Journal of Algebraic Systems
Vol. 2, No. 2, (2014), pp 147-151

ON THE GROUPS WITH THE PARTICULAR NON-COMMUTING GRAPHS

N. AHANJIDEH* AND H. MOUSAVI

Abstract. Let G be a non-abelian finite group. In this paper, we prove that $\Gamma(G)$ is K_4-free, if and only if $G \cong A \times P$, where A is an abelian group, P is a 2-group and $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, we show that $\Gamma(G)$ is $K_{1,3}$-free if and only if $G \cong \mathbb{S}_3$, D_8 or Q_8.

1. Introduction

For an integer $z > 1$, we denote by $\pi(z)$ the set of all prime divisors of z. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. Let G be a non-abelian finite group and $Z(G)$ be its center. For $x \in G$, suppose that $cl_G(x)$ denotes the conjugacy class in G containing x and $C_G(x)$ denotes the centralizer of x in G. We will associate a graph $\Gamma(G)$ to G which is called the non-commuting graph of G. The vertex set $V(\Gamma(G))$ is $G - Z(G)$ and the edge set $E(\Gamma(G))$ consists of (x, y) (we write $x \sim y$), where x and y are distinct non-central elements of G such that $xy \neq yx$. Here, we are considering simple graphs, i.e., graphs with no loops or directed or repeated edges. The non-commuting graphs of the non-abelian finite groups have been studied in some literatures. For example in [1], the authors classified non-abelian finite groups with Hamiltonian non-commuting graphs, regular non-commuting graphs and planner non-commuting graphs. Also, it has been shown in [1] that the non-commuting graph of a non-abelian group is connected. Note that for a graph H, the H-free graph L is a graph that does not have an

MSC(2010): Primary: 20D60; Secondary: 05C25
Keywords: Non-commuting graph, K_4-free graph, $K_{1,3}$-free graph.
Received: 14 September 2014, Revised: 8 February 2015.
*Corresponding author .

induced subgraph isomorphic to L. Because of the special properties of $K_{1,3}$-free graphs and K_4-free graphs, they have been studied in some papers. In this paper, we are going to study non-abelian finite groups which their non-commuting graphs are $K_{1,3}$-free and non-abelian finite groups which their non-commuting graphs are K_4-free. Throughout this paper, we will use the following notation: let G be a finite non-abelian group and $M(G)$ denote a set of the orders of maximal abelian subgroups G. A set of vertices of a graph Γ is called an independent set, if its elements are pairwise nonadjacent. The independent number of a graph Γ, which is denoted by $\alpha(\Gamma)$, is the cardinality of the largest its independent set.

2. Some Lemmas

In this section, we bring some lemmas which will be used in the proof of the main theorem:

Lemma 2.1. *If G is a finite group and H, K and L are distinct proper subgroups of G such that $G = H \cup K \cup L$, then $[G : H] = [G : K] = [G : L] = 2$ and $H \cap L = H \cap K = K \cap L = H \cap K \cap L$.*

Proof. It follows immediately by considering the order of G. □

Lemma 2.2. [3] *If for every $x \in G - Z(G)$, $|cl_G(x)| = m$, then m is a power of the prime p and $G = P \times A$, where P is a p-Sylow subgroup of G and A is abelian.*

Lemma 2.3. *For every $x \in G - Z(G)$, there is a triangular in $\Gamma(G)$ containing the vertex x.*

Proof. Since $x \notin Z(G)$, there exists $y \in G - Z(G)$ such that $xy \neq yx$. Thus $x \sim y$ in $\Gamma(G)$. Since $C_G(x) \cup C_G(y) \neq G$, we deduce that x, y, z form a triangular, where $z \in G - (C_G(x) \cup C_G(y))$. □

It follows from Lemma 2.3 that:

Corollary 2.4. *$\Gamma(G)$ contains a triangular.*

Lemma 2.5. *Let $p, q \in \pi(G)$.*

(i) *If $M(G) \subseteq \{p, q\}$, then $M(G) = \{p, q\}$ and G is the non-abelian group of order pq.*
(ii) *If $M(G) \subseteq \{p, p^2\}$, then G is a p-group, $|Z(G)| = p$ and $M(G) = \{p^2\}$.*

Proof. Since every maximal abelian subgroup of a finite non-abelian p-group has order at least p^2, we get that G is not a p-group and hence, G is a $\{p, q\}$-group such that every Sylow subgroup of G has prime order and its center is trivial. Thus $M(G) = \{p, q\}$, as claimed in (i). The same reasoning completes the proof of (ii). □

Lemma 2.6. *If $M(G) \subseteq \{2, 4\}$, then $G \cong D_8$ or Q_8.*

Proof. By Lemma 2.5, $M(G) = \{4\}$, $|Z(G)| = 2$ and G is a non-abelian 2-group. Thus there exists $x \in G - Z(G)$ such that $O(x) = 4$. Also, we can see that $G/Z(G)$ is a 2-elementary abelian group and $C_G(x) = \langle x \rangle$. So $\langle x \rangle / Z(G)$ is a normal subgroup of $G/Z(G)$ and hence, $\langle x \rangle$ is normal in G. Therefore, $G/\langle x \rangle = G/C_G(x) \hookrightarrow \mathrm{Aut}(\langle x \rangle) \cong \mathbb{Z}_2$. This forces $|G| = 8$ and hence, lemma follows. □

Lemma 2.7. *If $\alpha(\Gamma(G)) \leq 2$, then $G \cong \mathbb{S}_3$, D_8 or Q_8.*

Proof. By [1, Remak 2.5], we can see that M is a maximal abelian subgroup of G if and only if $M - Z(G)$ is a maximal independent set of $\Gamma(G)$. Thus "$\alpha(\Gamma(G)) \leq 2$" implies that for every maximal abelian subgroup M of G, $|M| - |Z(G)| \leq 2$. Thus $|Z(G)|(|M|/|Z(G)| - 1) \in \{1, 2\}$. This forces $(|M|, |Z(G)|) \in \{(2, 1), (3, 1), (4, 2)\}$. Thus by Lemmas 2.5 and 2.6, the result follows. □

2.1. **Main results.**

Theorem 2.8. *Let G be a non-abelian finite group.*

(i) *$\Gamma(G)$ is K_4-free if and only if $G \cong A \times P$, where A is an abelian group, P is a 2-group and $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

(ii) *$\Gamma(G)$ is $K_{1,3}$-free if and only if $G \cong \mathbb{S}_3$, D_8 or Q_8.*

Proof. (i) "$\Longrightarrow$ " Let x be an arbitrary element of $G - Z(G)$ and $y \in G - C_G(x)$. Then for every element $z \in G - (C_G(x) \cup C_G(y))$, $G = C_G(x) \cup C_G(y) \cup C_G(z)$. Fix $K = C_G(x) \cap C_G(y) \cap C_G(z)$. By Lemma 2.1, $C_G(x) \cap C_G(y) = K$ is a normal subgroup of G, $|G/K| = 4$ and G/K contains different elements xK, yK and zK of order 2. Thus $G/K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now for every $a, b \in C_G(x) - (C_G(x) \cap C_G(y)) = C_G(x) - K$, we have $a, b \in G - (C_G(y) \cup C_G(z))$ and hence, $G = C_G(a) \cup C_G(y) \cup C_G(z) = C_G(b) \cup C_G(y) \cup C_G(z)$. Thus $C_G(a) - K = C_G(b) - K$, so $a \in C_G(b)$. Consequently, $C_G(x) = \langle C_G(x) - K \rangle$ is abelian. Similarly, we can see that $C_G(y)$ and $C_G(z)$ are abelian and hence, we get that $K \leq Z(G)$. Since G is non-abelian and $|G/K| = 4$, we get $K = Z(G)$ and $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Now Lemma 2.2 completes the proof.

" $\Longleftarrow$ " Let $G/Z(G) = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $C_G(a)$, $C_G(b)$ and $C_G(ab)$ are abelian and $G = C_G(a) \cup C_G(b) \cup C_G(ab)$. For every subset T of $G - Z(G)$ with four elements, at least one of the sets $T \cap C_G(a)$, $T \cap C_G(b)$ and $T \cap C_G(ab)$ contains more than two elements.

This forces T not to form K_4, as desired.
(ii) If $G \cong \mathbb{S}_3$, D_8 or Q_8, then it is obvious that $\Gamma(G)$ is $K_{1,3}$-free. Now let $\Gamma(G)$ be $K_{1,3}$-free. Let $M = \{x_1, ..., x_t\}$ be a maximal independent set of $\Gamma(G)$ such that $|M| = \alpha(\Gamma(G))$. We continue the proof in the following cases:

(a) Let $\alpha(\Gamma(G)) \geq 3$ and $x_{i_1}, x_{i_2}, x_{i_3}$ be three arbitrary elements of M. Since $\Gamma(G)$ is $K_{1,3}$-free, we deduce that for every $y \in G - Z(G)$, $y \in C_G(x_{i_1})$, $C_G(x_{i_2})$ or $C_G(x_{i_3})$. This shows that $G = C_G(x_{i_1}) \cup C_G(x_{i_2}) \cup C_G(x_{i_2})$. Thus Lemma 2.1 shows that

$$\cap_{j=1}^{3} C_G(x_{i_j}) = C_G(x_{i_1}) \cap C_G(x_{i_2}). \tag{2.1}$$

Now let $z \in G - (M \cup Z(G))$. If there exists $1 \leq i, j \leq t$ such that $z \not\sim x_i$ and $z \not\sim x_j$, then by (2.1), for every $u \in \{1, .., t\} - \{i, j\}$, $z \not\sim x_u$. Thus $M \cup \{z\}$ is an independent set, which is a contradiction. Therefore, there exists at most one $i \in \{1, ..., t\}$ such that $z \not\sim x_i$ and hence, z is adjacent to every $x_j \in M - \{x_i\}$. This means that z has at least $t - 1$ neighbors in M. Since $\Gamma(G)$ is $K_{1,3}$-free, $t - 1 \leq 2$. But $\alpha(\Gamma(G)) \geq 3$ and hence, $\alpha(\Gamma(G)) = 3$. So [2, Lemma 2.4] shows that there exists a maximal abelian subgroup M' of G such that $|M' - Z(G)| = 3$ and for every maximal abelian subgroup M'' of G, we have $|M'' - Z(G)| \leq 3$. If $|Z(G)| = 3$, then $|M'| = 6$, which is an abelian subgroup. Assume that $M' = \langle a \rangle$. Then $a^3 \notin Z(G)$. Consequently for $b \in G - C_G(a^3)$, $\{b\} \cup (M' - Z(G))$ is the vertex set of $K_{1,3}$, a contradiction. Thus $|Z(G)| = 1$ and $|M'| = 4$. If G has a maximal abelian subgroup M'' of order 3, then for some $x \in M''$ of order 3, $C_G(x) = M''$. Consequently $(M' - Z(G)) \cup \{x\}$ is the vertex set of $K_{1,3}$, a contradiction. Thus $M(G) = \{2, 4\}$ and hence, by Lemma 2.6, $G \cong D_8$ or Q_8. So $\alpha(G) = 2$, which is a contradiction.

(b) If $\alpha(\Gamma(G)) \leq 2$, then Lemma 2.7 completes the proof.

□

Acknowledgments

The authors express their gratitude to the referee for carefully reading and several useful comments, which improved the manuscript, in particular, Theorem 2.8(ii). The first author was partially supported by the center of Excellence for Mathematics, University of Shahrekord, Iran.

References

1. A. Abdollahi, S. Akbari and H. R. Maimani, Non-commuting graph of a group, *J. Algebra* **298** (2006), 468-492.
2. N. Ahanjideh and A. Iranmanesh, On the relation between the non-commuting graph and the prime graph, *Int. J. Group Theory* **1** (2012), 25-28.
3. N. Ito, On finite groups with given conjugate type I, *Nagoya Math. J.* **6** (1953), 17-28.

Neda Ahanjideh
Department of pure Mathematics, Shahrekord University, P.O.Box 115, Shahrekord, Iran.
Email: ahanjideh.neda@sci.sku.ac

Hajar Mousavi
Department of pure Mathematics, Shahrekord University, P.O.Box 115, Shahrekord, Iran.
Email: h.sadat68@yahoo.com