Journal of Algebraic Systems
Vol. 2, No. 1, (2014), pp 61-65

LIFTING MODULES WITH RESPECT TO A PRERADICAL

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Abstract. Let M be a right module over a ring R, τ_M a preradical on $\sigma[M]$, and $N \in \sigma[M]$. In this note we show that if $N_1, N_2 \in \sigma[M]$ are two τ_M-lifting modules such that N_i is N_j-projective $(i, j = 1, 2)$, then $N = N_1 \oplus N_2$ is τ_M-lifting. We investigate when homomorphic image of a τ_M-lifting module is τ_M-lifting.

1. Introduction

Throughout this paper R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. For any module M, τ_M will denote a preradical in $\sigma[M]$. Recall that A is a *τ_M-cosmall submodule of B in N* if $B/A \subseteq \tau_M(N/A)$. According to [2], a module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$. In this note, we study some properties of τ_M-lifting modules, in particular, we survey when direct sum of τ_M-lifting modules is τ_M-lifting.

2. Main results

A module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$.

MSC(2010): Primary: 16D90; Secondary: 16D99
Keywords: Preradical, Hereditary, τ_M-lifting module.
Received: 28 September 2013, Revised: 20 April 2014.

Lemma 2.1. *Let $N \in \sigma[M]$. Then the following are equivalent:*

(1) *N is τ_M-lifting;*

(2) *For every submodule K of N, there is a direct summand A of N such that $A \subseteq K$ and $K/A \subseteq \tau_M(N/A)$;*

(3) *For every submodule K of N, there is a decomposition $N = A \oplus B$ such that $A \subseteq K$ and $B \cap K \subseteq \tau_M(N)$.*

Proof. See [2, Prposition 2.8]. □

Theorem 2.2. *Let $N_1, N_2 \in \sigma[M]$ be two τ_M-lifting modules such that N_i is N_j-projective ($i, j = 1, 2$). Then $N = N_1 \oplus N_2$ is τ_M-lifting.*

Proof. Let A be a submodule of N. Consider the submodule $N_1 \cap (A + N_2)$ of N_1. Since N_1 is τ_M-lifting, there exists a decomposition $N_1 = A_1 \oplus B_1$ such that $A_1 \leq N_1 \cap (A + N_2)$ and $N_1 \cap (A + N_2) \cap B_1 = B_1 \cap (A + N_2) \subseteq \tau_M(N_1)$. Then $N = N_1 \oplus N_2 = A_1 \oplus B_1 \oplus N_2 = A + (N_2 \oplus B_1)$. As $N_2 \cap (A + B_1) \leq N_2$ and N_2 is τ_M-lifting, there exists a decomposition $N_2 = A_2 \oplus B_2$ such that $A_2 \leq N_2 \cap (A + B_1)$ and $B_2 \cap (N_2 \cap (A + B_1)) = B_2 \cap (A + B_1) \subseteq \tau_M(N_2)$. Since $A_2 \leq A + B_1$, we have $N = A + B_1 \oplus N_2 = A + B_1 \oplus B_2$. But $N = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$ and $A_1 \oplus A_2$ is $(B_1 \oplus B_2)$-projective, thus there exists $A' \leq A$ such that $N = A' \oplus B_1 \oplus B_2$ by [3, 41.14]. Since $(B_1 \cap (A + B_2)) \oplus (B_2 \cap (A + B_1)) \subseteq \tau_M(N_1) \oplus \tau_M(N_2) = \tau_M(N)$ then $A \cap (B_1 \oplus B_2) \subseteq (B_1 \cap (A + B_2)) \oplus (B_2 \cap (A + B_1)) \subseteq \tau_M(N)$. Hence N is τ_M-lifting. □

Corollary 2.3. *Let $N_1, N_2 \in \sigma[M]$ be two projective τ_M-lifting modules. Then $N = N_1 \oplus N_2$ is τ_M-lifting.*

Proof. By Theorem 2.2. □

Recall that a submodule K of M is called *fully invariant* (denoted by $K \trianglelefteq M$) if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$. A module $N \in \sigma[M]$ is called a *duo module* provided every submodule of N is fully invariant.

Proposition 2.4. *Let $N \in \sigma[M]$ and $N = N_1 \oplus N_2$ be a duo module such that N_1 and N_2 are two τ_M-lifting modules. Then N is τ_M-lifting.*

Proof. Let A be a submodule of N. As A is fully invariant, we have $A = (A \cap N_1) \oplus (A \cap N_2)$. Since N_1 and N_2 are τ_M-lifting, there exist decompositions $A \cap N_1 = A_{11} \oplus A_{12}$ and $A \cap N_2 = A_{21} \oplus A_{22}$, where A_{11} is a direct summand of N_1, A_{21} is a direct summand of N_2, $A_{12} \subseteq \tau_M(N_1)$ and $A_{22} \subseteq \tau_M(N_2)$. Then $A_{11} \oplus A_{21}$ is a direct summand of N. Moreover, $A_{12} \oplus A_{22} \subseteq \tau_M(N_1) \oplus \tau_M(N_2) = \tau_M(N)$. Therefore N is τ_M-lifting. □

A preradical τ_M is called a *hereditary preradical* if for any submodule K of $N \in \sigma[M]$, $\tau_M(K) = \tau_M(N) \cap K$. But it is well known that if K is a direct summand of N, then $\tau_M(K) = \tau_M(N) \cap K$.

Proposition 2.5. *If τ_M is a hereditary preradical, then every submodule of a τ_M-lifting module is τ_M-lifting.*

Proof. Let N be τ_M-lifting and $K \leq N$. If $H \leq K \leq N$, then there exists a decomposition $H = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$. Thus A is a direct summand of K and $B \subseteq \tau_M(N) \cap H = \tau_M(H) \subseteq \tau_M(K)$. Therefore K is τ_M-lifting. □

Corollary 2.6. *If τ_M is hereditary, then every direct summand of a τ_M-lifting module is τ_M-lifting.*

Proof. By Proposition 2.5. □

Proposition 2.7. *Let τ_M be a hereditary preradical. Then the following conditions are equivalent:*
(i) *Every module is τ_M-lifting.*
(ii) *Every injective module is τ_M-lifting.*

Proof. By Proposition 2.5 and that every module is contained in an injective module. □

Proposition 2.8. *Let $N \in \sigma[M]$ be a τ_M-lifting module. Then:*
(i) *If K is a fully invariant submodule of N, then N/K is τ_M-lifting.*
(ii) *If K is a submodule of N such that the sum of K with any nonzero direct summand of N is a direct summand of N, then N/K is τ_M-lifting.*

Proof. (i) Let A/K be a submodule of N/K. Since N is τ_M-lifting, there exists a decomposition $N = H \oplus H'$ such that $A/H \subseteq \tau_M(N/H)$. As K is fully invariant, $N/K = (H + K)/K \oplus (H' + K)/K$. But $[A/K]/[(K + H)/K] \simeq [A/H]/[(K + H)/H] \subseteq [\tau_M(N/H)]/[(K + H)/H] \subseteq \tau_M([N/H]/[(K+H)/H]) \simeq \tau_M([N/K]/[(K+H)/K])$. Therefore N/K is τ_M-lifting.

(ii) Let A be a submodule of N such that $K \leq A$. If $A \subseteq \tau_M(N)$, then $A/K \subseteq \tau_M(N/K)$. Suppose that $A \not\subseteq \tau_M(N)$. Since N is τ_M-lifting, there exists a nonzero direct summand H of N such that $A/H \subseteq \tau_M(N/H)$. By hypothesis, $K + H$ is a direct summand of N since $K \neq 0$. It is obvious that $(K+H)/K$ is also a direct summand of N/K. Since $[A/K]/[(K + H)/K] \simeq [A/H]/[(K + H)/H]$, $[A/K]/[(K + H)/K]$ is isomorphism with a submodule of $\tau_M([N/K]/[(K+H)/K])$. Therefore N/K is τ_M-lifting. □

Theorem 2.9. *Let $N \in \sigma[M]$ and $N = N_1 \oplus N_2$. Assume that N_1 is τ_M-lifting and N_2 is N_1-projective. Then N is τ_M-lifting if and only if for every submodule K of N such that $K + N_1 \neq N$, there exists a direct summand H of N such that $K/H \subseteq \tau_M(N/H)$.*

Proof. $\Rightarrow$ Clear.

$\Leftarrow$ Let K be a submodule of N such that $K + N_1 = N$. Since N_2 is N_1-projective, there exists a submodule $K' \leq K$ such that $N = K' \oplus N_1$. Since $N/K' \simeq N_1$, N/K' is τ_M-lifting. Then there exists a direct summand H/K' of N/K' such that $K/H \simeq (K/K')/(H/K') \subseteq \tau_M([N/K']/[H/K']) \simeq \tau_M(N/H)$. Obviously, H is also a direct summand of N. Thus N is τ_M-lifting. □

Proposition 2.10. *An indecomposable module N in $\sigma[M]$ is τ_M-lifting if and only if for every proper submodule K of N we have $K \subseteq \tau_M(N)$.*

Proof. Clear. □

A preradical τ_M is called a *cohereditary preradical* if for any submodule K of $N \in \sigma[M]$, $\tau_M(N/K) = (\tau_M(N)+K)/K$. We denote $\chi_{\tau_M}(N)$ to be the sum of all submodules $K_i (i \in I)$ of N such that for every $i \in I$, there exists a submodule H_i of $\tau_M(N)$ with $K_i \simeq H_i$.

Proposition 2.11. *Let τ_M be a cohereditary preradical and $N \in \sigma[M]$ be a τ_M-lifting module. Then the module $N/\chi_{\tau_M}(N)$ is semisimple.*

Proof. Let $X = \chi_{\tau_M}(N)$. If K/X is a submodule of N/X, then there exist submodules H, H' of N such that $N = H \oplus H', H \subseteq K$ and $K/H \subseteq \tau_M(N/H)$. Hence $K = H \oplus (K \cap H')$ and $K \cap H' \simeq [H \oplus (K \cap H')]/H = K/H \subseteq \tau_M(N/H) = [\tau_M(N) + H]/H = [\tau_M(H') \oplus H]/H \simeq \tau_M(H')$. Thus $K \cap H' \subseteq X$, and so $N/X = (K/X) \oplus [(H' + X)/X]$. That is, K/X is a direct summand of N/X. Therefore N/X is semisimple. □

Recall that K is an *essential submodule* of M if, for all $0 \neq L \leq M, L \cap N \neq 0$.

Corollary 2.12. *Assume that τ_M is a cohereditary preradical. Let $N \in \sigma[M]$ be a τ_M-lifting module and for every simple submodule K of N we have $K \subseteq \tau_M(N)$. Then $\chi_{\tau_M}(N)$ is an essential submodule of N.*

Proof. Let $K \cap \chi_{\tau_M}(N) = 0$. Then K embeds in $N/\chi_{\tau_M}(N)$. By Proposition 2.11, N is semisimple and so by assumption $K \subseteq \chi_{\tau_M}(N)$. Hence $K = 0$. Thus $\chi_{\tau_M}(N)$ is an essential submodule of N. □

Lemma 2.13. *Let $N \in \sigma[M]$ be a τ_M-lifting module and K a submodule of N. Then either K contains a nonzero submodule H such that $H \subseteq \tau_M(N)$ or K is a semisimple direct summand of N.*

Proof. Suppose that K does not contain a nonzero submodule H such that $H \subseteq \tau_M(N)$. Let P be a submodule of K. Then $P = A \oplus B$ for some direct summand A of N and $B \subseteq \tau_M(N)$. But $B = 0$, and so $P = A$. Therefore K is a semisimple direct summand of N. □

Proposition 2.14. *Let $N \in \sigma[M]$ be a τ_M-lifting module. Then there exist a semisimple submodule N_1 and a submodule N_2 of N such that $N = N_1 \oplus N_2$ and every nonzero submodule of N_2 contains a nonzero submodule H such that H is isomorphism to a submodule of $\tau_M(N)$.*

Proof. By Zorn's Lemma, N contains a submodule N_1 maximal with respect to the property that it does not contain a nonzero submodule A such that $A \subseteq \tau_M(N)$. By Lemma 2.13, N_1 is a semisimple direct summand of N. Thus there exists a submodule N_2 such that $N = N_1 \oplus N_2$. Let P be a nonzero submodule of N_2. Then $N_1 \oplus P$ contains a nonzero submodule K such that $K \subseteq \tau_M(N)$, by the choice of N_1. But $K \cap N_1 \subseteq K \subseteq \tau_M(N)$ and so $K \cap N_1 = 0$. Therefore K embeds in P as that is required. □

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