rho(x) - \rho(x')| < \frac{\varepsilon}{3M||\mu||}\ , \qquad ||\delta_x * \mu - \delta_{x'} * \mu|| < \frac{\varepsilon}{3M}$$

for $x, x' \in U_i$.

The condition condition $P_1^*(\rho, M')$ ensures the existence of then $\mu \in M_a(S)$ with $\phi_\rho(\mu) = 1$, $||\mu|| \leq M$ and $||\delta_{x_i} * \mu - \rho(x_i)\mu|| < \frac{\delta}{3}$ for $i = 1, 2, ..., n$. For $x \in C$ there is a set U_j with $x \in U_j$. We have

$$\begin{aligned} ||\delta_x * \mu - \rho(x)\mu|| \ &\leq \ ||\ \delta_x * \mu - \delta_{x_j} * \mu|| \\ &+ \ ||\ \delta_{x_j} * \mu - \rho(x_j)\mu|| \\ &+ \ |\rho(x) - \rho(x')|\ ||\mu|| < \varepsilon, \end{aligned}$$

as required. □

Proposition 2.3. *Let S be a foundation semigroup with identity, $\rho \in \hat{S}$ and let the condition $P_1(\rho, M)$ be satisfied. Let $\{\mu_1, \mu_2, ..., \mu_n\}$ be a finite subset of $I_{0,\rho}(M_a(S))$. Then for every ε there is $\nu \in M_a(S)$ with $\phi_\rho(\nu) = 1$, $||\nu|| \leq M$ and $||\mu_i * \nu|| < \varepsilon$, $i = 1, ..., n$.*

Proof. Given $\varepsilon > 0$, there exists a compact subset K of S such that $|\mu_i|(K^c) < \varepsilon/4M$, $i = 1, ..., n$. Since the condition $P_1(\rho, M)$ satisfy, there is some $\nu \in M_a(S)$ with $\phi_\rho(\nu) = 1$, $||\nu|| \le M$ and

$$||\delta_x * \nu - \rho(x)\nu|| < \frac{\varepsilon}{2(1+N)}, \ (x \in K),$$

where $N = \max\{||\mu_1||, ..., ||\mu_n||\}$.

We note that $\mu_i \in I_{0,\rho}(M_a(S))$ and so $\phi_\rho(\mu_i) = \int_S \rho(x)\ d\mu_i(x) = 0$ for $i = 1, ..., n$. Now, let $f \in C_0(S)$ with $||f||_\infty = 1$, than we have

$$\begin{aligned} | < \mu_i * \nu,\ f > | &= | \int_S < \delta_y * \nu,\ f > \ d\mu_i(y)| \\ &= | \int_S < \delta_y * \nu,\ f > \ d\mu_i(y) \\ &- \int_S < \rho(y)\nu,\ f > \ d\mu_i(y)| \\ &= | \int_S < \delta_y * \nu \\ &- \rho(y)\nu,\ f > \ d\mu_i(y)| \\ &\le \int_S ||f||_\infty ||\delta_y * \nu - \rho(y)\nu||\ d|\mu_i|(y) \\ &\le \int_K ||\delta_y * \nu - \rho(y)\nu||\ d|\mu_i|(y) \\ &+ \int_{K^c} ||\delta_y * \nu - \rho(y)\nu||\ d|\mu_i|(y) \\ &\le \frac{\varepsilon}{2(1+N)} \int_K \ d|\mu_i|(y) \\ &+ ||\delta_y * \nu - \rho(y)\nu|| \int_{K^c} \ d|\mu_i|(y) \\ &\le \frac{\varepsilon}{2(1+N)} ||\mu_i|| + 2||\nu||\ |\mu_i|(K^c) < \varepsilon. \end{aligned}$$

This implies that $||\mu_i * \nu|| < \varepsilon$, $i = 1, ..., n$.

□

Theorem 2.4. *Let S be a foundation semigroup with identity and $\rho \in \hat{S}$. Then $I_{0,\rho}(M_a(S))$ has a bounded approximate identity bound by M if and only if $P_1(\rho, M')$ is satisfied, where M' depends only on M and ρ.*

Proof. In order to show that there is an approximate identity in the codimension one ideal $I_{0,\rho}(M_a(S))$, we have to prove that for every finite set $\{\mu_1, ..., \mu_n\} \subseteq I_{0,\rho}(M_a(S))$ and every $\varepsilon > 0$ there is a $\nu \in I_{0,\rho}(M_a(S))$ such that $||\mu_i * \nu - \mu_i|| < \varepsilon$ for $i = 1, ..., n$ (see [5], P.

3). Let $\{\mu_1, ..., \mu_n\} \subseteq I_{0,\rho}(M_a(S))$ be a finite set and $\varepsilon > 0$. By Proposition 2.3, there exists $\nu \in M_a(S)$ with $\phi_\rho(\nu) = 1$, $||\nu|| \leq M'$ and $||\mu_i * \nu|| < \varepsilon/2$, $i = 1, ..., n$.

Let (μ_α) be a bounded approximate identity for $M_a(S)$ bounded by one (see [6]), then there is a α_0 such that for $i = 1, ..., n$

$$||\mu_i * \mu_\alpha - \mu_i|| < \varepsilon/2 \quad \text{for all} \quad \alpha > \alpha_0.$$

We set $\lambda := \mu_{\alpha_0} - \nu * \mu_{\alpha_0}$. Since

$$\phi_\rho(\lambda) = \phi_\rho(\mu_{\alpha_0}) - \phi_\rho(\nu)\phi_\rho(\mu_{\alpha_0}) = 0,$$

the element λ is in $I_{0,\rho}(M_a(S))$. Furthermore we have for all $i = 1, ..., n$

$$||\mu_i*\lambda-\mu_i|| = ||\mu_i*\mu_{\alpha_0}-\mu_i*\nu*\mu_{\alpha_0}-\mu_i|| \leq ||\mu_i*\mu_{\alpha_0}-\mu_i||+||\mu_i*\nu|| < \varepsilon$$

and $||\lambda|| = ||\mu_{\alpha_0} - \nu * \mu_{\alpha_0}|| \leq 1 + M'$. This shows our assertion.

Conversely, Let (υ_α) be a bounded approximate identity in the codimension one ideal $I_{0,\rho}(M_a(S))$ with bound $M \geq 0$ and let $\nu \in M_a(S)$ be a measure with $\phi_\rho(\nu) = 1$. we define a net by $\nu_\alpha := \nu - \nu * \upsilon_\alpha$, $\alpha \in \Lambda$. It is clear that $\phi_\rho(\nu_\alpha) = 1$ and $||\nu_\alpha|| \leq ||\nu||(1 + M) := M'$.

Let $C \subseteq S$ be a given compact set $\varepsilon > 0$. By the continuity of the map $x \mapsto \delta_x*|\nu|$ from S into $M_a(S)$ and the continuity of semicharacter $\rho \in \hat{S}$, there exists $y_1, y_2, ..., y_n \in C$ and open neighbourhoods $U_i := U(y_i)$ of y_i, $i = 1, ..., n$ with $C \subseteq \bigcup U_i$ such that

$$||\delta_y * \nu - \delta_{y_i} * \nu|| < \frac{\varepsilon}{3(1 + M)}, \quad |\rho(y) - \rho(y_i)| < \frac{\varepsilon}{3M'}$$

for all $y \in U_i$, where $1 \leq i \leq n$. Set $\nu_y := \delta_y * \nu - \rho(y)\nu$ for all $y \in \{y_1, ..., y_n\}$. It is clear that $\nu_y \in I_{0,\rho}(M_a(S))$. Moreover, we have

$$\delta_y * \nu_\alpha - \rho(y)\nu_\alpha = \nu_y - \nu_y * \upsilon_\alpha.$$

Since (υ_α) is an approximate identity for $I_{0,\rho}(M_a(S))$ there is $\alpha_o \in \Lambda$ with $||\nu_{y_i} - \nu_{y_i} * \upsilon_{\alpha_0}|| < \varepsilon/3$ for all $i = 1, ..., n$, and so $||\delta_{y_i} * \nu_{\alpha_0} - \rho(y_i)\nu_{\alpha_0}|| < \varepsilon/3$ for all $i = 1, ..., n$. Applying the triangle inequality we end up with

$$||\delta_y * \nu_{\alpha_0} - \rho(y)\nu_{\alpha_0}|| < \varepsilon \qquad (y \in C).$$

This completes the proof. □

Before the following theorem, recall that a locally compact foundation semigroup with identity S is *left ρ-amenable* if there exists a mean m on $L^\infty(S; M_a(S))$ such that $m(\rho) = 1$ and $m({}_xf) = \rho(x)m(f)$ for all $x \in S$ and $f \in L^\infty(S; M_a(S))$.

Theorem 2.5. *Let S be a foundation semigroup with identity and $\rho \in \hat{S}$. There is a bounded approximate identity with bound $M > 0$ in $I_{0,\rho}(M_a(S))$ if and only if S be left ρ-amenable.*

Proof. Assume there is a bounded approximate identity with bound M in $I_{0,\rho}(M_a(S))$. Then by 2.4 the condition $P_1(\rho, M)$ is satisfied. For $\varepsilon > 0$ and a compact set $C \subseteq S$ let $\mu \in M_a(S)$ according to $P_1(\rho, M)$. We define the functional $m_{\varepsilon,C}$ on $L^\infty(S; M_a(S))$ by

$$m_{\varepsilon,C}(f) = \int_S f(x)\, d\mu(x).$$

We have $m_{\varepsilon,C}(\rho) = \phi_\rho(\mu) = 1$ and

$$||m_{\varepsilon,C}|| \leq ||\mu|| \leq M.$$

Hence the functionals $m_{\varepsilon,C}$ are uniformly bounded. Moreover, for $y \in S$ we have

$$m_{\varepsilon,C}({}_y f) = \int_S f(yx)\, d\mu(x) =< f, \delta_y * \mu > .$$

Thus

$$\begin{aligned} |m_{\varepsilon,C}({}_y f) - \rho(y) m_{\varepsilon,C}(f)| &\leq |\int_S < f, \delta_y * \mu > - < f, \rho(y)\mu > \\ &\leq ||f||_\infty ||\delta_y * \mu - \rho(y)\mu|| \leq \varepsilon ||f||_\infty . \end{aligned}$$

The family of $m_{\varepsilon,C}$ form a net, where the indices (ε, C) are partially ordered by

$$(\varepsilon, C) \leq (\varepsilon', C') \quad \text{if} \quad \varepsilon' \leq \varepsilon, C \subset C'.$$

Let m be an accumulation point of this net. Clearly $||m|| \leq M$, $m(\rho) = 1$ and $m({}_x f) = \rho(x) m(f)$ for all $x \in S$ and $f \in L^\infty(S; M_a(S))$.

Conversely assume that there exists $m \in L^\infty(S; M_a(S))^*$ such that $m(\rho) = 1$, $||m|| \leq M$ and $m({}_x f) = \rho(x) m(f)$ for all $x \in S$ and $f \in L^\infty(S; M_a(S))$. By the Goldstine theorem there is a net $(\mu_\alpha) \subseteq M_a(S)$ bounded by M, such that $\mu_\alpha \to m$ in the weak* topology. In particular, we have $< \mu_\alpha, \rho > \to < m, \rho >$. Since $m(\rho) = 1$ we can assume that $< \mu_\alpha, \rho >= 1$ for all α. Let $x \in S$ and $f \in L^\infty(S; M_a(S))$, then we have

$$< f,\, \delta_x * \mu_\alpha >=<_x f,\, \mu_\alpha > \to m({}_x f) = \rho(x) m(f).$$

Therefore

$$< f,\, \delta_x * \mu_\alpha - \rho(x)\mu_\alpha > \to 0.$$

Fix $x_1, ..., x_n \in S$ and set $F_{k,\alpha} := \delta_{x_k} * \mu_\alpha - \rho(x_k)\mu_\alpha$. The m-tuple

$$F_i = (F_{1,\alpha}, F_{2,\alpha}, ... F_{n,\alpha})$$

forms a net weakly convergent to 0 in the product space $M_a(S) \times M_a(S) \times ... \times M_a(S)$. It follows from Corollary 14, P. 422 of [3], that there is a sequence of convex combinations of F_i convergent to zero in norm. Hence for every $\varepsilon > 0$ there is a measure $\mu \in M_a(S)$, a convex linear combination of μ_α, such that $\phi_\rho(\mu) = 1$, $||\mu|| \leq M$ and

$$||\delta_x * \mu - \rho(x)\mu|| < \varepsilon \quad (i = 1, ..., n).$$

The proof is complete be Proposition 2.2. □

The next theorem will give us a sufficient condition for the existence of an approximate identity in $I_{0,\rho}(M_a(S))$ which is eventually unbounded.

Theorem 2.6. *Let S be a foundation semigroup with identity and $\rho \in \hat{S}$. Then $I_{0,\rho}(M_a(S))$ has an approximate identity if for any $\varepsilon > 0$ there exists some $\mu \in M_a(S)$ such that $\phi_\rho(\mu) = 1$ and $||\delta_y * \mu - \rho(y)\mu|| < \varepsilon$ for all $y \in S$.*

Proof. Let $\{\mu_1, ..., \mu_n\} \subseteq I_{0,\rho}(M_a(S))$ and every $\delta > 0$. Set $N := \max\{||\mu_1||, ..., ||\mu_n||\}$ and $\varepsilon = \frac{\delta}{1+N}$. Choose some $\mu \in M_a(S)$ such that $\phi_\rho(\mu) = 1$ and $||\delta_y * \mu - \rho(y)\mu|| < \varepsilon$ for all $y \in S$. Then an argument similar to Proposition 2.3 implies that $||\mu_i * \mu|| < \delta$ for $i = 1, ..., n$. Now we proceed with as in the proof of Theorem 2.4 Completes the proof.

□

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