GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

H.R. AFSHIN, S. BAGHERI AND M.A. MEHRJOOFARD*

ABSTRACT. The rank-k numerical range has a close connection to the construction of quantum error correction code for a noisy quantum channel. For a noisy quantum channel, a quantum error correcting code of dimension k exists, if and only if the associated joint rank-k numerical range is non-empty. In this paper, the notion of joint rank-k numerical range is generalized, and some statements of [2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.] are extended.

1. Introduction

Let $M_n$ be the set of $n \times n$ complex matrices, and $A \in M_n$. Furthermore, assume that $k \in \{1, \ldots, n\}$, $\alpha \subset \{1, \ldots, n\}$. Throughout this paper, the following notations are fixed:

\[ \omega_k = \exp \left( \frac{2\pi i}{k} \right) \]

\[ \Omega_k = \{ \omega_k^0, \omega_k^1, \ldots, \omega_k^{k-1} \} \]

Besides, the symbol $\sigma(A)$ stands for the spectrum of the matrix $A$, and $A\alpha$ refers to the principal submatrix of $A$ that lies in the rows and columns of $A$ indexed by $\alpha$.

Recently, the joint higher rank numerical range [5] has played a key role in finding quantum error correcting codes[4], and some researchers

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have taken this into consideration. The present article mainly con-
centrates on extending the notion of joint rank-k numerical range of
\( A = (A_1, ..., A_m) \in M^n_m \), i.e. the set of all \( (a_1, ..., a_m) \in \mathbb{C}^m \), such that
there exists the orthogonal projector \( P \) of rank \( k \) that satisfies
\[ PA_jP = a_jP \forall j. \]

**Definition 1.1.** Let \( A = (A_1, ..., A_m) \in M^n_m, B \in M_k \), and \( k \leq n \). Then the set
\[ B_{\Lambda_k}(A) = \{(a_1, ..., a_m) \in \mathbb{C}^m : \exists U \in M_{n,k}, s.t., U^*U = I_k, U^*A_jU = a_jB \forall j \} \]
is called the joint matrix higher rank numerical range.

When \( B = \text{diag}(b_1, ..., b_k) \), we abbreviate \( B_{\Lambda_k}(A) \) as \( b_1, ..., b_k \Lambda_k(A) \), and in the case \( b_1 = k = 1 \), the joint numerical range of \( A \) is defined
as \( W(A) = b_1, ..., b_k \Lambda_k(A) \).

**Definition 1.2.** Let \( A = (A_1, ..., A_m) \in M_n^m, k \leq n \). The kth joint
matrix numerical range of \( A \) is the set
\[ W_k(A) = \{(U^*A_1U, U^*A_2U, ..., U^*A_mU) : U \in M_{n,k}, U^*U = I_k \} \]

In [3], The authors have introduced ”k-generalized projector”. They
have said that \( A \in M_n \) is the k-generalized projector, if \( A^k = A^* \) and
\( k > 1 \).

**Theorem 1.3.** [3] Let \( A \in M_n \), and \( k \in \mathbb{N}, k > 1 \). Then the following
statements are equivalent:

(a) \( A \) is a k-generalized projector.
(b) \( A \) is a normal matrix, and \( \sigma(A) \subset \{0\} \cup \Omega_{k+1} \).

Now, it is natural to extend ”joint higher rank numerical range” as
follows:

**Definition 1.4.** Let \( A = (A_1, ..., A_m) \in M_n^m \), and \( k \) and \( k' \) are positive
integers. Then the set
\[ \{(a_1, ..., a_m) \in \mathbb{C}^m : \exists k' - \text{generalized projector of rank } k (P), \ s.t., PA_jP = a_jP \forall j \} \]
is called the \( k' \)-generalized joint rank-k numerical range, and is abbrevi-
ated as \( G_{\Lambda_{k',k}}(A) \).

Notice that the recent definition is an obvious extension of ”gen-
eralized higher rank numerical range,” which has been defined in [1].
2. Main results

The proof of the following results is elementary and hence, we leave it to the interested reader.

**Proposition 2.1.** Let $A = (A_1, \cdots, A_m) \in M_n^m$, $k \leq n$, and $1 < k'$. The following statements are equivalent:

(i) $a = (a_1, a_2, \cdots, a_m) \in GA_{k', k} (A_1, A_2, \cdots, A_m)$.

(ii) There exist $b_1, \cdots, b_k \in \Omega_{k'+1}$ and unitary matrix $U \in M_n$, such that for any $j \in \{1, \cdots, m\}$,

\[ (U^*A_jU) \{1, 2, \cdots, k\} = a_j \text{diag} \left( \{b_i\}_{i=1}^k \right). \]

(iii) There exist $b_1, \cdots, b_k \in \Omega_{k'+1}$, and $X = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \in M_{n,k}$, such that $X^*X = I_k$, and

\[ \forall j \in \{1, \cdots, m\}, X^*A_jX = a_j \text{diag} \left( \{b_i\}_{i=1}^k \right). \]

(iv) There exists $b_1, \cdots, b_k \in \Omega_{k'+1}$, and orthonormal vectors $u_1, \cdots, u_k \in \mathbb{C}^n$, such that

\[ \forall r \in \{1, \cdots, m\} \forall i, j \in \{1, \cdots, k\}, \langle A_r u_i, u_j \rangle = a_r b_i \delta_{ij}. \]

**Proof.** One can deduce, from Theorem 1.3, the equivalency of (i) and (ii). Notice that $P$ is a $k'$–generalized projector of rank $k$, if and only if there exists a unitary matrix $U$, and numbers $b_1, \cdots, b_k \in \Omega_{k'+1}$ such that

\[ P = U^* \text{diag} \left( b_1, \cdots, b_k, 0, \cdots, 0 \right)_{n-k, 0^n} U. \]

Equivalence of parts (ii), (iii), and (iv) is obvious. \(\square\)

**Corollary 2.2.** Let $A = (A_1, \cdots, A_m) \in M_n^m$. Then:

(i) $\Lambda_k (A) \subset GA_{k', k} (A)$;

(ii) $GA_{k', k} (A) \subset GA_{k', k} ((A_1, \cdots, A_{m-1})) \times \mathbb{C}$;

(iii) $GA_{k', k+1} (A) \subset GA_{k', k} (A)$;

**Proof.** (i) is trivial, since every orthogonal projector of rank $k$ is a $k'$–generalized projector of rank $k$.

(ii) is obvious.

(iii) Assume that $(\lambda_1, \cdots, \lambda_m) \in GA_{k', k+1} (A)$. Then there exist orthonormal vectors $u_1, \cdots, u_{k+1} \in \mathbb{C}^n$, and $b_1, \cdots, b_{k+1} \in \Omega_{k'+1}$, such that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$, for $r \in \{1, \cdots, m\}$, $1 \leq i, j \leq k + 1$. Therefore, by considering the orthonormal vectors $u_1, \cdots, u_k \in \mathbb{C}^n$ and $b_1, \cdots, b_k \in \Omega_{k'+1}$, we see that $\langle A_r u_i, u_j \rangle = \lambda_r b_i \delta_{ij}$ for $r \in \{1, \cdots, m\}$, $1 \leq i, j \leq k$, and therefore, $(\lambda_1, \cdots, \lambda_m) \in GA_{k', k} (A)$. \(\square\)
Corollary 2.3. Let \( A = (A_1, \ldots, A_m) \in M^m_n \), and \( k', k > 1 \). If \( n \geq (k - 1)(m + 1)^2 \), then \( G_{A;k'}(A) \neq \emptyset \).

Proof. It suffices to consider [5, Proposition 2.4], and Corollary 2.2(i). \( \square \)

Proposition 2.4. Let \( k' > 1 \), and \( A \in M^m_n \). Then:

(i) \( G_{k', k}(A) = \bigcup_{b_1, \ldots, b_k \in \Omega_{k'+1}} A_k(A) \);

(ii) \( G_{k', 1}(A) = \bigcup_{b \in \Omega_{k'+1}} bW(A) \).

Proof. By definition, (i), and (ii) can readily be verified. \( \square \)

Corollary 2.5. Consider the Pauli matrices:

\[
I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

and let \( A_1, \ldots, A_m \in \{I, X, Y, Z\} \), and \( k' > 1 \). Then:

\[
G_{k', 1}(I_{2^n}, A_1 \otimes \cdots \otimes A_n) =
\begin{cases}
\bigcup_{b \in \Omega_{k'+1}} \{(b, b)\} : \{A_1, \ldots, A_m\} = \{I\} \\
\bigcup_{b \in \Omega_{k'+1}} b \{(1, a) : a \in [-1, 1]\} : \text{elsewhere}
\end{cases}
\]

Proof. It suffices to note that Pauli matrices are normal, and:

\[
\sigma(X) = \sigma(Y) = \sigma(Z) = \{-1, 1\}.
\]

\( \square \)

The proofs of the next two results are straightforward, and thus are omitted.

Corollary 2.6.

\[
G_{k', 1}(\{\text{diag}(a_{i1}, \ldots, a_{in})\}_{i=1}^m) = \bigcup_{b \in \Omega_{k'+1}} b \text{conv} \left( \{(a_{1j}, \ldots, a_{mj})\}_{j=1}^n \right)
\]

Lemma 2.7. Let \( A_j = \text{diag}(a_{1j}, \ldots, a_{nj}) \), \( j = 1, \ldots, m \). Then:

\[
G_{k', n}(A_1, \ldots, A_m) \subset \left( \bigcup_{b_1, \ldots, b_n \in \Omega_{k'+1}} \left\{ c_1 : c_1 = \frac{a_{11}}{b_1} = \cdots = \frac{a_{1n}}{b_n} \right\} \right) \times \left( \bigcup_{b_1, \ldots, b_n \in \Omega_{k'+1}} \left\{ c_2 : c_2 = \frac{a_{12}}{b_1} = \cdots = \frac{a_{2n}}{b_n} \right\} \right) \times \cdots \times \left( \bigcup_{b_1, \ldots, b_n \in \Omega_{k'+1}} \left\{ c_m : c_m = \frac{a_{1m}}{b_1} = \cdots = \frac{a_{nm}}{b_n} \right\} \right)
\]
Definition 2.8. [2] Let $S$ be a convex set, and $R := S^k = \{ z \in \mathbb{C} : z^k \in S \}$. Then $R$ is called the convex kth root set.

Corollary 2.9. Let $A_j = \text{diag}(a_{1j}, \ldots, a_{nj})$, $j \in \{1, \ldots, m\}$, $k' > 1$, and there exists $i_1, i_2 \in 1, \ldots, n$, such that $\frac{a_{i_1j}}{a_{i_2j}} \in \mathbb{C} \setminus \left( \mathbb{R} \cup \mathbb{R}^{\frac{k'}{k}} \right)$. Then:
$$G\Lambda_{k^r,n}(A_1, \ldots, A_m) = \emptyset.$$

Now, we extend [5, Proposition 2.5]:

Proposition 2.10. Let $A = (A_1, \ldots, A_m) \in M_{n}^m, B \in M_k$, and $1 \leq r < k \leq n$. Then:
$$B\Lambda_k(A) \subset \bigcap_{X \in M_{n,n-r}} \left\{ (a_1, \ldots, a_m) \in \mathbb{C}^m : \left( W_{k-r}(a_1B, \ldots, a_mB) \cap W_{k-r}(X^*A_1X, \ldots, X^*A_mX) \right) \neq \emptyset \right\}$$

Proof. Let $a = (a_1, \ldots, a_m) \in B\Lambda_k(A)$, and $X \in M_{n,n-r}$ be such that $X^*X = I_{n-r}$. Then there exists $U \in M_{n,k}$, such that:
$$U^*U = I_k, U^*A_jU = a_jB \ \forall j.$$ We can choose the orthonormal vectors $x_1, \ldots, x_{k-r} \in (\mathbb{C}^{n-r}) \cap (U\mathbb{C}^k)$, and therefore, there exist $Y = [y_1, \ldots, y_{k-r}] \in M_{k,k-r}, Z = [z_1, \ldots, z_{k-r}] \in M_{n-r,k-r}$, such that $XZ = [x_1, \ldots, x_{k-r}] = UY$, and $Y^*Y = I_{k-r} = Z^*Z$. Therefore,
$$Z^*X^*A_jXZ = a_jY^*BY \ \forall j,$$
and the proof is completed. \hfill \Box

The following lemma can directly follow from the definition.

Lemma 2.11. Let $A = (A_1, \ldots, A_m) \in M_{n_1}^m, C = (C_1, \ldots, C_m) \in M_{n_2}^m, B \in M_{k, k \leq \min\{n_2, n_1\}}$, and there exists the matrix $V \in M_{n_1,n_2}$, such that $V^*V = I_{n_2}$, and for any $1 \leq j \leq m$, $C_j = V^*A_jV$. Then:
$$B\Lambda_k(C) \subset B\Lambda_k(A)$$

Lemma 2.12. Let $A = (A_1, \ldots, A_m) \in M_{n_1}^m, C = (C_1, \ldots, C_m) \in M_{n_2}^m, B \in M_{k, k \leq n_1 \leq n_2}$, and for any $j$, $A_j = C_j \{1, \ldots, n_1\}$. Then:
$$B\Lambda_k(A) \subset B\Lambda_k(C).$$

Proof. Let $(a_1, \ldots, a_m) \in B\Lambda_k(A)$. Therefore, there exist $X \in M_{n_1,k}$, such that $X^*X = I_k$, and $X^*A_jX = a_jB$, for all $j \in \{1, \ldots, m\}$. Now, define $Y = \left[ \begin{array}{c} X \\ 0 \end{array} \right]_{n_2 \times k}$. Then, one can see that $Y^*Y = I_k$, and $Y^*C_jY = a_jB$, for all $j \in \{1, \ldots, m\}$. \hfill \Box
Corollary 2.13. Let \( A = (A_1, \ldots, A_m) \in M_{n_1}^m, C = (C_1, \ldots, C_m) \in M_{n_2}^m, k \leq n_1 \leq n_2, \) and for any \( j, A_j = C_j \{1, \ldots, n_1\}, 1 < k' \); and there exists the matrix \( V \in M_{n_1,n_2}, \) such that \( V^*V = I_{n_2}, \) and for any \( 1 \leq j \leq m, D_j = V^*A_jV. \) Then:

\[
\Gamma A_{k',k}(D) \subset \Gamma A_{k',k}(A) \subset \Gamma A_{k',k}(C).
\]

The following theorem is an extension of [5, Theorem 3.1].

Theorem 2.14. Let \( A = (A_1, \ldots, A_m) \in M_{n_1}^m, \hat{k} \geq (m + 2)k, B \in M_{k}, (0, \ldots, 0) \in B\Lambda_k(A), \) and \( (a_1, \ldots, a_m) \in B\Lambda_k(A). \) Then for any \( t \in [0,1], \)

\[
t(a_1, \ldots, a_k) \in B\Lambda_k(A).
\]

Proof. Assume that there exist \( X \in M_{n,k}, \) and \( V \in M_{n,(m+2)k}, \) such that:

\[
X^*X = I_k, \forall j, X^*A_jX = a_jB, \\
V^*V = I_{(m+2)k}, \forall j, V^*A_jV = 0_{(m+2)k}.
\]

By the Lemma 2.11 and Lemma 2.12, it suffices to show that there is the non-singular matrix \( Z \in M_{n,(m+2)k}, \) such that:

\[
\begin{cases}
Z^*Z = I_{(m+2)k}, \\
\forall j, Z^*A_jZ = \left[ \begin{array}{ccc} a_jB & 0_k & * \\ 0_k & 0_k & * \\ * & * & (m+2)k \times (m+2)k \end{array} \right].
\end{cases}
\]

Because, in this case, we have:

\[
\begin{aligned}
& B\Lambda_k \left( \begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \\ \vdots & \vdots \\ a_mB & 0_k \\ 0_k & 0_k \end{bmatrix} \right) \\
\subset & B\Lambda_k \left( Z^*A_1Z, \ldots, Z^*A_mZ \right) \\
\subset & B\Lambda_k \left( A_1, \ldots, A_m \right)
\end{aligned}
\]

and

\[
\forall t \in [0,1], t(a_1, \ldots, a_k) \in B\Lambda_k \left( \begin{bmatrix} a_1B & 0_k \\ 0_k & 0_k \\ \vdots & \vdots \\ a_mB & 0_k \\ 0_k & 0_k \end{bmatrix} \right)
\]

(\text{Note that for any } t \in [0,1] \text{ there exists } U = \begin{bmatrix} \sqrt{t}I_k \\ \sqrt{1-t}I_k \end{bmatrix} \text{ such that for all } j, U^* \begin{bmatrix} a_jB & 0_k \\ 0_k & 0_k \end{bmatrix} U = ta_jB.)

Now, we want to select \( Y \in M_{n,k}, \) and \( W \in M_{n,mk}, \) such that their columns selected from columns of \( V \) and \( Z = \begin{bmatrix} X & Y & W \end{bmatrix} \) satisfy in 2.1. But, 2.1 is equivalent to:

\[
\begin{cases}
Y^*Y = I_k, W^*W = I_{mk}, X^*Y = 0_k, X^*W = 0_{k, mk}, Y^*W = 0_{mk}, \\
\forall j, X^*A_jY = Y^*A_jX = Y^*A_jY = 0_k.
\end{cases}
\]
Thus, in order to find the $Y$, it is sufficient to find the $k$ columns of columns of $V$, such that they lie in the space $H^\perp$, such that:

$$H = \text{span} \left( \{ \text{columns of } X \} \cup \{ \text{columns of } A_1X \} \cup \cdots \cup \{ \text{columns of } A_mX \} \right).$$

But $\dim(H) \leq (m+1)k$, while $V$ has $(m+2)k$ columns. Therefore, we can construct $Y$. Now, $\text{span}(\{ \text{columns of } X \} \cup \{ \text{columns of } Y \})$ is a space with dimension $2k$ and so we can find the $mk$ columns of $V$, such that they lie not in this space, and assume $W$, such that their columns are constructed by those columns.

Also, we can extend [5, proposition 2.1], as follows:

**Proposition 2.15.** Suppose $A = (A_1, \cdots, A_m) \in M_n^m, k \leq n, 1 < k'$ and $S = (s_{ij})$ is an $m \times n$ matrix. If $B_j = \sum_{i=1}^{m} s_{ij}A_i$, for $j = 1, \cdots, n$, then:

$$\{aT : a \in GA_{k',k} (A) \} \subset GA_{k',k} (B).$$

Equality holds, if $\{A_1, \cdots, A_m \}$ is linearly independent and:

$$\text{span} \{ A_1, \cdots, A_m \} = \text{span} \{ B_1, \cdots, B_n \}.$$
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GENERALIZED JOINT HIGHER-RANK NUMERICAL RANGE

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برد عددی رتیه-بالاتر توأم تعمیم یافته

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برد عددی $k$ رتیه رابطه ی نزدیکی با ساختن کد تصحیح خطای کوانتونومی برای کنال کوانتونومی
بر پارازیت دارد. در کنال کوانتونومی بر پارازیتی، کد تصحیح خطای کوانتونومی از بعد $k$ وجود دارد
اگر و تنها اگر عددی $k$ رتیه ی توأم مربوط با آن، $A$ ناحیه باشد. در این مقاله مفهوم عددی
رتبه ی توأم تعمیم داده شده و برخی گزاره های مقاله
[2011, Generalized numerical ranges and quantum error correction, J. Operator Theory, 66: 2, 335-351.]

توسعه داده شده است.

کلمات کلیدی: تصویرگر تعمیم یافته، برد عددی رتیه بالاتر توأم، برد عددی ماتریسی توأم، برد
عددی رتیه بالاتر ماتریسی توأم، برد عددی رتیه بالاتر توأم تعمیم یافته.