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# COGENERATOR AND SUBDIRECTLY IRREDUCIBLE IN THE CATEGORY OF $S$-POSETS

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ABSTRACT. In this paper we study the notions of cogenerator and subdirectly irreducible in the category of $S$-posets. First we give some necessary and sufficient conditions for an $S$-poset to be a cogenerator. Then we see that under some conditions, regular injectivity implies generator and cogenerator. Recalling Birkhoff's Representation Theorem for algebras, we study subdirectly irreducible $S$-posets and prove this theorem for the category of ordered right acts over an ordered monoid. Among other things, we present the relationship between cogenerators and subdirectly irreducible $S$-posets.

## 1. Introduction and Preliminaries

Laan [8] studied the generators in the category of right $S$-posets, where $S$ is a pomonoid. Also Knauer and Normak [7] gave a relation between cogenerators and subdirectly irreducibles in the category of right $S$-acts. The main objective of this paper is to study cogenerators and subdirectly irreducible $S$-posets. Some properties of the category of $S$-posets have been studied in many papers, and recently in [2, 3, 5]. Now we give some preliminaries about $S$-act and $S$-poset needed in the sequel. A *pomonoid* is a monoid $S$ equipped with a partial order relation $\leq$ which is compatible with the monoid operation, in the sense that, if $s \leq t$ then $su \leq tu$, $us \leq ut$, for all $s, t, u \in S$. Let

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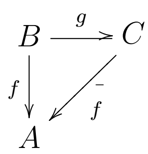
**Pos** denote the category of all partially ordered sets with order preserving ( monotone ) maps. A poset is said to be *complete* if each of its subsets has an infimum and a supremum. Recall that each poset can be embedded into a complete poset, its *Dedekind MacNeile completion* (see[1]). For a pomonoid $S$, a *right $S$-poset* is a poset $A$ with a function $\alpha : A \times S \to A$, called the action of $S$ on $A$, such that for $a, b \in A$, $s, t \in S$ (denoting $\alpha(a, s)$ by $as$) (1) $a(st) = (as)t$, (2) $a1 = a$ , (3)$a \leq b \Rightarrow as \leq bs$, (4) $s \leq t \Rightarrow as \leq at$. If $A$ satisfies conditions (1) and (2) only then it is called a *right $S$-act*. For two $S$-posets $A$ and $B$, an *$S$-poset morphism* is a map $f : A \to B$ such that $f(as) = f(a)s$ and $a \leq b$ implies $f(a) \leqslant f(b)$, for each $a, b \in A,\ s \in S$. We denote the category of all right $S$-poset, with $S$-poset morphisms between them by $\mathbf{Pos_S}$. For a pomonoid $T$, *left $T$-posets* can be defined analogously. A left $T$-poset $A$ which is also a right $S$-poset is called a $(T, S)$-*biposet* (and is denoted by ${}_TA_S$ ) if $(ta)s = t(as)$ for all $a \in A,\ t \in T,\ s \in S$. By $\mathbf{{}_TPos}$ and $\mathbf{{}_TPos_S}$ we mean the category of all left $T$-poset and the category of all $(T, S)$-biposets respectively. Recall from [3] that in the category $\mathbf{Pos_S}$ monomorphisms are exactly the one to one morphisms and also the epimorphisms and the onto morphisms coincide. A *regular monomorphism* or *embedding* is an $S$-poset morphism $f : A \to B$ such that $a \leq b$ if and only if $f(a) \leq f(b)$, for each $a, b \in A$. An $S$-poset morphism $f : A \to B$ is called a *retraction (coretraction)* provided that there exist some $S$-poset morphism $g : B \to A$ such that $fg = id_B$ ( $gf = id_A$). If there exists such a retraction, then $B$ ($A$) will be called a *retract (coretract)* of $A$ ($B$). It is easy to see that every coretraction is regular monomorphism. An $S$-poset $A$ is called *regular injective* if for each regular monomorphism $g : B \to C$ and each $S$-poset morphism $f : B \to A$ there exists an $S$-poset morphism $\bar{f} : C \to A$ such that $\bar{f}g = f$. That is the following diagram is commutative.




Let $A$ be a poset. Recall that [5], the right $S$-poset $A^{(S)} = Map(S, A)$ consisting of all monotone maps from $S$ into $A$ is a cofree $S$-poset on $A$.

**Proposition 1.1.** *Let $A$ be a complete $S$-poset. Then $A$ is regular injective if and only if it is a retract of the cofree $S$-poset $A^{(S)}$.*

*Proof.* It is easy to see that the $S$-poset map $\gamma_A : A \to A^{(S)}$ given by $a \mapsto \varphi_a$ with $\varphi_a : S \to A$ defined by $\varphi_a(s) = as$ is an order embedding.

Then since $A$ is regular injective, there exist a morphism $\pi_A : A^{(S)} \to A$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\gamma_A} & A^{(S)} \\ {\scriptstyle id_A}\downarrow & \swarrow{\scriptstyle \pi_A} & \\ A & & \end{array}$$

that is $\pi_A \circ \gamma_A = id_A$. Conversely since $A$ is complete, $A^{(S)}$ is a regular injective (see Theorem 3.3. of [5]). Now $A$ being a retract of a regular injective, is a regular injective. □

## 2. COGENERATORS

An object $A$ in the category $\mathbf{Pos_S}$ is called a cogenerator if the functor $Pos_S(-, A)$ is faithful, that is if for any $X, Y \in Pos_S$ and any $f, g \in Pos_S(X, Y)$ with $f \neq g$ there exists $\beta \in Pos_S(Y, A)$ such that

$$\beta f = Pos_S(f, A)(\beta) \neq Pos_S(g, A)(\beta) = \beta g.$$

That is, if $f \neq g$ then one has $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{\beta} A$ with $\beta f \neq \beta g$.

The following results are true in each category (see Proposition I.7.32., II.4.13., and II.4.14. of [6]).

**Proposition 2.1.** *Let $A$ be a cogenerator in a category $\mathcal{C}$. If $A \to A'$ is a monomorphism, then $A'$ is also a cogenerator in $\mathcal{C}$.*

**Proposition 2.2.** *Let $\mathcal{C}$ be a concrete category and $A \in \mathcal{C}$ be $|I|$-cofree, for $|I| \geq 2$. Then $A$ is a cogenerator in $\mathcal{C}$.*

**Lemma 2.3.** *If $A \in Pos_S$ is a cogenerator then $Pos_S(X, A) \neq \emptyset$ for all $X \in Pos_S$.*

Recall that an element $a$ in an $S$-poset $A$ is called a *zero element* if $as = a$ for all $s \in S$. Also $\Theta = \{\theta\}$ with the action $\theta s = \theta$ for all $s \in S$ and order $\theta \leq \theta$ is called the *one element* $S$-poset. Recall from [3] that coproducts in $Pos_S$ are disjoint unions.

The following result is an analogue of Proposition II.4.17. of [6] and the proof is the same as that.

**Proposition 2.4.** *If an $S$-poset $A$ is a cogenerator then $A$ contains two different zero elements.*

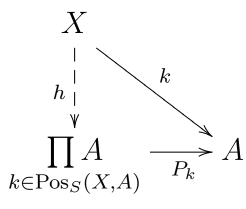
*Proof.* Consider $\Theta \underset{u_1}{\overset{u_2}{\rightrightarrows}} \Theta \bigsqcup \Theta$ in $\mathbf{Pos_S}$, where $u_1,\ u_2$ are the injections of the coproduct. Now, since $u_1 \neq u_2$ and $A$ is cogenerator, there exists $\Theta \bigsqcup \Theta \xrightarrow{\beta} A$ such that $\beta u_1 \neq \beta u_2$. Therefore $\beta u_1(\theta), \beta u_2(\theta)$ are two different zero elements in $A$. □

In the two next theorems we characterize cogenerators.

**Theorem 2.5.** *The following assertions are equivalent for a right $S$-poset $A$ in the category of all $S$-posets with regular monomorphism between them:*

(1) *for all $X, Y \in Pos_S$ and two morphisms $f, g : X \to Y, f \leq g$ whenever $\beta \circ f \leq \beta \circ g$ for all $\beta : Y \to A$;*
(2) *$A$ is a cogenerator;*
(3) *for every $X \in Pos_S$ there exists a set $I$ and a regular monomorphism $h : X \to \prod_I A$ in $Pos_S$;*
(4) *for every $X \in Pro_S$ there exists a set $I$ and a regular monomorphism $\pi : X^{(S)} \to \prod_I A$;*
(5) *if $X$ is a complete $S$-poset then the cofree object $X^{(S)}$ is a retract of $\prod_I A$ for some set $I$.*

*Proof.*
(1) ⇒ (2) Let for any morphisms $f, g : X \to Y$ and $\beta : Y \to A$, $\beta \circ f = \beta \circ g$. Now $\beta \circ f \leq \beta \circ g$ implies $f \leq g$, and also $\beta \circ g \leq \beta \circ f$ implies $g \leq f$, thus $f = g$.
(2) ⇒ (3) Let $A$ be a cogenerator. Then by Lemma 2.3, $\mathbf{Pos}_S(X, A) \neq \emptyset$, for every $X \in Pos_S$. By the universal property of products, there exists a unique regular monomorphism $h : X \to \prod_{k \in Pos_S(X,A)} A$ such that $p_k \circ h = k$ for every morphism $k : X \to A$ where $p_k : \prod_{k \in Pos_S(X,A)} A \to A$ are the projections maps.




(3) ⇒ (4) It is obvious.
(4) ⇒ (5) Let $X$ be a complete $S$-poset. Then $X^{(S)}$ is regular injective (see Theorem 3.3. of [5]). Since $\gamma : X^{(S)} \to \prod_I A$ is a regular

monomorphism, there exist an S-Poset morphism $\pi : \prod_I A \to X^{(S)}$ such that $\pi \circ \gamma = id_{X^{(S)}}$

$$\begin{array}{ccc} X^{(S)} & \xrightarrow{\gamma} & \prod_I A \\ {\scriptstyle id_{X^{(S)}}}\downarrow & \swarrow{\scriptstyle \pi} & \\ X^{(S)} & & \end{array}$$

that is $X^{(S)}$ is a retract of $\prod_I A$.

(5) $\Rightarrow$ (1) Let $f, g : X \to Y$ and $f \nleq g$. Then $f(x_0) \nleq g(x_0)$ for some $x_0 \in X$. We have to show that there exists a morphism $k : Y \to A$ such that $k \circ f \nleq k \circ g$. We know that each $S$-poset can be regularly embedded into a regular injective $S$-poset (see Theorem 2.11. of [5]) as follow:

$$\begin{array}{rcl} J : Y & \longrightarrow & \overline{Y}^{(S)} \\ y & \longmapsto & L_y \end{array}, \qquad \begin{array}{rcl} L_y : S & \longrightarrow & \overline{Y} \\ s & \longmapsto & \downarrow (y.s) \end{array}$$

where $\overline{Y}$ is the MacNeile completion of $Y$ (see[1]). Since $f(x_0) \nleq g(x_0)$ and $J$ is embedding we get $J(f(x_0)) \nleq J(g(x_0))$. Hence $J \circ f \nleq J \circ g$.

Now $\overline{Y}$ is a complete $S$-poset and hence, by assumption $\overline{Y}^{(S)}$ is a retract of $\prod_I A$. Consequently there exist morphisms $\pi : \overline{Y}^{(S)} \to \prod_I A$ and $\gamma : \prod_I A \to \overline{Y}^{(S)}$ such that $\gamma \circ \pi = id_{\overline{Y}^{(S)}}$. Now we have the following diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{J} \bar{Y}^{(S)} \underset{\gamma}{\overset{\pi}{\rightleftarrows}} \prod_I A\ .$$

Hence $\pi \circ J \circ f \nleq \pi \circ J \circ g$. It is because if $\pi \circ J \circ f \leq \pi \circ J \circ g$ then

$$\begin{aligned} J \circ f &= id_{\overline{Y}^{(S)}} \circ (J \circ f) = \gamma \circ \pi \circ J \circ f \leq \gamma \circ \pi \circ J \circ g \\ &= id_{\overline{Y}^{(S)}} \circ (J \circ g) = J \circ g \end{aligned}$$

which contradicts the fact $J \circ f \nleq J \circ g$. So there exists $j \in I$ such that

$$p_j \circ \pi \circ J \circ f \nleq p_j \circ \pi \circ J \circ g$$

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{J} \bar{Y}^{(S)} \underset{\gamma}{\overset{\pi}{\rightleftarrows}} \prod_I A \xrightarrow{p_j} A.$$

This is because if for every $j \in I$, $\rho_j \circ \pi \circ J \circ f \leq \rho_j \circ \pi \circ J \circ g$ then

$$\forall j \in I \quad p_j(\pi(J(f(x_0))) \leq p_j(\pi(J(g(x_0))).$$

Therefore $\pi(J(f(x_0)) \leq \pi(J(g(x_0))$. Since $\pi$ is a coretraction, it is a regular monomorphism. Now the mappings $\pi, J$ are regular monomorphism, thus we have $f(x_0) \leq g(x_0)$ which contradict the fact $f(x_0) \nleq g(x_0)$. Therefore there exists $j \in I$ such that $p_j \circ \pi \circ J \circ f \nleq p_j \circ \pi \circ J \circ g$, that is, there exists aregular monomorphism $k = p_j \circ \pi \circ J : Y \to A$ such that $k \circ f \nleq k \circ g$. □

**Corollary 2.6.** *If $A \in$ **Pos$_\mathbf{S}$** is a cogenerator then each $S$-poset $X$ can be regularly embedded into power of $A$.*

**Theorem 2.7.** *In the category* **Pos$_\mathbf{S}$***, a power of cogenerator is a cogenerator.*

*Proof.* Let $A \in$ **Pos$_\mathbf{S}$** be a cogenerator. By Proposition 2.2, $A^{(S)} \in$ **Pos$_\mathbf{S}$** is cogenerator. Now by Theorem 2.5, there exists a regular monomorphism and hence a monomorphism, $\alpha : A^{(S)} \to \prod\limits_I A$. Consequently by Proposition 2.1, $\prod\limits_I A$ is a cogenerator. □

**Definition 2.8.** Let $X, Y \in$ **Pos$_\mathbf{S}$**. Define the cotrace of $Y$ in $X$ by

$$cotr_X(Y) = \bigcap_{\beta \in Pos_S(X,Y)} \ker \beta = \cap\{(x, x') \in X \prod X \mid \beta(x) = \beta(x')\}.$$

**Theorem 2.9.** *A right $S$-poset $A$ is a cogenerator if and only if $cotr_Y(A) = \Delta_Y$ for all $Y \in$* **Pos$_\mathbf{S}$**.

*Proof.* Let $A$ be a cogenerator and $y \neq y'$, $y, y' \in Y$. Now for projections $p_1, p_2$ from $Y \prod Y$ we have $p_1|_{<(y,y')>}(y, y') = y \neq y' = p_2|_{<(y,y')>}(y, y')$, where $<(y, y')> = \{(ys, y's)|s \in S\}$ is a sub $S$-poset of $Y \prod Y$ generated by $(y, y')$, that is $p_1|_{<(y,y')>} \neq p_2|_{<(y,y'>)}$. Since $A$ is cogenerator, there exist $\beta \in Pos_S(Y, A)$ such that $\beta p_1|_{<(y,y')>} \neq \beta p_2|_{<(y,y')>}$ hence $\beta(y) \neq \beta(y')$, since otherwise if $\beta(y) = \beta(y')$ then $\beta(ys) = \beta(y's)$ for all $s \in S$, that is $\beta p_1|_{<(y,y')>} = \beta p_2|_{<(y,y')>}$ which is a contradiction. Conversely let $f, g : X \to Y$ and $f \neq g$. Then $f(x) \neq g(x)$ for some $x \in X$, by $\underset{\beta \in Pos_S(Y,A)}{\cap} ker\beta = \Delta$, there exists $\beta \in Pos_S(Y, A)$ such that $\beta(f(x)) \neq \beta(g(x))$ which implies $\beta f \neq \beta g$, that is, $A$ is cogenerator. □

Recall from [8] that a biposet ${}_TA_S$ is called faithful (regularly faithful, faithfully balanced) if the pomonoid homomorphisms $\lambda : T \to End(A_S)$

and $\rho : S \to End({}_TA)$ are injective (order reflecting, isomorphisms) where $End(A_S) = Pos_S(A, A)$ is a pomonoid with respect to composition and pointwise order also $End({}_TA) ={}_T Pos(A, A)$ is a pomonoid with multiplication $f.g = g \circ f$ for $f, g \in_T Pos(A, A)$. Also an $S$-poset $A$ is called faithful (regularly faithful, faithfully balanced) if the biposet ${}_{End(A_S)}A_S$ is faithful (regularly faithful, faithfully balanced).

**Theorem 2.10.** *Let ${}_TA_S \in {}_\mathbf{T}\mathbf{Pos}_\mathbf{S}$ be a faithfully balanced biposet and $\varphi : S \to A^{(S)}$, $\psi : T \to A^{(T)}$ be isomorphisms. If $A$ as a right $S$-poset is regular injective then ${}_TA \in {}_\mathbf{T}\mathbf{Pos}$, as a left $T$-poset, is a cogenerator and a generator.*

*Proof.* By assumption, $T \cong Pos_S(A, A)$ and $T \cong A^{(T)}, S \cong A^{(S)}$. Since $A$ is a regular injective there exists $A^{(S)} \underset{\gamma}{\overset{\pi}{\to}} A$ such that $\pi \circ \gamma = id_A$. Applying the functor $\text{Pos}(-, A)$ we get:

$$T \cong Pos_S(A, A) \underset{Pos_S(\gamma,A)=\gamma'}{\overset{Pos_S(\pi,A)=\pi'}{\rightleftarrows}} \text{Pos}(A^{(S)}, A) \cong Pos_S(S, A)$$

$Pos_S(\gamma, A) \circ Pos_S(\pi, A) = Pos_S(\pi \circ \gamma, A) = Pos_S(id_A, A) = id_{Pos_S(A,A)} = id_T$. But $Pos_S(S, A) \cong_T A$ (see Lemma 1.1. of [8]). Therefore we have

$${}_TT \underset{\gamma'}{\overset{\pi'}{\to}} Pos_S(S, A) \cong_T A \quad (*)$$

such that $\gamma' \circ \pi' = id_T$, hence $A$ is a generator (see Theorem 2.1 in [8]). But by Proposition 2.2, $A^{(T)}$ is a cogenerator and consequently, by $(*)$ we have:

$$A^{(T)} \cong_T T \overset{\pi'}{\to}_T A, \gamma' \circ \pi' = id_T$$

that is $\pi'$ is a monomorphism. Therefore, by Proposition 2.1, $A$ is a cogenerator. □

In the following we grt the relation between cogenerator and regularly faithful.

**Proposition 2.11.** *If an $S$-poset $A$ is a cogenerator then it is regularly faithful.*

*Proof.* We have to show that $\rho : S \to End({}_{End(A_S)}A)$ is an order reflecting. Since $A$ is a cogenerator, by Theorem 2.5, the morphism $g : S_S \to \prod_I A$ is a regular monomorphism. By Proposition $II.1.4$ in [6], for each $i \in I$, $p_i : \prod_I A \to A$ are retractions and hence there exists

$S$-poset morphism $q_i \in Pos_S(A, \prod_I A)$ such that $p_i q_i = id_A$. Therefore we have:

$$S \xrightarrow{g} \prod_I A \underset{q_i}{\overset{p_i}{\rightleftarrows}} A; \quad p_i q_i = id_A$$

Now let $\rho_s \leq \rho_{s'}$, where $s, s' \in S$ hence we have:

$$\begin{aligned} \forall i \in I, \rho_s(p_i(g(1))) &\leq \rho_{s'}(p_i(g(1))) \Rightarrow p_i(g(s)) = p_i(g(1.s)) = p_i(g(1)).s \\ &= \rho_s(p_i(g(1))) \leq \rho_{s'}(p_i(g(1))) = p_i(g(1)).s' = p_i(g(s')). \end{aligned}$$

Consequently for all $i \in I$, $p_i(g(s)) \leq p_i(g(s'))$ thus $g(s) \leq g(s')$. Since $g$ is an order embedding, hence $s \leq s'$. Therefore we get that $\rho$ is order reflecting. □

## 3. Subdirectly irreducible

In this section we first characterize subdirectly irreducible $S$-posets, then write the Birkhoff's Representation Theorem for this category, and finally we will give the relation between subdirectly irreducible and cogenerator $S$-posets. Although the proof of these theorems are the same as for $S$-acts (see [6]), we try to write a short proof for them. Recall that an equivalence relation $\theta$ on an $S$-act $A$ is called a *congruence* on $A$, if $a\theta a'$ implies $(as)\theta(a's)$ for $a, a' \in A$, $s \in S$. A congruence on an $S$-poset $A$ is a congruence $\theta$ on the $S$-act $A$ with the property that the $S$-act $A/\theta$ can be made into an $S$-poset in such a way that the natural map $A \to A/\theta$ is an $S$-poset morphism. We denote the set of all congruences on $A$ by $ConA$.

**Definition 3.1.** An $S$-poset $A$ is a subdirect product of an indexed family $(A_i)_{i \in I}$ of $S$-posets if $A$ is a sub $S$-poset of $\prod_{i \in I} A_i$ and $p_i(A) = A_i$ for each $i \in I$, where $p_i$'s are the restrictions to $A$ of projections from $\prod_{i \in I} A_i$.

*Remark* 3.2. For a right $S$-poset $A$ and each $a, b \in A$, $a \neq b$ we denote the maximal congruence on $A$ such that $a$ and $b$ are not related, by $\overline{\rho_{(a,b)}}$. This congruence exist by Zorn's Lemma. Consider $P = \{\theta \in ConA_S : (a, b) \notin \theta\}$. Then $(P, \subseteq)$ is a partially order set and $\Delta \in P$. For any chain $\{\theta_i\}_{i \in I}$ in $P$ the join $\bigvee_{i \in I} \theta_i$ is an upper bound and hence, by Zorn's Lemma, there exists $\overline{\rho_{(a,b)}}$.

A right $S$-poset $A$ is called *subdirectly irreducible* if $\bigcap_{i \in I} \rho_i \neq \Delta$ for all congruences $\rho_i$ on $A$ with $\rho_i \neq \Delta$. If $A$ is not subdirectly irreducible then it is called *subdirectly reducible* (see [4, 6]). Notice that for each

$S$-poset $A$ with $|A| = 2$ there exist only two congruences $\Delta$ and $\bigtriangledown$ on $A$ and so these $S$-posets are subdirectly irreducible.

**Theorem 3.3.** *Let $A$ be an $S$-poset and $a, b \in A$, $a \neq b$. Then $A/\overline{\rho_{(a,b)}}$ is subdirectly irreducible.*

*Proof.* Let $A/\overline{\rho_{(a,b)}}$ be subdirectly reducible. Hence $\sigma = \bigcap_{i \in I} \rho_i = \Delta$ where the elements of $\{\rho_i : i \in I\}$ are all non diagonal congruences on $A/\overline{\rho_{(a,b)}}$. Therefore there exists $i \in I$ such that $([a], [b]) \notin \rho_i$. But we know $\rho_i = \rho/\overline{\rho_{(a,b)}}$ where $\rho \in ConA$ and $\overline{\rho_{(a,b)}} \subseteq \rho$ that is we get a congruence $\rho$ on $A$ such that $(a, b) \notin \rho$ and $\overline{\rho_{(a,b)}} \subseteq \rho$ which contradicts the maximality of $\overline{\rho_{(a,b)}}$. Hence $A/\overline{\rho_{(a,b)}}$ is subdirectly irreducible. □

Now, similar to Birkhoff's Representation Theorem for algebra (see [4, 6]), we have:

**Theorem 3.4.** *(Birkhoff's Theorem for $S$-posets) Any nontrivial $S$-poset $A$ is a subdirect product of subdirectly irreducible $S$-posets of the form $A/\overline{\rho_{(a,b)}}$ for $a, b \in A$, $a \neq b$.*

**Corollary 3.5.** *A nontrivial $S$-poset $A$ is subdirectly irreducible if and only if $A \simeq A/\overline{\rho(a, b)}$ for some $a, b \in A$, $a \neq b$.*

*Proof.* Let $A$ be a nontrivial subdirectly irreducible $S$-poset. By Birkhoff's Theorem it is subdirect product of subdirectly irreducible$S$-posets of the form $A \simeq A/\overline{\rho_{(a,b)}}$ for $a, b \in A$, $a \neq b$. Since the intersection of the kernels of all restriction of the projections of the direct product is diagonal, and $A$ is subdirectly irreducible, therefore one of the kernel must be diagonal. Thus $A \simeq A/\overline{\rho(a, b)}$ for some $a, b \in A$, $a \neq b$. The converse is Theorem 3.3. □

We close the paper by the following proposition which gives the relation between cogenerators and subdirectly irreducible $S$-posets.

**Proposition 3.6.** *An $S$-poset $C$ is a cogenerator if and only if every subdirectly irreducible $S$-poset can be embedded into $C$.*

*Proof.* By Corollary 3.5 any nontrivial subdirectly irreducible $S$-poset is of the form $A/\overline{\rho_{(a,b)}}$ for some $a, b \in A$, $a \neq b$. Consider the two homomorphism $f_1, f_2 : S_S \to A/\overline{\rho_{(a,b)}}$ with $f_1(1) = [a] \neq [b] = f_2(1)$. As $C$ is a cogenerator, there exists a homomorphism $h : A/\overline{\rho_{(a,b)}} \to C$ such that $h([a]) \neq h([b])$. To prove that $h$ is a monomorphism, let $h([x]) = h([y])$ for $x, y \in A$ with $[x] \neq [y]$. Let $\rho$ be a relation on $A$ defined by

$$u\rho v \Leftrightarrow h([u]) = h([v]) \text{ } for \text{ } any \text{ } u, v \in A$$

That is $\rho$ is induced by the kernel congruence of $h$ and is itself a congruence on $A$. Since $h([x]) = h([y])$, $x\rho y$ and therefore $\overline{\rho(a, b)} \subsetneq \rho$. But

$h([a]) \neq h([b])$ that is $(a,b) \notin \rho$ which contradicts the maximality of $\overline{\rho(a,b)}$. Hence $h$ is a monomorphism. Conversely let $f,g : B \to A$ be two $S$-poset morphisms such that $f(b) \neq g(b)$ for some $b \in B$. Now for $\pi : A \to A/\overline{\rho(f(b),g(b))}$ and the embedding $h$ from subdirectly irreducible $A/\overline{\rho(f(b),g(b))}$ into $C$, we have $h\pi f \neq h\pi g$. Hence $C$ is a cogenerator. □

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