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BEST APPROXIMATION IN QUASI TENSOR PRODUCT SPACE AND DIRECT SUM OF LATTICE NORMED SPACES

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Abstract. We study the thoery of best approximation in tensor product space and the direct sum of some lattice normed spaces $\mathbb{X}_i$. We introduce quasi tensor product space and discuss about the relation between tensor product space and this new space which we denote it by $\mathbb{X} \boxtimes \mathbb{Y}$. We investigate best approximation in direct sum of lattice normed spaces by elements which are not necessarily downward or upward and we call them I_m-quasi downward or I_m-quasi upward. We show that these sets can be interpreted as downward or upward sets. The relation of these sets with downward and upward subsets of the direct sum of lattice normed spaces $\mathbb{X}_i$ is discussed. This will be done by homomorphism functions. Finally, we introduce the best approximation of these sets.

1. Introduction

The theory of best approximation by elements of convex sets in the normed linear spaces, which has many important applications in mathematics and some other sciences, is well developed. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets. In this direction, Rubinov and Singer [7, 8] developed a theory of best approximation by elements of so-called normal sets in the non-negative orient $\mathbb{R}^I_+$, of a finite-dimensional coordinate space $\mathbb{R}^I$ endowed with the

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max- norm. Martinez-Legaz, Rubinov and Singer in [3] have developed a theory of best approximation of downward subsets of the space $\mathbb{R}^I$. Downward sets play an important role in some parts of mathematical economics (see e.g., [2]) and game theory. Also Mohebi and Rubinov [5] generalized these concepts and developed the theory of best approximation by closed normal and downward subsets of a Banach lattice $\mathbb{X}$ with a strong unit $\mathbf{1}$. Therefore study of these concepts in more detail and also examination of the effect of some special operators on normal and downward subsets of Banach lattice spaces, are useful for mathematicians. We use the concept of best approximation by downward subsets of Banach lattice $\mathbb{X}$, to introduce a theory of best approximation in two new spaces which we call them, quasi tensor product space and direct sum of lattice normed spaces. The structure of the paper is as follows: In Section 3 we present some preliminary results. In Section 2 we investigate best approximation in quasi tensor product of lattice normed spaces by elements of downward sets. In particular, we show that the least element of the set of best approximations exists. In Section 4 we investigate best approximation in direct sum of lattice normed spaces by elements which are called "I_m-quasi downward sets". Then we discuss about the relation of I_m-quasi downward sets, downward sets and upward sets. In Section 5 we define positive I_m-quasi downward sets and discuss about its relations to I_m-quasi downward sets.

2. Preliminaries

Let $\mathbb{X}$ be a normed vector space. For a nonempty subset $\mathbb{W}$ of $\mathbb{X}$ and $x \in \mathbb{X}$, define

$$d(x, \mathbb{W}) = inf_{w \in \mathbb{W}} \|x - w\|. \tag{1}$$

Recall that a point $w_0 \in \mathbb{W}$ is called a best approximation for $x \in \mathbb{X}$ if

$$\|x - w_0\| = d(x, \mathbb{W}). \tag{2}$$

If each $x \in \mathbb{X}$ has at least one best approximation $w_0 \in \mathbb{W}$, then $\mathbb{W}$ is called a proximinal subset of $\mathbb{X}$. Let $\mathbb{W} \subseteq \mathbb{X}$ and $x \in \mathbb{X}$, we denote by $\mathbf{P}_{\mathbb{W}}(x)$, the set of all best approximations of x in $\mathbb{W}$. Therefore

$$\mathbf{P}_{\mathbb{W}}(x) := \{w \in \mathbb{W} : \ \|x - w\| = d(x, \mathbb{W})\}. \tag{3}$$

It is well-known that if $\mathbb{W}$ is closed then $\mathbf{P_W}(x)$ is a closed and bounded subset of $\mathbb{X}$. If $x \in \mathbb{X}$ then $\mathbf{P}_{\mathbb{W}}(x)$ is located in the boundary of $\mathbb{W}$. Let $\mathbb{X}$ be a lattice vector space with the strong unit $\mathbf{1}$. Using $\mathbf{1}$, we define a norm on $\mathbb{X}$ by

$$\|x\| := inf\{\lambda \geq 0 :\mid x \mid \leq \lambda\mathbf{1}\}, \tag{4}$$

and notice the ball

$$B(x,r) = \{y \in \mathbb{X} : x - r\mathbf{1} \leq y \leq x + r\mathbf{1}\}. \tag{5}$$

It is clear that

$$\mid x \mid \leq \|x\|\mathbf{1} \qquad \forall x \in \mathbb{X}. \tag{6}$$

Example 2.1. Let X be a vector lattice with a strong unit $\mathbf{1}$. The latter means that for each $x \in \mathbb{X}$ there exists $\lambda \in \mathbb{R}$ such that $|x| \leq \lambda\mathbf{1}$ and define

$$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda\mathbf{1}\}.$$

It is well known (see, for example, [10]) that each vector lattice X with a strong unit is isomorphic as a vector ordered space to the space $C(Q)$ of all continuous functions defined on a compact topological space Q. For a given strong unit $\mathbf{1}$ the corresponding isomorphism ψ can be chosen in such a way that $\psi(\mathbf{1})(q) = 1$ for all $q \in Q$. The cone $\psi(K)$ coincides with the cone of all nonnegative functions defined on Q. If $X = C(Q)$ and $\mathbf{1}(q) = 1$ for all q, then

$$p(x) = \max_{q \in Q} x(q) \textit{ and } \|x\| = \max_{q \in Q} |x(q)|.$$

A well-known example of a vector lattice with a strong unit is the space $L^\infty(S, \Sigma, \mu)$ of all essentially bounded functions defined on a measure space (S, Σ, μ). Assume that $\mathbf{1}(s) = 1$ for all $s \in S$, then we have $p(x) = \text{ess sup}_{s \in S} x(s)$ and $\|x\| = \text{ess sup}_{s \in S}|x(s)|$.

Example 2.2. Let $\mathbb{X} = \mathbb{R} \times \mathbb{Y}$, where $\mathbb{Y}$ is a Banach space with a norm $\|\cdot\|$, and let $\mathbb{K} \subset \mathbb{X}$ be the epigraph of the norm: $\mathbb{K} = \{(\lambda, x) : \lambda \geq \|x\|\}$. The cone $\mathbb{K}$ is closed solid convex and pointed. It is easy to check and well known that $\mathbf{1} = (1, 0)$ is an interior point of $\mathbb{K}$. For each $(c, y) \in \mathbb{X}$ we have

$$\begin{aligned} p(c,y) &= \inf\{\lambda \in \mathbb{R} : (c,y) \leq \lambda\mathbf{1}\} \\ &= \inf\{\lambda \in \mathbb{R} : (\lambda, 0) - (c,y) \in \mathbb{K}\} \\ &= \inf\{\lambda \in \mathbb{R} : (\lambda - c, -y) \in \mathbb{K}\} \\ &= \inf\{\lambda \in \mathbb{R} : \lambda - c \geq \|-y\|\} = c + \|y\|. \end{aligned}$$

Hence

$$\|(y,c)\| = \max\{p(y,c), p(-(y,c))\} = \max\{c + \|y\|, -c + \|y\|\} = |c| + \|y\|.$$

Moreover, we consider the set of all bounded linear functionals from $\mathbb{X}$ to complex field $\mathbb{C}$, dual space of $\mathbb{X}$, which is denoted by $\mathbb{X}^*$.

Let $\mathbb{X}, \mathbb{Y}$ be two Lattice Banach algebras and denote their duals by $\mathbb{X}^*$ and $\mathbb{Y}^*$, respectively. We recall (see [1]) that the uncompleted tensor product of $\mathbb{X}$ and $\mathbb{Y}$ is the set of all formal expressions $\sum_{i=1}^n x_i \otimes y_i$, where $x_i \in \mathbb{X}$ and $y_i \in \mathbb{Y}$ and $n \in \mathbb{N}$. We regard such an expression as defining an operator $\mathbb{A} : \mathbb{X}^* \to \mathbb{Y}$, given by

$$\mathbb{A}(\phi) = \sum_{i=1}^{n} \phi(x_i) y_i \quad \phi \in \mathbb{X}^*. \tag{7}$$

Amongst all these formal expressions, we introduce the relation

$$\sum_{i=1}^{n} x_i \otimes y_i \sim \sum_{i=1}^{m} a_i \otimes b_i,$$

if both expressions define the same operator from $\mathbb{X}^*$ to $\mathbb{Y}$. This relation is an equivalence relation on the set of all such formal expressions. We shall denote the set of all such equivalence classes by $\mathbb{X} \otimes \mathbb{Y}$. We shall abuse notation in the usual way by referring to the expression $\sum_{i=1}^{n} x_i \otimes y_i$ as a member of $\mathbb{X} \otimes \mathbb{Y}$ when we intend to refer to the equivalence classes of expression containing $\sum_{i=1}^{n} x_i \otimes y_i$. We define multiples of $\sum_{i=1}^{n} x_i \otimes y_i$ with any $\alpha \in \mathbb{R}$, by $\sum_{i=1}^{n} \alpha x_i \otimes y_i$. Similarly, we define addition by

$$\sum_{i=1}^{n} x_i \otimes y_i + \sum_{i=n+1}^{m} x_i \otimes y_i = \sum_{i=1}^{m} x_i \otimes y_i.$$

We recall that a complex algebra is a vector space $\mathbb{A}$ over the complex field $\mathbb{C}$ in which a multiplication is defined by $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which satisfies

$$x(yz) = (xy)z, \tag{8}$$

$$(x + y)z = xz + yz, \ x(y + z) = xy + xz, \tag{9}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{10}$$

for all x, y and z in $\mathbb{A}$ and all scalars α. If in addition, $\mathbb{A}$ is a Banach space with respect to a norm which satisfies the multiplicative inequality

$$\|xy\| \le \|x\| \|y\| \ (x, y \in \mathbb{A}) \tag{11}$$

and if $\mathbb{A}$ contains an element e such that $\|e\| = 1$ and

$$xe = ex = x \ (x \in \mathbb{A}), \tag{12}$$

then $\mathbb{A}$ is called a unital Banach algebra. Let $\mathbb{Y}$ be a lattice Banach algebra with the strong unit $\mathbf{1}_{\mathbb{Y}}$. We using the order relation on $\mathbb{Y}$ to define a partially order relation on $\mathbb{X} \otimes \mathbb{Y}$ as follows:

$$\sum_{i=1}^{n} x_i \otimes y_i \ll \sum_{i=1}^{m} a_i \otimes b_i \Leftrightarrow \sum_{i=1}^{n} \phi(x_i) y_i \le \sum_{i=1}^{m} \phi(a_i) b_i \quad (\forall \phi \in \mathbb{X}^*). \tag{13}$$

We recall (see [1]) that it is possible to construct various norms on $\mathbb{X} \otimes \mathbb{Y}$ using the norms in $\mathbb{X}$ and $\mathbb{Y}$. The most obvious way to introduce a norm which is independent to its representation, is to assign to $\sum_{i=1}^{n} x_i \otimes y_i$ its norm when regarded as an operator from $\mathbb{X}^*$ to $\mathbb{Y}$. We define the norm $\|.\|$ by:

$$\|\sum_{i=1}^{n} x_i \otimes y_i\| = \sup\{\|\sum_{i=1}^{n} \phi(x_i)y_i\|, \|\phi\| = 1, \phi \in \mathbb{X}^*\} \tag{14}$$

3. Downward sets and their Best Approximations in Quasi Tensor Product spaces

Definition 3.1. Let $\mathbb{X}, \mathbb{Y}$ be two Banach Algebras. A homomorphism from $\mathbb{X}$ to $\mathbb{Y}$ is a map $F : \mathbb{X} \to \mathbb{Y}$ which satisfies the following statements:

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \quad (\forall \alpha, \beta \in \mathbb{R}), \tag{15}$$

$$F(xy) = F(x)F(y). \tag{16}$$

We use the notion $\mathbb{X}^\times$ to denote the set of all non- zero homomorphisms from the Banach algebra $\mathbb{X}$ to the Banach algebra $\mathbb{C}$. By Theorem (1.3.3 [6]) if $\mathbb{X}$ is a unital abelian Banach algebra then $\mathbb{X}^\times \neq \varnothing$ and for all $f \in \mathbb{X}^\times$, we have $\|f\| = 1$. Therefor if $\mathbb{X}$ is a unital abelian Banach algebra then $\mathbb{X}^\times \subseteq \mathbb{X}^*$ and in expression (7), we can replace $\mathbb{X}^*$ with $\mathbb{X}^\times$. We denote the representation of each new equivalence class by the form $\sum_{i=1}^{n} x_i \boxtimes y_i$. Also we call the new space, quasi tensor product space and denote it by $\mathbb{X} \boxtimes \mathbb{Y}$. We define a norm $\|.\|_\boxtimes$ on $\mathbb{X} \boxtimes \mathbb{Y}$ by

$$\|\sum_{i=1}^{n} x_i \boxtimes y_i\|_\boxtimes = \sup_{\phi \in \mathbb{X}^\times} \|\sum_{i=1}^{n} \phi(x_i)y_i\|. \tag{17}$$

Lemma 3.2. *Consider that $\mathbb{X}$ is a Banach algebra with unit element $e_\mathbb{X}$. Then $f(e_\mathbb{X}) = 1$, for each $0 \neq f \in \mathbb{X}^\times$.*

Proof. Since $e_\mathbb{X} = e_\mathbb{X} e_\mathbb{X}$ and $f \in \mathbb{X}^\times$, we have

$$f(e_\mathbb{X}) = f(e_\mathbb{X} e_\mathbb{X}) = f(e_\mathbb{X})f(e_\mathbb{X}) \tag{18}$$

then $f(e_\mathbb{X}) = 1$ since $f \neq 0$. □

Corollary 3.3. *Let $\mathbb{X}$ be a unital abelian Banach algebra and $\mathbb{Y}$ be a Banach space. Let $z = \sum_{i=1}^{n} x_i \otimes y_i$ and $z_o = \sum_{i=1}^{n} x_i \boxtimes y_i$, then $\|z\| \geq \|z_o\|_\boxtimes$.*

Proof. Since $\mathbb{X}^\times \subseteq \mathbb{X}^*$. We have

$$\begin{aligned} \|z_o\|_{\boxtimes} &= \sup_{\phi\in\mathbb{X}^\times} \|\sum_{i=1}^{n}\phi(x_i)y_i\| \\ &\leq \sup\{\|\sum_{i=1}^{n}\phi(x_i)y_i\|, \|\phi\|=1, \phi\in\mathbb{X}^*\} = \|z\|. \end{aligned}$$

□

Corollary 3.4. *Let $\mathbb{X}$ be a unital abelian Banach algebra with unit element $e_{\mathbb{X}}$ and $\mathbb{Y}$ be a lattice Banach algebra with the strong unit $\mathbf{1}_{\mathbb{Y}}$, then $\|e_{\mathbb{X}}\otimes\mathbf{1}_{\mathbb{Y}}\| = \|e_{\mathbb{X}}\boxtimes\mathbf{1}_{\mathbb{Y}}\|_{\boxtimes} = 1$.*

Proof. Suppose $\phi\in\mathbb{X}^\times$. By Lemma 3.2, $\phi(e_{\mathbb{X}})=1$. Thus we get

$$\|e_{\mathbb{X}}\boxtimes\mathbf{1}_{\mathbb{Y}}\|_{\boxtimes} = \sup_{\phi\in\mathbb{X}^\times}\|\phi(e_{\mathbb{X}})\mathbf{1}_{\mathbb{Y}}\| = \|\mathbf{1}_{\mathbb{Y}}\| = 1,$$

and

$$\begin{aligned} \|e_{\mathbb{X}}\otimes\mathbf{1}_{\mathbb{Y}}\| &= \sup\{\|\phi(e_{\mathbb{X}})\mathbf{1}_{\mathbb{Y}}\|, \|\phi\|=1, \phi\in\mathbb{X}^*\} \\ &= \|\mathbf{1}_{\mathbb{Y}}\|\sup\{\|\phi(e_{\mathbb{X}})\|, \|\phi\|=1, \phi\in\mathbb{X}^*\} \\ &= \|\mathbf{1}_{\mathbb{Y}}\|\|e_{\mathbb{X}}\| = 1. \end{aligned}$$

This completes the proof. □

We define an order relation $\ll$ on $\mathbb{X}\boxtimes\mathbb{Y}$ as follows:

$$\sum_{i=1}^{n}x_i\boxtimes y_i \ll \sum_{i=1}^{n}a_i\boxtimes b_i \Leftrightarrow \sum_{i=1}^{n}\phi(x_i)y_i \leq \phi(a_i)b_i \ \ \forall\phi\in\mathbb{X}^\times. \tag{19}$$

Definition 3.5. (see [7] ,[9]) *A set $\mathbb{U}\subseteq\mathbb{X}$ is said to be downward if $u\in\mathbb{U}$ and $x\leq u$ implies $x\in\mathbb{U}$.*

Definition 3.6. (see [7] ,[9]) *A set $\mathbb{U}\subseteq\mathbb{X}$ is said to be upward if $u\in\mathbb{U}$ and $x\geq u$ implies that $x\in\mathbb{U}$.*

By definition 3.5, we get the following results for $\mathbb{X}\boxtimes\mathbb{Y}$, where $\mathbb{X}$ is a unital abelian Banach algebra and $\mathbb{Y}$ is a lattice Banach algebra with the strong unit $\mathbf{1}_{\mathbb{Y}}$.

Proposition 3.7. *For each downward subset $\mathbb{U}$ of $\mathbb{Z} := \mathbb{X}\boxtimes\mathbb{Y}$, the following assertions are true:*

(1) *If* $\sum_{i=1}^{m}x_i\boxtimes y_i\in\mathbb{U}$ *then* $\sum_{i=1}^{m}x_i\boxtimes y_i-\varepsilon e_{\mathbb{X}}\boxtimes\mathbf{1}_{\mathbb{Y}}\in\operatorname{int}\mathbb{U}$ *for each* $\varepsilon>0$.

$$(2)\operatorname{int}\mathbb{U} = \{\sum_{i=1}^{m} a_i \boxtimes b_i \in \mathbb{Z} : \sum_{i=1}^{m} a_i \boxtimes b_i + \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U} \text{ for some } \varepsilon > 0\}.$$

Proof. (1). Let $\varepsilon > 0$ be given and $\sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{U}$. Then it is clear that $\sum_{i=1}^{m} x_i \boxtimes y_i - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}$ is an element of $\mathbb{X} \boxtimes \mathbb{Y}$. Consider $\mathcal{N}$ be an open neighborhood of $\sum_{i=1}^{m} x_i \boxtimes y_i - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}$, thus :

$$\mathcal{N} = \{\sum_{i=1}^{n} a_i \boxtimes b_i \in \mathbb{X} \boxtimes \mathbb{Y} : \| \sum_{i=1}^{n} a_i \boxtimes b_i - (\sum_{i=1}^{m} x_i \boxtimes y_i - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}) \|_{\boxtimes} < \varepsilon\}.$$

Now by (6) and (17), we have

$$| \sum_{i=1}^{n} \phi(a_i) b_i - (\sum_{i=1}^{m} \phi(x_i) y_i - \varepsilon \mathbf{1}_{\mathbb{Y}}| \leq \varepsilon \mathbf{1}_{\mathbb{Y}} \quad (\forall \phi \epsilon \mathbb{X}^{\times}),$$

and by (5), we get $\mathcal{N}$ is the set of all $\sum_{i=1}^{n} a_i \boxtimes b_i \in \mathbb{X} \boxtimes \mathbb{Y}$ where

$$\sum_{i=1}^{m} \phi(x_i) y_i - 2\varepsilon \mathbf{1}_{\mathbb{Y}} \ll \sum_{i=1}^{n} \phi(a_i) b_i \ll \sum_{i=1}^{m} \phi(x_i) y_i.$$

By (19) we have $\sum_{i=1}^{m} a_i \boxtimes b_i \ll \sum_{i=1}^{m} x_i \boxtimes y_i$. Since $\mathbb{U}$ is a downward set and $\sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{U}$, it follows that $\mathcal{N} \subseteq \mathbb{U}$. This shows that $\sum_{i=1}^{m} x_i \boxtimes y_i - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \operatorname{int}\mathbb{U}$.
(2). Let $\sum_{i=1}^{m} x_i \boxtimes y_i \in \operatorname{int}\mathbb{U}$. Then there exists $\varepsilon_0 > 0$ such that the closed ball $B(\sum_{i=1}^{m} x_i \boxtimes y_i, \varepsilon_0)$ is a subset of $\mathbb{U}$. In view of (17) and (5), we get $\sum_{i=1}^{m} x_i \boxtimes y_i + \varepsilon_0 e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}$.

Conversely, if there exists $\varepsilon > 0$ such that $\sum_{i=1}^{m} x_i \boxtimes y_i + \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}$, by part (1), $\sum_{i=1}^{m} x_i \boxtimes y_i = (\sum_{i=1}^{m} x_i \boxtimes y_i + \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}) \in \operatorname{int}\mathbb{U}$, which completes the proof. ☐

Corollary 3.8. *Let $\mathbb{U}$ be a downward subset of $\mathbb{X} \boxtimes \mathbb{Y}$. Then $\mathbb{U}$ is proximinal in $\mathbb{X} \boxtimes \mathbb{Y}$.*

Proof. For an arbitrary element $\sum_{i=1}^{m} x_i \boxtimes y_i$ of $\mathbb{X} \boxtimes \mathbb{Y} \backslash U$, we get:

$$r = d(\sum_{i=1}^{m} x_i \boxtimes y_i, \mathbb{U}) = \inf_{\sum_{i=1}^{n} u_i \boxtimes v_i \in \mathbb{U}} \| \sum_{i=1}^{m} x_i \boxtimes y_i - \sum_{i=1}^{n} u_i \boxtimes v_i \|_{\boxtimes}.$$

This implies for $\varepsilon > 0$, there exists an element $\sum_{i=1}^{n} u_i^{\varepsilon} \boxtimes v_i^{\varepsilon}$ of $\mathbb{U}$ such that $\| \sum_{i=1}^{m} x_i \boxtimes y_i - \sum_{i=1}^{n} u_i^{\varepsilon} \boxtimes v_i^{\varepsilon} \|_{\boxtimes} < r + \varepsilon$. Then by (17) we get

$$| \sum_{i=1}^{m} \phi(x_i) y_i - \sum_{i=1}^{n} \phi(u_i^{\varepsilon}) v_i^{\varepsilon} | \leq (\varepsilon + r) \mathbf{1}_{\mathbb{Y}} \quad (\forall \phi \epsilon \mathbb{X}^{\times}).$$

Therefore by (5) we get

$$-(r+\varepsilon)\mathbf{1}_{\mathbb{Y}} \leq \sum_{i=1}^{n}\phi(u_i^{\varepsilon})v_i^{\varepsilon} - \sum_{i=1}^{m}\phi(x_i)y_i \leq (r+\varepsilon)\mathbf{1}_{\mathbb{Y}}. \tag{20}$$

Let $\sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0 = \sum_{i=1}^{m} x_i \boxtimes y_i - re_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}$, then , we have

$$\|\sum_{i=1}^{m} x_i \boxtimes y_i - \sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0\| = r = d(\sum_{i=1}^{m} x_i \boxtimes y_i, \mathbb{U}) \tag{21}$$

and so by (19), and (20) we have

$$\sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0 - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} = \sum_{i=1}^{m} x_i \boxtimes y_i - (r+\varepsilon)e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \ll \sum_{i=1}^{n} u_i^{\varepsilon} \boxtimes v_i^{\varepsilon} \tag{22}$$

As $\mathbb{U}$ is a downward set and $\sum_{i=1}^{n} u_i^{\varepsilon} \boxtimes v_i^{\varepsilon} \in \mathbb{U}$; for each $\varepsilon > 0$, we get

$$\sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0 - \varepsilon e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}.$$

Since $\mathbb{U}$ is closed, we have $\sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0 \in \mathbb{U}$, and so by (21) and (3) we get

$$\sum_{i=1}^{m+1} u_i^0 \boxtimes v_i^0 \in \mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i).$$

This shows that $\mathbb{U}$ is proximinal. □

Proposition 3.9. *Let $\mathbb{U} \subset \mathbb{Z} := \mathbb{X}\boxtimes\mathbb{Y}$ be a closed downward set, then if $\sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{Z}\backslash\mathbb{U}$, there exists the least element $z_0 = \min \mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i)$ of the set $\mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i)$; namely, $z_0 = \sum_{i=1}^{m} x_i \boxtimes y_i - re_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}$; where $r := d(\sum_{i=1}^{m} x_i \boxtimes y_i, \mathbb{U})$.*

Proof. If $\sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{U}$, the result holds. Let $\sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{Z}\backslash\mathbb{U}$ and $z_0 = \sum_{i=1}^{m} x_i \boxtimes y_i - re_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}$. Then by the proof of Corollary 3.8, we have $z_0 \in \mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i)$. Thus by equality $\|\sum_{i=1}^{m} x_i \boxtimes y_i - z_0\| = r$ and applying (17), (5), we get $z \geq z_0$ for each $z \in B(\sum_{i=1}^{m} x_i \boxtimes y_i, r)$. Thus z_0 is the least element of the closed ball $B(\sum_{i=1}^{m} x_i \boxtimes y_i; r)$. Now Let $z' \in \mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i)$. Then we have $\|\sum_{i=1}^{m} x_i \boxtimes y_i - z'\|_{\boxtimes} = r$, and so $z' \in B(\sum_{i=1}^{m} x_i \boxtimes y_i, r)$. Therefore $z' \geq z_0$. Hence z_0 is the least element of the set $\mathbf{P}_{\mathbb{U}}(\sum_{i=1}^{m} x_i \boxtimes y_i)$. □

Corollary 3.10. *Let $\mathbb{U}$ be a closed downward subset of $\mathbb{X} \boxtimes \mathbb{Y}$ and $\sum_{i=1}^{m} x_i \boxtimes y_i$ be an element of $Z \setminus \mathbb{U}$. Then*

$$d(\sum_{i=1}^{m} x_i \boxtimes y_i, \mathbb{U}) = \min\{\lambda \geq 0 | \sum_{i=1}^{m} x_i \boxtimes y_i - \lambda e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}\}.$$

Proof. Assume that $A = \{\lambda | \lambda \geq 0 \ , \ \sum_{i=1}^{m} x_i \boxtimes y_i - \lambda e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}\}$. If $x := \sum_{i=1}^{m} x_i \boxtimes y_i \in \mathbb{U}$ then we get $(\sum_{i=1}^{m} x_i \boxtimes y_i - 0 e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}}) \in \mathbb{U}$ and so $\min A = 0 = d(\sum_{i=1}^{m} x_i \boxtimes y_i, \mathbb{U})$.
Now let $x \notin \mathbb{U}$ then $r = d(\sum_{i=1}^{m} x_i \boxtimes y_i; \mathbb{U}) > 0$. Let $\lambda > 0$ be such that $\sum_{i=1}^{m} x_i \boxtimes y_i - \lambda e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}$. Thus we have

$$\lambda = \|x - (x - \lambda e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}})\|_{\boxtimes} \geq d(x; \mathbb{U}) = r.$$

By Proposition 3.9, we have $\sum_{i=1}^{m} x_i \boxtimes y_i - r e_{\mathbb{X}} \boxtimes \mathbf{1}_{\mathbb{Y}} \in \mathbb{U}$, and therefore $r \in A$. Hence $\min A = r$, which completes the proof. □

4. Im-quasi downward sets in direct sum of lattice normed spaces with applications

Now let I be a finite set of indices, and $(\mathbb{X}_i)_{i \in I}$ be a collection of lattice normed spaces with the strong unit $\mathbf{1}_i$, we use the notation $\sum_{i \in I} \mathbb{X}_i$ for direct sum of lattice normed spaces $\mathbb{X}_i$. Also for each $x, y \in \sum_{i \in I} \mathbb{X}_i$, we define

$$x + y := (x_i + y_i)_{i \in I},$$

where $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I}$. If $(\mathbb{X}_i, \|.\|_i)_{i \in I}$ be a collection of lattice normed spaces, we define a norm $\|.\|$, on the space $\sum_{i \in I} \mathbb{X}_i$ as follows:

$$\|x\| := \max_{i \in I} \|x_i\|_i \ \ for\ each \ \ x \in \sum_{i \in I} \mathbb{X}_i. \tag{23}$$

We use the notation $\mathbf{1}_{\bigoplus}$ for vector $y = (\mathbf{1}_i)_{i \in I} \in \sum\limits_{i \in I} \mathbb{X}_i$, and define a partial ordered relation on the direct sum of lattice normed spaces $\mathbb{X}_i$, as follows: For each x, y in $\sum\limits_{i \in I} \mathbb{X}_i$,

$$x \leq y \Leftrightarrow x_i \leq y_i \ (\forall i \in I). \tag{24}$$

Let $I_m = \{i_1, i_2, ... i_m\}$ be an arbitrarily subset of I and $x = (x_i)_{i \in I}$ be an arbitrary element of $\sum_{i \in I} \mathbb{X}_i$. We define the following useful sets:

$$(\sum_{i \in I} \mathbb{X}_i)_x^{I_m} := \{y = (y_i)_{i \in I} \in \sum_{i \in I} \mathbb{X}_i\}$$

where

$$\begin{cases} x_i \geq y_i & \text{if i} \in \mathrm{I_m} \\ x_i \leq y_i & \text{if i} \notin \mathrm{I_m} \end{cases}$$

and

$$(co \sum_{i \in I} \mathbb{X}_i)_x^{I_m} := \{y = (y_i)_{i \in I} \in \sum_{i \in I} \mathbb{X}_i\}$$

where

$$\left\{ \begin{array}{ll} x_i \le y_i & \text{if i} \in \mathrm{I_m} \\ x_i \ge y_i & \text{if i} \notin \mathrm{I_m} \end{array} \right\},$$

and define

$$((\sum_{i\in I}\mathbb{X}_i)_x^{I_m})_+ := (\sum_{i\in I}\mathbb{X}_i)_+ \bigcap (\sum_{i\in I}\mathbb{X}_i)_x^{I_m},$$

where $(\sum_{i\in I}\mathbb{X}_i)_+ = \big\{y|y=(y_i)_{i\in I}\in\sum_{i\in I}\mathbb{X}_i : y_i \ge 0\ (\forall i\in I)\big\}$. We use the notation $\mathbf{1}_{\bigoplus}^{I_m}$ for the vector $y=(y_i)_{i\in I}$ where

$$y_i = \left\{ \begin{array}{ll} 1 & \text{if } \ i \in I_m \\ -1 & \text{if } \ i \in I\setminus I_m. \end{array} \right. \tag{25}$$

Also we define $coPr^{I_m}(x)$ as follows:

$$(coPr^{I_m}(x))_i = \left\{ \begin{array}{ll} x_i & \text{if } \ i \in I_m \\ 0 & \text{if } \ i \in I\setminus I_m \end{array} \right. \tag{26}$$

Definition 4.1. A set $\mathbb{U} \subseteq \sum_{i\in I}\mathbb{X}_i$ is called I_m-quasi downward if $(\sum_{i\in I}\mathbb{X}_i)_u^{I_m} \subseteq \mathbb{U}$ for each $u\in\mathbb{U}$.

In particular, an I_m-quasi downward set $\mathbb{U}$ is downward, if $I_m = I$ and is upward, if $I_m = \varnothing$.

Proposition 4.2. *Consider $\mathbb{U}$ as an I_m-quasi downward subset of $\sum_{i\in I}\mathbb{X}_i$, and let $x\in\sum_{i\in I}\mathbb{X}_i$. Then the following assertions are true:*

(1) *If $x\in\mathbb{U}$, then $x-\varepsilon\mathbf{1}_{\bigoplus}^{I_m}\in \mathrm{int}\mathbb{U}$ for all $\varepsilon>0$.*

(2) $\mathrm{int}\mathbb{U} = \{\mathrm{x}\in\sum_{\mathrm{i}\in\mathrm{I}}\mathbb{X}_\mathrm{i} : \mathrm{x}+\varepsilon\mathbf{1}_{\bigoplus}^{\mathrm{I_m}}\in\mathbb{U} \text{ for some } \varepsilon>0\}$.

Proof. (1). Let $\varepsilon>0$ and $x\in\mathbb{U}$ be given. Consider $\mathcal{N}$ as an open neighborhood of $x-\varepsilon\mathbf{1}_{\bigoplus}^{I_m}$ i.e

$$\mathcal{N} := \{y\in\sum_{i\in I}\mathbb{X}_i : \|y-(x-\varepsilon\mathbf{1}_{\bigoplus}^{I_m})\| \le \varepsilon\}.$$

Let

$$\mathcal{N}_1 = \{y\in\sum_{i\in I}\mathbb{X}_i : x_i - 2\varepsilon \le y_i \le x_i\ (\forall i\in I_m)\}$$

and

$$\mathcal{N}_2 = \{y\in\sum_{i\in I}\mathbb{X}_i : x_i \le y_i \le x_i + 2\varepsilon\ (\forall i\in I\setminus I_m)\}.$$

By (5) we have,

$$\mathcal{N} = \mathcal{N}_1\cap\mathcal{N}_2.$$

By definition of $(\sum_{i\in I}\mathbb{X}_i)_u^{I_m}$ and that $\mathbb{U}$ is an I_m-quasi downward set, it follows that $\mathcal{N}\subset\mathbb{U}$, and so $x-\varepsilon\mathbf{1}_{\bigoplus}^{I_m}\in\mathrm{int}\mathbb{U}$.

(2). Let $x \in \mathrm{int}\mathbb{U}$. Then there exists $\varepsilon_0 > 0$ such that $B(x, \varepsilon_0) \subset \mathbb{U}$. In view of (5), we get $x + \varepsilon_0 \mathbf{1}_{\bigoplus}^{I_m} \in \mathbb{U}$.

Conversely, suppose that there exists $\varepsilon > 0$ such that $x + \varepsilon \mathbf{1}_{\bigoplus}^{I_m} \in \mathbb{U}$. By part (1) we have $x = (x + \varepsilon \mathbf{1}_{\bigoplus}^{I_m}) - \varepsilon \mathbf{1}_{\bigoplus}^{I_m} \in \mathrm{int}\mathbb{U}$, which completes the proof. □

Proposition 4.3. *Each downward subset $\mathbb{U}$ of $\sum_{i\in I} \mathbb{X}_i$ is proximinal in $\sum_{i\in I} \mathbb{X}_i$.*

Proof. Let $x_0 \in \sum_{i\in I} \mathbb{X}_i \setminus \mathbb{U}$ and, $r = d(x_0, \mathbb{U}) = inf_{u\in\mathbb{U}} \|x_0 - u\|$, this implies, for $\varepsilon > 0$ there exists $u_\varepsilon \in \mathbb{U}$ such that $\|x_0 - u_\varepsilon\| < r + \varepsilon$. Then by (23) we have

$$\|(x_0)_i - (u_\varepsilon)_i\|_i \leq r + \varepsilon \quad (\forall i \in I),$$

and by (5) we get

$$-(r + \varepsilon)\mathbf{1}_i < (u_\varepsilon)_i - (x_0)_i < (r + \varepsilon)\mathbf{1}_i, \ (\forall i \in I). \tag{27}$$

Clearly when $u_0 := x_0 - r\mathbf{1}_{\bigoplus}$, we have $\|x_0 - u_0\| = r = d(x_0, \mathbb{U})$ and so by (27) and (24), $u_0 = x_0 - r\mathbf{1}_{\bigoplus} - \varepsilon\mathbf{1}_{\bigoplus} \leq u_\varepsilon$. As $\mathbb{U}$ is downward and $u_\varepsilon \in \mathbb{U}$, it follows that $u_0 = x_0 - r\mathbf{1}_{\bigoplus} - \varepsilon\mathbf{1}_{\bigoplus} \in \mathbb{U}$ and thus $u_0 \in \mathbf{P}_{\mathbb{U}}(x_0)$, i.e $\mathbf{P}_{\mathbb{U}}(x_0) \neq \emptyset$. □

Corollary 4.4. *Let $\mathbb{U}$ be a closed downward subset of $\sum_{i\in I} \mathbb{X}_i$ and $x_0 \in \sum_{i\in I} \mathbb{X}_i \setminus \mathbb{U}$. The least element $u_0 = \min \mathbf{P}_{\mathbb{U}}(x_0)$ of the set $\mathbf{P}_{\mathbb{U}}(x_0)$ exists. Where $u_0 = x_0 - r\mathbf{1}_{\bigoplus}$ and $r := d(x_0, \mathbb{U})$.*

Proof. If $x_0 \in \mathbb{U}$, the result holds. Assume $x_0 \in \sum_{i\in I} \mathbb{X}_i \setminus \mathbb{U}$ and $u_0 = x_0 - r\mathbf{1}_{\bigoplus}$. By proposition 4.3, we have $u_0 \in \mathbf{P}_{\mathbb{U}}(x_0)$. By applying (23), (5) and the equality $\|x_0 - u_0\| = r$, we get $y \geq x_0 - r\mathbf{1}_{\bigoplus}$ for each $y \in B(x_0, r)$. This implies u_0 is the least element of the closed ball $B(x_0, r)$. Now, $\|x_0 - u\| = r$ for an arbitrary element $u \in \mathbf{P}_{\mathbb{U}}(x_0)$ and so $u \in B(x_0, r)$. This shows that $u \geq u_0$. Hence u_0 is the least element of the set $\mathbf{P}_U(x_0)$. □

In the following we define two useful maps:

$$T_m : \sum_{i\in I} \mathbb{X}_i \to \sum_{i\in I} \mathbb{X}_i$$

by

$$T_m(x) = y = (y_i)_{i\in I}$$

where:

$$y_i = (\mathbf{1}_{\bigoplus}^{I_m})_i . x_i \tag{28}$$

and

$$(coT)_m := \sum_{i\in I} \mathbb{X}_i \to \sum_{i\in I} \mathbb{X}_i$$

by

$$(coT)_m(x) = z = (z_i)_{i\in I}$$

where

$$z_i = -(\mathbf{1}_{\bigoplus}^{I_m})_i.x_i \tag{29}$$

Lemma 4.5. *The maps T_m and $(coT)_m$ defined by (28) and (29) are diffeomorphism.*

Proof. The proof is trivial. □

Theorem 4.6. *Let $\mathbb{U} \subset \sum_{i\in I} \mathbb{X}_i$ be an I_m-quasi downward set, then $T_m(\mathbb{U})$ is downward, and $(coT)_m(\mathbb{U})$ is upward, where T_m and coT_m be the maps defined by (28) and (29).*

Proof. By definition, $T_m(\mathbb{U})$ is downward if and only if the hypothesis $h \in T_m(\mathbb{U})$, $x \in \sum_{i\in I} \mathbb{X}_i$ and $x \leq h$, implies that $x \in T_m(\mathbb{U})$. Let $h \in T_m(\mathbb{U})$, By Lemma 4.5 there exists $u \in \mathbb{U}$ such that $T_m(u) = h$. As $x \leq h$ then for each $i \in I$, $x_i \leq h_i$. Then by (28) we have $x_i \leq u_i$ if $i \in I_m$ and $-x_i \geq u_i$ if $i \in I \setminus I_m$. As $u \in \mathbb{U}$ and $\mathbb{U}$ is I_m-quasi downward, we conclude $w = (w_i)_{i\in I} \in \mathbb{U}$, where $(w_i)_{i\in I}$ is defined by $w_i = (\mathbf{1}_{\bigoplus}^{I_m})_i.x_i$. Then $T_m(\mathbb{U})$ is downward since $x = T_m(w) \in T_m(\mathbb{U})$. Similarly $(coT)_m(\mathbb{U})$ is downward.This completes the proof. □

Definition 4.7. *A set $U \subset \sum_{i\in I} \mathbb{X}_i$ is called I_m-quasi upward if its compliment be an I_m-quasi downward.*
$\big(i.e: (co\sum_{i\in I}\mathbb{X}_i)_u^{I_m} \subseteq U;\ for\ all u \in \mathbb{U}\big)$

Now by (28) and (29) we conclude the following proposition:

Proposition 4.8. *Consider $\mathbb{U}$ as a subset of $\sum_{i\in I} \mathbb{X}_i$ which is closed I_m-quasi downward or I_m-quasi upward set and $x \in \sum_{i\in I} \mathbb{X}_i$. Set $r := d(x, \mathbb{U})$, $r' := d(T_m(x), T_m(\mathbb{U}))$, $r'' := d((coT)_m(x), (coT)_m(\mathbb{U}))$, then $r = r' = r''$.*

Proof.

$$\begin{aligned}
\|T_m(x) - T_m(\mathbb{U})\| &= \max_{i\in I} \|\big(T_m(x)\big)_i - \big(T_m(u)\big)_i\|_i \\
&= \max\{\max_{i\in I_m} \|x_i - u_i\|_i, \max_{i\in I\setminus I_m} \|u_i - x_i\|_i\} \\
&= \max_{i\in I} \|x_i - u_i\|_i = \|x - u\|.
\end{aligned}$$

By taking infimum we get $r = r'$. Similarly $r = r''$. This completes the proof. □

Proposition 4.9. *Consider* $\mathbb{U} \subset \sum_{i\in I} \mathbb{X}_i$ *as a closed* I_m-*quasi downward set,* $x \in \sum_{i\in I} \mathbb{X}_i$ *and* $r := d(x, \mathbb{U})$. *Then* $u_m = x - r\mathbf{1}_{\oplus}^{I_m} \in \mathbf{P}_{\mathbb{U}}(x)$.

Proof. Let $x \in \sum_{i\in I} \mathbb{X}_i$. By Theorem 4.6, $T_m(\mathbb{U})$ is a downward set. Hence by Corollary 4.4 $w_0 = \min \mathbf{P}_{T_m(\mathbb{U})}\big(T_m(x)\big)$ exists and thus we get $w_0 = T_m(x) - r\mathbf{1}_{\oplus}$. Thus we get $u_m = T_m^{-1}(w_0) \in \mathbf{P}_{\mathbb{U}}(x)$. □

Proposition 4.10. Consider $\mathbb{U}$ as a closed I_m-quasi downward subset of $\sum_{i\in I} \mathbb{X}_i$. Let $x \in \sum_{i\in I} \mathbb{X}_i$ and T_m as in (28). Then the following assertions are true:

i) $\mathbf{P}_{\mathbb{U}}(x) = \{u \in \mathbb{U} : \ T_m(u) \in \mathbf{P}_{T_m(\mathbb{U})}\big(T_m(x)\big)\}$.

ii) $d(x, \mathbb{U}) = \min\{\lambda \geq 0 : T_m(x) - \lambda\mathbf{1}_{\oplus} \in T_m(\mathbb{U})\}$.

Proof. (i) It follows from Lemma 4.5 and Proposition 4.8. (ii) It follows from Propositions 4.8 and 4.9. □

5. The relation of Positive I_m-quasi Downward sets and I_m-quasi Downward sets

Definition 5.1. *A set* $\mathbb{V} \subseteq (\sum_{i\in I} \mathbb{X}_i)_+$ *is called a positive* I_m-*quasi downward if* $(\sum_{i\in I} \mathbb{X}_i)_v^{I_m})_+ \subseteq \mathbb{V}$ *for each* $v \in \mathbb{V}$.

Downward hull of a positive I_m-quasi downward set $\mathbb{V} \subset (\sum_{i\in I} \mathbb{X}_i)_+$ is defined as follows:

Definition 5.2. *Let* $\mathbb{V} \subset (\sum_{i\in I} \mathbb{X}_i)_+$ *be a positive* I_m-*quasi downward set. The intersection of all* I_m-*quasi downward sets which contains* $\mathbb{V}$ *is an* I_m-*quasi downward set, which is called* I_m-*quasi downward hull of* $\mathbb{V}$ *and denoted by* $\mathbb{V}_*$.

In the following we see some properties of I_m-quasi downward hull of a positive I_m-quasi downward set :

Proposition 5.3. *Let* $\mathbb{V}_*$ *be* I_m-*quasi downward hull of* $\mathbb{V} \subset (\sum_{i\in I} \mathbb{X}_i)_+$, *then*

(1) $\mathbb{V}_* = \{x \in \sum_{i\in I} \mathbb{X}_i : coP^{I_m}(x) \in \mathbb{V}$ *and* $x^+ \in \mathbb{V}\}$,
(2) $\mathbb{V} = \mathbb{V}_* \bigcap (\sum_{i\in I} \mathbb{X}_i)_+$.

Proof. Let $A = \{x \in \sum_{i\in I} \mathbb{X}_i : coPr^{I_m}(x) \in \mathbb{V}$, and $x^+ \in \mathbb{V}\}$. We first prove that A is I_m-quasi downward. Let $a \in A$, there exists $x \in \sum_{i\in I} \mathbb{X}_i$ such that $x_i \leq a_i$, if $i \in I_m$ and $x_i \geq a_i$, if $i \in I \setminus I_m$. We are going to show that $x \in A$. We have $coPr^{I_m}(a)$, $a^+ \in \mathbb{V}$ since $a \in A$. Thus Let $y = coPr^{I_m}(x)$. By (26) we get

$$y_i = \begin{cases} 0 & \text{if } \ i \in I_m \\ x_i & \text{if } \ i \in I \setminus I_m. \end{cases} \tag{30}$$

On the other hand we have

$$y_i = \begin{cases} x_i^+ \leq a_i^+ & \text{if } i \in I_m \\ x_i^+ \geq a_i^+ & \text{if } i \in I \setminus I_m. \end{cases}$$

Now we get $x^+, y \in \mathbb{V}$ since $\mathbb{V}$ is positive I_m-quasi downward. Thus $x \in A$. This shows A is I_m-quasi downward. As $\mathbb{V} \subset A$, hence $\mathbb{V}_* \subset A$.

Let $x \in A$, thus $x^+ \in \mathbb{V}$ and $coPr^{I_m}(x) = coPr^{I_m}(x^+)$ and $x_i^+ \geq x_i$ for each $i \in I$, As $\mathbb{V}_*$ is I_m-quasi downward, we get $x \in \mathbb{V}_*$. Thus $A \subset \mathbb{V}_*$, which completes the proof.

(2). It is immediately a consequence of the first part. $\square$

Proposition 5.4. *Let $\mathbb{V}_*$ be the closed I_m-quasi downward hull of $V \subseteq (\sum_{i\in I} \mathbb{X}_i)_+$, and $x \in (\sum_{i\in I} \mathbb{X}_i)_+$, then $d(x, V) = d(x, \mathbb{V}_*)$.*

Proof. It is clear that $\mathbb{V} \subset \mathbb{V}_*$. For each $v \in \mathbb{V}_*$, we have

$$\begin{aligned} \|x - v\| &= \max_{i\in I} \|x_i - v_i\|_i = \max\{\max_{i\in I_m} \|x_i - v_i\|_i, \max_{i\in I\setminus I_m} \|x_i - v_i\|_i\} \\ &\geq max\{\max_{i\in I_m} \|x_i - v_i^+\|_i, \max_{i\in I\setminus I_m} \|x_i - v_i\|_i\} = \|x - v^+\| \\ &\geq d(x, V). \end{aligned}$$

Therefore $\inf_{v\in\mathbb{V}_*} \|x - v\| = d(x, \mathbb{V}_*) \geq d(x, V)$, which completes the proof. $\square$

Theorem 5.5. *Let $\mathbb{U}$ be an I_m-quasi upward subset of $\sum_{i\in I} \mathbb{X}_i$, then $T_m(\mathbb{U})$ is upward and $coT_m(\mathbb{U})$ is downward .*

Proof. By definition, $T_m(\mathbb{U})$ is upward if and only if $h \in T_m(\mathbb{U})$ and $x \in \sum_{i\in I} \mathbb{X}_i$ and $x \geq h$ implies that $x \in T_m(\mathbb{U})$. Let $h \in T_m(\mathbb{U})$, By Lemma 4.5 there exists $u \in \mathbb{U}$ such that $T_m(u) = h$. As $x \geq h$ then for each $i \in I,\ x_i \geq h_i$. By (28) we have $x_i \geq u_i$ if $i \in I_m$ and $-x_i \leq u_i$ if $i \in I \setminus I_m$. As $u \in \mathbb{U}$ and $\mathbb{U}$ is I_m-quasi upward, we conclude $w = (w_i)_{i\in I} \in \mathbb{U}$, where $(w_i)_{i\in I}$ is defined by $w_i = (\mathbf{1}_{\bigoplus}^{I_m})_i.x_i$. Then $x = T_m(w) \in T_m(\mathbb{U})$ and hence $T_m(\mathbb{U})$ is upward. Similarly it can be shown that $(coT)_m(\mathbb{U})$ is also downward.This completes the proof. $\square$

Corollary 5.6. Let $U \subset \sum_{i\in I} \mathbb{X}_i$ be closed I_m-quasi upward and $x \in \sum_{i\in I} \mathbb{X}_i$, then

$$\mathbf{P}_{\mathbb{U}}(x) = \{u \in \mathbb{U} :\ T_m(u) \in \mathbf{P}_{T_m(\mathbb{U})}(T_m(x))\}.$$

Proof. This follows by Lemma 4.5 and proposition 4.8. $\square$

Proposition 5.7. *Let $\mathbb{U} \subset \sum_{i\in I} \mathbb{X}_i$ be a closed I_m-quasi upward set, $x \in \sum_{i\in I} \mathbb{X}_i$ and $r := d(x, \mathbb{U})$ then $u_m = x + r\mathbf{1}_{\bigoplus}^{I_m} \in \mathbf{P}_{\mathbb{U}}(x)$.*

Proof. Suppose $\mathbb{U} \subset \sum_{i\in I} \mathbb{X}_i$ be a closed I_m-quasi upward set and $x \in \sum_{i\in I} \mathbb{X}_i$. By Theorem 5.5, $T_m(\mathbb{U})$ is an upward set. Since $-T_m(\mathbb{U})$ is downward, by Corollary 4.4, $w_0 = \max \mathbf{P}_{T_m(\mathbb{U})}(T_m(x))$ exists and $w_0 = T_m(x)+r\mathbf{1}_{\bigoplus}$. Then by Corollary 5.6, $u_m = T_m^{-1}(w_0) \in \mathbf{P}_{\mathbb{U}}(x)$. $\square$

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