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ON ANNIHILATOR PROPERTIES OF INVERSE SKEW POWER SERIES RINGS

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Abstract. Let α be an automorphism of a ring R. The authors [On skew inverse Laurent-serieswise Armendariz rings, *Comm. Algebra* **40**(1) (2012) 138-156] applied the concept of Armendariz rings to inverse skew Laurent series rings and introduced skew inverse Laurent-serieswise Armendariz rings. In this article, we study on a special type of these rings and introduce strongly Armendariz rings of inverse skew power series type. We determine the radicals of the inverse skew Laurent series ring $R((x^{-1};\alpha))$, in terms of those of R. We also prove that several properties transfer between R and the inverse skew Laurent series extension $R((x^{-1};\alpha))$, in case R is a strongly Armendariz ring of inverse skew power series type.

1. Introduction

Throughout this article, R denotes an associative ring with unity. For a non-empty subset X of R, $l_R(X)$ and $r_R(X)$ denote the left annihilator and the right annihilator of X in R, respectively.

Let R be a ring equipped with an automorphism α. We denote by $R((x^{-1};\alpha))$ the inverse skew Laurent series ring over the coefficient ring R formed by formal series

$$f(x) = \sum_{i=-\infty}^{n} a_i x^i,$$

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where x is a variable, n is an integer and $a_i \in R$. In the ring $R((x^{-1};\alpha))$ addition is defined as usual and multiplication is defined with respect to the relation

$$\forall i \qquad x^i a = \alpha^i(a)x^i.$$

In 1974 Armendariz noted in [3] that whenever the product of two polynomials over reduced rings (i.e. rings without non-zero nilpotent elements) is zero, then the products of their coefficients are all zero. In fact for a reduced ring R, the polynomial ring $R[x]$ satisfies the following condition:

$$\forall\ f(x) = \sum_{i=0}^{m} a_i x^i\ ,\ g(x) = \sum_{j=0}^{n} b_j x^j \in R[x],$$

$$\text{if } f(x)g(x) = 0, \text{ then } a_i b_j = 0, \forall\, i, j.$$

Nowadays the above condition is known as the *Armendariz condition*, and a ring R which satisfies this condition is called *Armendariz*. The systematic study of Armendariz rings was initiated by Rege and Chhawchharia in 1997 in [35]. In [20] Hong, Kim and Kwak extended the Armendariz property of rings to skew polynomial rings $R[x;\alpha]$: For an endomorphism α of a ring R, R is called an α-*skew Armendariz* ring if for polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ in $R[x;\alpha]$, $f(x)g(x) = 0$ implies that $a_i\alpha^i(b_j) = 0$, for each i, j.

Anderson and Camillo [2, Theorem 1] proved that a ring R is Armendariz if and only if so is $R[x]$. But Rege and Bhuphang in [34] give an example of a commutative Armendariz ring R whose power series ring $R[[x]]$ is not Armendariz. In 2006 Kim et al. in [23] called a ring R *power-serieswise Armendariz*, if $a_i b_j = 0$, for all i, j, whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$.

Recently, in [30] the authors studied the ring $R((x^{-1};\alpha))$ and introduced skew inverse Laurent-serieswise Armendariz ring as a generalization of the standard Armendariz condition from polynomials to skew inverse Laurent series. A ring R is said to be *skew inverse Laurent-serieswise Armendariz* ring, if for each $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\alpha))$, $f(x)g(x) = 0$, it implies $a_i\alpha^i(b_j) = 0$, for each $i \leq n$ and $j \leq m$. They study relations between the set of annihilators in R and the set of annihilators in $R((x^{-1};\alpha))$. Among applications, they show that a number of interesting properties of a skew inverse Laurent-serieswise Armendariz ring R such as the Baer, p.p. and the α-quasi Baer property transfer to its skew inverse Laurent series extensions $R((x^{-1};\alpha))$ and vice versa.

The purpose of this article is to introduce and investigate a special type of skew inverse Laurent-serieswise Armendariz rings. We say R is a *strongly Armendariz ring of inverse skew power series type* (or simply, *strongly* ISP-*Armendariz ring*), if for each $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j \in R((x^{-1};\alpha))$, $f(x)g(x) = 0$ if and only if $a_i b_j = 0$ for each $i \leq m$ and $j \leq n$. We study radical properties of R and $R((x^{-1};\alpha))$ for strongly ISP-Armendariz rings; and show that if R is a strongly ISP-Armendariz ring, then both R and $R((x^{-1};\alpha))$ are semicommutative rings and satisfy the Köthe's conjecture. We also prove that for a strongly ISP-Armendariz ring R, R is symmetric (respectively reversible, zip, prime, semiprime) if and only if so is $R((x^{-1};\alpha))$.

We use $N_0(R)$, $N_*(R)$, $L\text{-}rad(R)$, $N^*(R)$, $N_l^*(R)$, $N(R)$ and $J(R)$ to denote the Wedderburn radical (i.e., sum of all nilpotent ideals), the lower nil radical (i.e., the prime radical), the Levitzky radical (i.e., sum of all locally nilpotent ideals), the upper nil radical (i.e., sum of all nil ideals), the sum of all nil left ideals, the set of all nilpotent elements and the Jacobson radical of R, respectively. Note that the sum of all nil left ideals of R coincides with the sum of all nil right ideals of R, since Ra is nil implies that Rxr is also nil, for any $a \in R$.

2. Main results

Definition 2.1. [30, Definition 2.2] A ring R is called a *skew inverse Laurent-serieswise Armendariz* (or simply, *SIL-Armendariz*) *ring*, if for each elements $f(x) = \sum_{i=-\infty}^{n} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{m} b_j x^j$ in $R((x^{-1},\alpha))$, $f(x)g(x) = 0$ implies that $a_i \alpha^i(b_j) = 0$, for each $i \leq n$ and $j \leq m$.

As a first result of this article, in the following theorem, we state several equivalent definitions for SIL-Armendariz rings.

Theorem 2.2. *Let R be a ring with an automorphism α and $A = R((x^{-1};\alpha))$. Then the following statements are equivalent:*

(i) *R is SIL-Armendariz.*

(ii) *For each $f(x) = \sum_{i=-\infty}^{0} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{0} b_j x^j$ in A, if $f(x)g(x) = 0$, it implies $a_i \alpha^i(b_j) = 0$, for each $i, j \leq 0$.*

(iii) *For each $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j$ in A, if $f(x)g(x) = 0$, it implies $a_0 b_j = 0$, for each $j \leq n$.*

(iv) *For each $f(x) = \sum_{i=-\infty}^{0} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{0} b_j x^j$ in A, if $f(x)g(x) = 0$, it implies $a_0 b_j = 0$, for each $j \leq 0$.*

Proof. We only need to prove (iv) $\Rightarrow$ (i). For this purpose, let $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j$ be series of $R((x^{-1};\alpha))$ with

$f(x)g(x) = 0$. We have

$$\begin{aligned} f(x)g(x) &= (\textstyle\sum_{i=-\infty}^{m} a_i x^i)(\sum_{j=-\infty}^{n} b_j x^j) \\ &= (\textstyle\sum_{i=-\infty}^{m} a_i x^{i-m})x^m(\sum_{j=-\infty}^{n} b_j x^j) \\ &= (\textstyle\sum_{i=-\infty}^{m} a_i x^{i-m})(\sum_{j=-\infty}^{n} \alpha^m(b_j)x^{j+m}) = 0. \qquad (*) \end{aligned}$$

By multiplying x^{-m-n} from the right-hand side of Eq. $(*)$, we obtain

$$(\sum_{i=-\infty}^{m} a_i x^{i-m})(\sum_{j=-\infty}^{n} \alpha^m(b_j)x^{j-n}) = 0.$$

So $a_m\alpha^m(b_j) = 0$, for each $j \leq n$, by (vi). This implies that

$$\begin{aligned} f(x)g(x) &= (\textstyle\sum_{i=-\infty}^{m-1} a_i x^i)(\sum_{j=-\infty}^{n} b_j x^j) \\ &= (\textstyle\sum_{i=-\infty}^{m-1} a_i x^{i-m+1})x^{m-1}(\sum_{j=-\infty}^{n} b_j x^j) \\ &= (\textstyle\sum_{i=-\infty}^{m-1} a_i x^{i-m+1})(\sum_{j=-\infty}^{n} \alpha^{m-1}(b_j)x^{j+m-1}) = 0. \, (**) \end{aligned}$$

By multiplying x^{-m-n+1} from the right-hand side of Eq. $(**)$, we have

$$(\sum_{i=-\infty}^{m-1} a_i x^{i-m+1})(\sum_{j=-\infty}^{n} \alpha^{m-1}(b_j)x^{j-n}) = 0.$$

Hence $a_{m-1}\alpha^{m-1}(b_j) = 0$, for each $j \leq n$, by (vi). By continuing in this way, we get $a_i\alpha^i(b_j) = 0$, for each $i \leq m$ and $j \leq n$ and the result follows. □

Definition 2.3. Let R be a ring with an automorphism α. We say that R is a *strongly Armendariz ring of inverse skew power series type* (or simply, *strongly* ISP*-Armendariz ring*), if R satisfies the following condition.

$$\forall\ f(x), g(x) \in R((x^{-1};\alpha))\ f(x)g(x) = 0 \ \Leftrightarrow\ ab = 0\ \forall\, a \in C_f, b \in C_g,$$

where C_f and C_g are the sets of all coefficients of elements $f(x)$ and $g(x)$, respectively.

Recall that a ring R is said to be *semicommutative* if for all $a, b \in R$ we have $ab = 0 \Rightarrow aRb = 0$. It is obvious that every commutative ring is semicommutative. Thus, semicommutative rings provide a sort of bridge between commutative and noncommutative ring theory. On the one hand, the semicommutative condition forces a noncommutative ring to have certain affinities with its commutative cousins (e.g., it must be Dedekind-finite, it cannot be a full matrix ring, etc.). Semicommutative rings are studied in papers of Du [12], Hirano [19], Huh, Lee and Smoktunowicz [22] and Nielsen [32]. In Bell's paper [5] semicommutative is called the *insertion-of-factors-property*, or (*I.F.P.*). Clearly, reduced rings are semicommutative. In [15, Section 2], the authors constructed a rich source of examples of non reduced semicommutative rings.

Theorem 2.4. *Let α be an automorphism of a ring R. If R is strongly* ISP*-Armendariz, then we have the following statements:*

(i) *R is semicommutative.*
(ii) *$N_0(R) = N_*(R) = L\text{-}rad(R) = N^*(R) = N_l^*(R) = N(R)$.*

Proof. (i) Let a and b be two elements of R with $ab = 0$ and r be an arbitrary element of R. Suppose $f(x) = arx^{-1} - a$ and $g(x) = \cdots + r\alpha^{-1}(r)\alpha^{-2}(r)\alpha^{-3}(b)x^{-3} + r\alpha^{-1}(r)\alpha^{-2}(b)x^{-2} + r\alpha^{-1}(b)x^{-1} + b$ be elements of $R((x^{-1};\alpha))$. Theefore $f(x)g(x) = 0$ and so $arb = 0$, since R is a strongly ISP-Armendariz ring. Hence R is semicommutative and the result follows.
(ii) It is clear, by [23, Lemma 2.3]. □

According to Krempa [24], an endomorphism α of a ring R is said to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be *α-rigid* if there exists a rigid endomorphism α of R. In [17], the authors introduced α-compatible rings and studied its properties. A ring R is *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Clearly, this may only happen when the endomorphism α is injective. Also, by [17, Lemma 2.2], a ring R is α-compatible and reduced if and only if R is α-rigid. Thus α-compatible rings are generalization of α-rigid rings to the more general case, where R is not assumed to be reduced.

Lemma 2.5. *Let α be an automorphism of a ring R. If R is strongly* ISP*-Armendariz, then we have the following statements:*

(i) *R is an α-compatible ring.*
(ii) *If $ab = 0$, then $a\alpha^k(b) = \alpha^k(a)b = 0$ for all integers k.*
(iii) *If $\alpha^k(a)b = 0$ for some integer k, then $ab = 0$.*

Proof. (i) Let a and b be elements of R. If $ab = 0$, then $axb = 0$, since R is strongly ISP-Armendariz. Thus $a\alpha(b) = 0$. Now, let $a\alpha(b) = 0$. So $axb = 0$ and consequently $ab = 0$, since R is strongly ISP-Armendariz. Therefore R is an α-compatible ring.
(ii) and (iii) are clear, by [17, Lemma 3.2], since α is an automorphism. □

Note that for an α-compatible rings two concepts SIL-Armendariz and strongly ISP-Armendariz over a ring R is coincide. Also all of the examples of SIL-Armendariz rings which construct in [30] are α-compatible. So we propose the following important question.

Question. Does a ring R with an automorphism α exist which R is SIL-Armendariz but is not α-compatible?

Theorem 2.6. *Let α be an automorphism of a ring R. Then R is α-rigid if and only if R is reduced and strongly ISP-Armendariz.*

Proof. Let α be a rigid endomorphism of R and a an element of R with $a\alpha^{-1}(a) = 0$. Thus $\alpha(a)a = 0$ and hence $a\alpha(a) = 0$, since R is reduced. Therefore $a = 0$, since R is α-rigid. This implies that R is α-rigid if and only if R is α^{-1}-rigid. Hence R is α-rigid if and only if R is reduced and strongly ISP-Armendariz, by Theorem 2.10 of [30]. □

Proposition 2.7. *Let α be an automorphism of a ring R and $A = R((x^{-1};\alpha))$. Then the following conditions are equivalent.*

(i) *R is strongly ISP-Armendariz.*
(ii) *For each $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j$ in A, $f(x)g(x) = 0$ if and only if $a_i b_j = 0$, for each $i \leq m$ and $j \leq n$.*

Proof. (i) $\Rightarrow$ (ii) Let R be an strongly ISP-Armendariz ring and $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j$ be elements of $R((x^{-1};\alpha))$. If $f(x)g(x) = 0$, then $a_i\alpha^i(b_j) = 0$ for each $i \leq m$ and $j \leq n$, by Theorem 2.2 and consequently $a_i b_j = 0$ for each $i \leq m$ and $j \leq n$, since R is α-compatible by Lemma 2.5. Now, suppose $a_i b_j = 0$ for each $i \leq m$ and $j \leq n$. So $a_i\alpha^i(b_j) = 0$, since R is α-compatible, by Lemma 2.5 and consequently $f(x)g(x) = 0$.
(ii) $\Rightarrow$ (i) It is straightforward. □

Theorem 2.8. *Let α be an automorphism of a ring R. If R is strongly ISP-Armendariz, then we have the following statements:*

(i) *$R((x^{-1};\alpha))$ is semicommutative.*
(ii) *$N_0(A) = N_*(A) = L\text{-}rad(A) = N^*(A) = N_l^*(A) = N(A)$, where $A = R((x^{-1};\alpha))$.*

Proof. (i) Let $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ and $g(x) = \sum_{j=-\infty}^{n} b_j x^j$ be elements of A with $f(x)g(x) = 0$ and $h(x) = \sum_{k=-\infty}^{t} c_k x^k$ be an arbitrary element of A. Since R is strongly ISP-Armendariz, we have $a_i b_j = 0$ for each $i \leq -m$ and $j \leq n$. Also R is a semicommutative ring, by part (i) of Theorem 2.4. Thus $a_i c_k b_j = 0$ for each $k \leq t$ and so $a_i\alpha^i(c_k)\alpha^{i+k}(b_j) = 0$, for each $i \leq m$, $k \leq t$ and $j \leq n$, by Lemma 2.5.

This implies that $f(x)h(x)g(x) = 0$. Hence $R((x^{-1};\alpha))$ is a semicommutative ring and the result follows.
(ii) It is similar to the proof of part (ii) of Theorem 2.4. □

Theorem 2.9. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. Then*

$$\mathfrak{Rad}(R((x^{-1};\alpha))) \cap R = \mathfrak{Rad}(R),$$

where $\mathfrak{Rad}(\text{-})$ is one the radicals $N_0(\text{-}), N_*(\text{-})$, L-$rad(\text{-})$, $N^*(\text{-})$, $N_l^*(\text{-})$ or $N(\text{-})$.

Proof. Note that $N(R((x^{-1};\alpha))) \cap R = N(R)$. So the proof is clear, by Theorems 2.4 and 2.8. □

Lemma 2.10. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz and n a positive integer. If $f_1, \ldots, f_n$ are elements of $R((x^{-1};\alpha))$ with $f_1 \cdots f_n = 0$, then $a_1 \cdots a_n = 0$, where $a_i \in C_{f_i}$ for $1 \leq i \leq n$.*

Proof. Let $f_i(x) = \sum_{l=-\infty}^{m_i} a_l^{(i)} x^l$, for each $i = 1, \ldots, n$. The proof is by induction on n. The proof of the case $n = 2$ is clear, by Proposition 2.7. Now, let $n > 2$. Since R is strongly ISP-Armendariz and $f_1(f_2 \cdots f_n) = 0$, we have $a_1 b = 0$ for each $a_1 \in C_{f_1}$, where b is a coefficient of $f_2 \cdots f_n$. So $(a_1 f_2) f_3 \cdots f_k = 0$. Thus $a_1 a_2 \cdots a_n = 0$ for each $a_i \in C_{f_i}$ and we are done. □

Corollary 2.11. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. Then $N(R((x^{-1};\alpha))) \subseteq N(R)((x^{-1};\alpha))$.*

Recall that a ring R is said to have *bounded index of nilpotency* if there exists a positive integer n such that $r^n = 0$ for each nilpotent element r of R.

Proposition 2.12. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. If $N(R)((x^{-1};\alpha)) \subseteq N(R((x^{-1};\alpha)))$, then R has bounded index of nilpotency.*

Proof. To the contrary, let for each $i \in \mathbb{N}$ there exists $a_i \in N(R)$ such that $a_i^i \neq 0$. Consider

$$f(x) = a_1 x^{-1} + a_2 x^{-2!} + a_3 x^{-3!} + \cdots \in N(R)((x^{-1};\alpha)).$$

Since $N(R)((x^{-1};\alpha)) \subseteq N(R((x^{-1};\alpha)))$, there exists $t \geq 2$ with $f^t = 0$. Let $n \geq t$. We have

$$f(x) = (a_1 x^{-1} + a_2 x^{-2!} + \cdots + a_n x^{-n!}) + x^{-(n+1)!} h(x).$$

The coefficient of $x^{-tn!}$ in f^t is equal to the coefficient of $x^{-tn!}$ in $a_1x^{-1}+a_2x^{-2!}+\cdots+a_nx^{-n!}$ which is

$$a_n\alpha^{-n!}(a_n)\alpha^{-2n!}(a_n)\cdots\alpha^{-(t-1)n!}(a_n).$$

So $a_n^t=0$, by α-compatibility of R which is impossible. Hence R has bounded index of nilpotency and we are done. $\square$

Theorem 2.13. *Let α be an automorphism of a ring R. If R is a strongly* ISP*-Armendariz ring which satisfies any one of the following conditions:*

(i) *R has the ACC and DCC on left annihilators;*
(ii) *R has the ACC on ideals;*
(iii) *R is left or right Goldie;*
(iv) *R has right Krull dimension.*

then

$$\mathfrak{Rad}(R((x^{-1};\alpha)))=\mathfrak{Rad}(R)((x^{-1};\alpha)),$$

where $\mathfrak{Rad}(R)$ is one the radicals $N_0(R), N_*(R)$, L-$rad(R)$, $N^*(R)$, $N_l^*(R)$ or $N(R)$.

Proof. First we prove that if R is a strongly ISP-Armendariz ring, then $N(R((x^{-1};\alpha)))\subseteq N_0(R)((x^{-1};\alpha))$. For this goal, let $f(x)=\sum_{i=-\infty}^{n}a_ix^i\in N(R((x^{-1};\alpha)))$. Thus the ideal generated by $f(x)$ in $R((x^{-1};\alpha))$ is nil, by Theorem 2.8. So Ra_iR is nil ideal in R, by Propositon 2.10 and consequently $a_i\in N^*(R)=N_0(R)$, for each $i\leq n$. Conversely, let $g(x)=\sum_{i=-\infty}^{n}a_ix^i\in N_0(R)((x^{-1};\alpha))$. Since R satisfies one of the above conditions, $N^*(R)=N_0(R)$ is a nilpotent ideal of R, by [18, Theorem 1], [25, Theorem 4.12], [27, Theorem 1] and [28], respectively. So there exists a positive integer t such that $(N_0(R))^t=0$. Therefore $(p(x)g(x)q(x))^t=0$ for each $p(x),q(x)\in R((x^{-1};\alpha))$, by Propositon 2.10. Hence $g(x)\in N^*(R((x^{-1};\alpha)))=N(R((x^{-1};\alpha)))$, by Theorem 2.8. So $N_0(R)((x^{-1},\alpha))\subseteq N(R((x^{-1},\alpha)))$ and the proof is complete $\square$

Proposition 2.14. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. Then we have the following statements:*

(i) *R is prime if and only if so is $R((x^{-1};\alpha))$.*
(ii) *R is semiprime if and only if so is $R((x^{-1};\alpha))$.*

Proof. (i) Let R be a prime ring and $f(x)R((x^{-1};\alpha))g(x)=0$, for some $f(x)$ and $g(x)$ in $R((x^{-1};\alpha))$. Suppose a and b are the leading coefficients $f(x)$ and $g(x)$, respectively. Thus $arb=0$ for each $r\in R$, since R is α-compatible by Lemma 2.5. This implies that $a=0$ or $b=0$, since R is prime and consequently $f(x)=0$ or $g(x)=0$.

Therefore $R((x^{-1};\alpha))$ is a prime ring. Conversely, suppose $R((x^{-1};\alpha))$ is prime and a,b are elements of R with $aRb=0$. So $aR((x^{-1};\alpha))b=0$, by Lemma 2.5. Thus $a=0$ or $b=0$, since $R((x^{-1};\alpha))$ is a prime ring. Hence R is prime and the result follows.
(ii) The proof is similar to that of part (i). □

Perhaps the greatest unsolved problem in noncommutative ring theory today is the Köthe's conjecture, which posits that a ring with no non-zero nil ideals has no non-zero nil one-sided ideals. (i.e., if $N^*(R)=0$, then R has no non-zero nil one-sided ideal; where $N^*(R)$ is the upper nil radical of R.) One can see more discussion of the Köthe conjecture and various related problems, in [33]. The Köthe's conjecture has been resolved in several special cases, including for rings with Krull dimension, for PI rings, and for algebras over uncountable fields. We will presently add strongly `ISP`-Armendariz rings to this list.

Theorem 2.15. *Let α be an automorphism of a ring R. Suppose R is strongly ISP-Armendariz. Then we have the following statements:*

(i) *R satisfies the Köthe's conjecture;*
(ii) *$R((x^{-1};\alpha,\delta))$ satisfies the Köthe's conjecture;*
(iii) $J(R[x])=N_0(R)[x]=N_*(R)[x]=L\text{-}rad(R)[x]=N^*(R)[x]=N_\ell^*(R)[x]$;
(iv) $N_*(M_n(R))=M_n(N_*(R))=M_n(N^*(R))=N^*(M_n(R))$;
(v) $J(A[y])=N_0(A)[y]=N_*(A)[y]=L\text{-}rad(A)[y]=N^*(A)[y]=N_\ell^*(A)[y]$;
(vi) $N_*(M_n(A))=M_n(N_*(A))=M_n(N^*(A))=N^*(M_n(A))$;

where $A=R((x^{-1};\alpha))$.

Proof. (i) Let $N^*(R)=0$. Thus R is semiprime and so, similar to the proof of [29, Proposition 2.18(i)], R has no non-zero nil one sided ideal. Hence R satisfies the Köthe's conjecture.
(ii) Let $N^*(R((x^{-1};\alpha)))=0$. Thus $R((x^{-1};\alpha))$ is semiprime an so R is semiprime, by Proposition 2.14. Similar to the proof of [29, Proposition 2.18(i)], one can see that $R((x^{-1};\alpha))$ has no non-zero nil one sided ideal. So $R((x^{-1};\alpha))$ satisfies the Köthe's conjecture.
(iii) and (iv) follow from part (i) and [25, Exercise 10.25].
(v) and (vi) follow from part (ii) and [25, Exercise 10.25]. □

Note that Amitsur [1] proved that any ring R satisfies the following inclusion:

$$J(R[x])\cap R\subseteq N(R).$$

But it remains an open question whether $J(R((x^{-1};\alpha)))\cap R$ is nil. Concerning this problem, we have the following theorem:

Theorem 2.16. *Let R be a ring with an endomorphism α and A_1, A_2 be two subrings of $R((x^{-1};\alpha))$, as follows:*

$$A_1 = \{f(x) = \sum_{i=-m}^{n} a_i x^i \ : a_i \in R\,, m, n \in \mathbb{N}\},$$

$$A_2 = \{f(x) = \sum_{i=0}^{n} a_i x^i \ : a_i \in R\,, n \in \mathbb{N}\}.$$

If R is a strongly `ISP`*-Armendariz ring, then $J(A_1) \cap R$ and $J(A_2) \cap R$ are nil.*

Proof. Let $a \in R$ be an element of $J(A_1)$. Then $1 - ax$ is an invertible element in A_1. So there exists an element $\sum_{i=-m}^{n} b_i x^i$ in A_1 such that

$$(1 - ax)(\sum_{i=-m}^{n} b_i x^i) = 1.$$

Therefore we have:

$$\begin{aligned}
&b_{-m} = 0;\\
&b_{-m+1} - a\alpha(b_{-m}) = 0;\\
&\vdots\\
&b_{-1} - a\alpha(b_{-2}) = 0;\\
&b_0 - a\alpha(b_{-1}) = 1;\\
&b_1 - a\alpha(b_0) = 0;\\
&\vdots\\
&b_n - a\alpha(b_{n-1}) = 0;\\
&a\alpha(b_n) = 0.
\end{aligned}$$

So by replacing $b_{-m} = 0$ in Equation $b_{-m+1} - a\alpha(b_{-m}) = 0$, we get $b_{-m+1} = 0$. By continuing in this way, we obtain $b_{-m} = \cdots = b_{-1} = 0$ and $b_0 = 1$. Now, by replacing $b_0 = 1$ in Equation $b_1 - a\alpha(b_0) = 0$, we have $b_1 = a$. So $b_2 - a\alpha(b_1) = 0$ implies that $b_2 = a\alpha(a)$. Continuing in this way, we get $a\alpha(a)\alpha^2(a)\cdots\alpha^n(a) = 0$ and hence $a^{n+1} = 0$, since R is α-compatible by Lemma 2.5.Thus $a \in N(R)$ and so $J(A_1) \cap R$ is nil. Similarly, we can show that $J(A_2) \cap R$ is nil and the proof is complete. ☐

Proposition 2.17. *Let α be an automorphism of a ring R. Suppose R is strongly* `ISP`*-Armendariz, $U = \{l_R(I)|\ I$ is a subset of $R\}$, $V = \{l_{R((x^{-1};\alpha))}(J)|\ J$ is a subset of $R((x^{-1};\alpha))\}$, $\phi : U \to V$ and $\psi : V \to U$, given by $\phi(L) = L((x^{-1};\alpha))$ and $\psi(L') = L' \cap R$, respectively; then $o\phi = id_U$.*

Proof. Let I be a subset of R and $A = R((x^{-1};\alpha))$. First we prove that $l_R(I)((x^{-1};\alpha)) = l_A(I)$. For this purpose, suppose that $f(x) = \sum_{i=-\infty}^{n} r_i x^i \in l_R(I)((x^{-1};\alpha))$. Thus $r_i \in l_R(I)$ for each $i \leq n$ and so $r_i\alpha^i(I) = 0$, by Lemma 2.5. So $f(x) \in l_A(I)$ and hence $l_R(I)((x^{-1};\alpha)) \subseteq l_A(I)$. Now, assume $g(x) = \sum_{j=-\infty}^{m} s_j x^j \in l_A(I)$. So $(\sum_{j=-\infty}^{m} b_j x^j)I = 0$. Therefore $b_j\alpha^j(I) = 0$ and hence $b_jI = 0$, by Lemma 2.5. Thus $b_j \in l_R(I)$ for each $j \leq m$. Hence $g(x) \in l_R(I)((x^{-1};\alpha))$ and consequently $l_R(I)((x^{-1};\alpha)) = l_A(I)$. Therefore ϕ is well defined. Next assume that J is a subset of A. Clearly, $l_A(J) \cap R = l_R(C_J)$. So ψ is well defined. Therefore $\psi o\phi(U) = U((x^{-1};\alpha,\delta)) \cap R = U$ and we are done. □

Theorem 2.18. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. Then R satisfies the ascending chain condition on left (resp., right) annihilators if and only if so does $R((x^{-1};\alpha))$.*

Proof. It is clear, by Proposition 2.17. □

According to Cohn [11], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for $a, b \in R$. Prior to Cohn's work, reversible rings were studied under the name *completely reflexive* by Mason in [31] and under the name *zero commutative*, or *zc*, by Habeb in [14]. In his monograph [36] on distributive lattices arising in ring theory, Tuganbaev investigates a property called *commutative at zero*, which is equivalent to the reversible condition on rings. Reversible rings are between commutative rings and semicommutative rings. A stronger condition than reversible was defined by Lambek in [26]. A ring R is called *symmetric* if for all $a, b, c \in R$ we have $abc = 0 \Rightarrow bac = 0$.

Theorem 2.19. *Let α be an automorphism of a ring R. If R is a strongly* ISP*-Armendariz ring, then we have the following statements:*

(i) *R is symmetric if and only if so is $R((x^{-1};\alpha))$.*
(ii) *R is reversible if and only if so is $R((x^{-1};\alpha))$.*

Proof. (i) If $R((x^{-1};\alpha))$ is symmetric, then clearly so is R. Now let $fgh = 0$ with $f(x)$, $g(x)$, $h(x)$ in $R((x^{-1};\alpha))$. Thus $abc = 0$ for each $a \in C_f$, $b \in C_g$ and $c \in C_h$, by Proposition 2.10. Since R is symmetric, $acb = 0$. Therefore $a\alpha^p(c)\alpha^q(b) = 0$ for each integers p and q, by Lemma 2.5. This implies that $fhg = 0$ and hence $R((x^{-1};\alpha))$ is symmetric.
(ii) It is similar to part (i). □

The concept of zip rings initiated by Zelmanowitz [37]. Zelmanowitz stated that any ring R satisfying the descending chain condition on right annihilators satisfies the following condition:

$$\forall\, \varnothing \neq X \subseteq R$$

$$\text{if } r_R(X) = 0, \text{ then } \exists\, Y \subseteq X \text{ with } r_R(Y) = 0 \text{ and } Y \text{ is finite.}$$

Faith in [13] called a ring R right *zip* if R satisfies the following condition. Zelmanowitz [37] noted that any ring satisfying the descending chain condition on right annihilators is right zip, and he also showed that there exist commutative zip rings which do not satisfy the descending chain condition on (right) annihilators. Beachy and Blair [4] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ is a zip ring. Hong et al. [21, Theorem 11] proved that R is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring, when R is a Armendariz ring.

Now we turn our attention to the relationship between the zip property of a ring R and these of the inverse skew Laurent series ring $R((x^{-1};\alpha))$.

Theorem 2.20. *Let α be an automorphism of a ring R. Suppose R is strongly ISP-Armendariz. Then R is a right zip ring if and only if so is $R((x^{-1};\alpha))$*

Proof. Let $A = R((x^{-1};\alpha))$ is right zip and X a non-empty subset of R such that $r_R(X) = 0$. We show that $r_A(X) = 0$. To see this, let $f(x) = \sum_{i=-\infty}^{n} a_i x^i$ be an element of $r_A(X)$. So $a_i \in r_R(X) = 0$ and so $f(x) = 0$. Hence there exists a finite set $Y \subseteq X$ such that $r_A(Y) = 0$. On the other hand, $r_R(Y) = r_A(Y) \cap R$. Therefore $r_R(Y) = 0$ and so R is right zip ring. Conversely, suppose R is right zip and let U be a non-empty subset of A such that $r_A(U) = 0$. Suppose $U_0 = C_U$ be the set of all coefficients of elements in U. If $a \in r_R(U_0)$, then $f(x)a = 0$ for any $f(x) \in U$, by Lemma 2.5 and so $a \in r_A(U) = 0$. That is, $r_R(U_0) = 0$. Therefore, there exists a finite set $V_0 \subseteq U_0$ such that $r_R(V_0) = 0$. For each $a \in V_0$, there exists $g_a(x) \in U$ such that some of coefficients of $g_a(x)$ is a. Let V be a minimal subset of U such that $g_a(x) \in U$, for each $a \in V_0$. Then V is a non-empty finite subset of U. Let W_0 be the set of all coefficients of elements in V. Then $V_0 \subseteq W_0$ and so $r_R(W_0) = 0$. If $f(x) = \sum_{i=-\infty}^{m} a_i x^i$ be elements of $r_A(V)$, then $g(x)f(x) = 0$, for each $g(x) \in V$ and so $a_i \in r_R(W_0) = 0$, for each $i \leq m$. Therefore $f(x) = 0$ and hence $r_A(V) = 0$. So $R((x^{-1};\alpha))$ is right zip and the result follows. $\square$

Corollary 2.21. *Let R be an α-rigid ring. Then R is right zip if and only if so is $R((x^{-1};\alpha))$.*

Proof. It is clear, by Theorem 2.6 $\square$

We finish the article, by considering the quasi Baer and p.q.-Baer properties of rings R and $R((x^{-1};\alpha))$.

A ring R is called *quasi Baer* if the left annihilator of every ideal of R is generated, as a left ideal, by an idempotent of R. Clark [10] introduced the quasi Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. The definitions of quasi Baer rings is left-right symmetric. E. Armendariz [3, Theorem B] proved that for a reduced ring R, $R[x]$ is a quasi Baer ring if and only if so is R. Birkenmeier et al. [7, Theorems 1.2 and 1.5] showed that for many polynomial extensions, a ring R is quasi Baer if and only if the polynomial extension over R is quasi Baer. In [30, Theorem 3.1] the authors proved that if R is a quasi Baer ring with an automorphism α, then $R((x^{-1};\alpha))$ is a quasi Baer ring. But, the following example shows that the converse of this result is not true, in general.

Example 2.22. Let S be a prime ring which is not simple and assume that I is a non-trivial ideal of S. Consider the ring

$$R = \{(a,b) \in S \oplus S \,:\, b - a \in I\}$$

and the automorphism α of R given by $\alpha((a,b)) = (b,a)$, for each $(a,b) \in R$. In [16, Example 2.9] it is shown that R is not quasi Baer. But similar to the proof of Example 3.2 of [30], one can see that $R((x^{-1};\alpha))$ is quasi Baer.

The following theorem shows that if R is a strongly `ISP`-Armendariz ring, then the converse of [30, Theorem 3.1] is indefeasible.

Theorem 2.23. *Let α be an automorphism of a ring R. Suppose R is strongly `ISP`-Armendariz. If $R((x^{-1};\alpha))$ is quasi Baer, then so is R.*

Proof. Assume that I is an ideal of R and $A = R((x^{-1};\alpha))$. Since A is a quasi Baer ring, we have $l_A(AIA) = Ae(x)$, for some idempotent $e(x) \in A$. Note that $e(x) = e \in R$, by Proposition 2.6 of [30]. Thus $Re \subseteq l_R(I)$. Now, let $r \in l_R(I)$. We show that $r \in l_A(AIA)$. To see this, let $a \in I$ and $f(x) = \sum_{j=-\infty}^{n} b_j x^j \in A$. Thus we have $rf(x)a = \sum_{i=-\infty}^{n} rb_j\alpha^j(a)x^j$. On the other hand $ra = 0$ implies that $r\alpha^j(a) = 0$ for each $j \leq n$, since R is α-compatible by Lemma 2.5. So $rb_j\alpha^j(a) = 0$ for each $j \leq n$, since R is semicommutative, by Theorem 2.4. Thus $rf(x)a = 0$ and hence $r \in l_A(AIA) = Ae$. This implies that $r = re$. Therefore $l_R(I) = Re$ and so R is quasi Baer. $\square$

In [8] Birkenmeier, Kim and Park, defined a ring to be called *left* (resp., *right*) *principally quasi Baer* (or simply *left* (resp., *right*) *p.q.-Baer*) if the left (resp., right) annihilator of every principal left (resp., right) ideal of R is generated by an idempotent, as a left (resp., right) ideal of R. Equivalently, R is left (resp., right) p.q.-Baer if R modulo the left (resp., right) annihilator of any principal left (resp., right) ideal is projective. A ring is called *p.q.-Baer* if it is both right and left p.q.-Baer. The class of p.q.-Baer rings includes all biregular rings, all quasi Baer rings and is closed under direct products and Morita invariance. Notice that the definition of a p.q.-Baer ring is not symmetric, however, if R is a semiprime ring, then R is right p.q.-Baer if and only if R is left p.q.-Baer.

Recall from [6], an idempotent $e \in R$ is *left* (resp., *right*) *semicentral* in R if $ere = re$ (resp., $ere = er$), for all $r \in R$. Equivalently, an idempotent $e \in R$ is left (resp., right) semicentral if Re (resp., eR) is an ideal of R. The set of all left semicentral idempotents of R is denoted by $S_l(R)$.

Definition 2.24. [9] Let $E = \{e_0, e_1, \ldots\}$ be a non-empty countable subset of $S_l(R)$. Then E is said to have a *generalized countable join* e if for a given $a \in R$, there exists $e \in S_l(R)$ such that

(i) $ee_i = e_i$, for all positive integers i;
(ii) If $ae_i = e_i$ for all positive integers i, then $ae = e$.

Finally, we state the sufficient condition on a ring R that p.q.-Baer property of R is also preserved by the inverse skew Laurent series extension $R((x^{-1}; \alpha))$.

Theorem 2.25. *Let α be an automorphism of a ring R. Suppose R is strongly* ISP*-Armendariz. Then $R((x^{-1}; \alpha))$ is a left p.q.-Baer ring if and only if R is left p.q.-Baer and every countable subset of $S_l(R)$ has a generalized countable join in R.*

Proof. It is clear by Theorem 3.18 of [30], since R is α-compatible by Lemma 2.5. □

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