ON THE VANISHING OF DERIVED LOCAL HOMOLOGY MODULES

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Abstract. Let $R$ be a commutative Noetherian ring, $a$ be an ideal of $R$, and $\mathcal{D}(R)$ denote the derived category of $R$-modules. For any homologically-bounded complex $X$, we conjecture that $\sup L\Lambda^a(X) \leq \text{mag}_R X$. We prove this in several cases.

1. Introduction

Throughout this paper, $R$ is a commutative Noetherian ring with non-zero identity, and $\mathcal{D}(R)$ denotes the derived category of $R$-modules. The full subcategory of homologically-bounded complexes is denoted by $\mathcal{D}^b(R)$, and that for complexes homologically-bounded to the right (resp. left) is denoted by $\mathcal{D}^b_+(R)$ (resp. $\mathcal{D}^b_-(R)$). Also $\mathcal{D}^b_{\text{f}}(R)$ (resp. $\mathcal{D}^b_{\text{Art}}(R)$) consists of homologically-bounded complexes with finitely-generated (resp. Artinian) homologies. The symbol $\simeq$ denotes an isomorphism in the category $\mathcal{D}(R)$. For any complex $X$ in $\mathcal{D}^b_+(R)$ (resp. $\mathcal{D}^b_-(R)$), there is a bounded to the right (resp. left) complex $U$ of projective (resp. injective) $R$-modules such that $U \simeq X$. A such
complex $U$ is called a projective (resp. injective) resolution of $X$. The left-derived tensor product functor $- \otimes_R^L \sim$ is computed by taking a projective resolution of the first argument or of the second one. Also, the right-derived homomorphism functor $R \text{Hom}_R(-, \sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one.

Let $a$ be an ideal of $R$, and $C_0(R)$ denote the full sub-category of $R$-modules. It is known that the $a$-adic completion functor 

$$\Lambda^a(-) = \lim_{\rightarrow n} (R/a^n \otimes_R -) : C_0(R) \to C_0(R)$$

is not right exact, in general. The left-derived functor of $\Lambda^a(-)$ exists in $D(R)$, and so, for any complex $X \in D_{\triangle}(R)$, the complex $L\Lambda^a(X) \in D_{\triangle}(R)$ is defined by $L\Lambda^a(X) := \Lambda^a(P)$, where $P$ is a (every) projective resolution of $X$. Let $X \in D_{\triangle}(R)$. For any integer $i$, the $i$-th local homology module of $X$ with respect to $a$ is defined by

$$H^a_i(X) := H_i(L\Lambda^a(X)).$$

First E. Matlis [12], in 1974, studied the theory of the local homology. Next Simon in [18] and [19] continued the study of this theory. Later, J.P.C. Greenlees and J.P. May [9] defined local homology groups of a module $M$ using a new approach. Then came the works of Alonso Tarrío, Jeremías López and Lipman [1]. After the works of [17], [4], [5], [8], and [14], started a new era in the study of local homology.

The most essential vanishing result for the local cohomology modules $H^a_i(M)$ is Grothendieck’s Vanishing Theorem, which asserts that $H^a_i(M) = 0$ for all $i > \dim_R M$. Letting $X \in D_{\square}(R)$, Foxby generalized this result for derived local cohomology modules $H^a_i(X)$. He proved that $H^a_i(X) = 0$ for all $i > \dim_R X$ [7, Theorem 7.8, Corollary 8.29]. We intend to establish the dual of this result for the derived local homology modules. Let $\check{C}(\mathfrak{a})$ denote the Čech complex of $R$ on a set $\mathfrak{a}$ of generators of $\mathfrak{a}$. By [1, (0.3), aff,p.4] (see also [17, Section 4] for corrections),

$$L\Lambda^a(X) \simeq R \text{Hom}_R(\check{C}(\mathfrak{a}), X).$$
Using this isomorphism Frankild [8, Theorem 2.11] proved that
\[ \inf L(a; X) = \text{width}_R(R/a \otimes R X). \]

The aim of this work was to find an upper bound for \( \sup L(a; X) \). Finding a good upper bound for \( \sup L(a; X) \) was considered in [17] and [8]. In fact, we conjecture that \( H_i^a(X) = 0 \) for all \( i > \text{mag}_R X \). Our investigation on this conjecture is the core of this paper. We show the correctness of this conjecture in several cases. Namely, we prove that if for all \( i \in \mathbb{Z} \), either:

1. \( \text{Coass}_R H_i(X) = \text{Att}_R H_i(X) \),
2. \( H_i(X) \) is finitely-generated, Artinian or Matlis reflexive,
3. \( H_i(X) \) is linearly-compact,
4. \( R \) is complete local, and \( H_i(X) \) has finitely many minimal coassociated prime ideals; or:
   - \( R \) is complete local with the maximal ideal \( \mathfrak{m} \), and \( \mathfrak{m}^n H_i(X) \) is minimax for some integer \( n \geq 0 \),
   - then \( H_i^a(X) = 0 \) for all \( i > \text{mag}_R X \).

First, Sazeedeh [16] studied connections between the Gorenstein injective modules and the local cohomology modules. The Gorenstein flat dimension of \( X \) is defined by

\[ \text{Gfd}_R X := \inf \{ \sup \{ l \in \mathbb{Z} | Q_l \neq 0 | Q \text{ is a bounded to the right complex of Gorenstein flat } R\text{-modules and } Q \simeq X \} \}. \]

For more details on the theory of Gorenstein homological dimensions for complexes, we refer the reader to [2].

2. Results

In what follows, we denote the faithful exact functor,

\[ \text{Hom}_R(-, \bigoplus_{m \in \text{Max}_R} E(R/m)) \]

by \((-)^\vee\). Let \( M \) be an \( R \)-module. A prime ideal \( \mathfrak{p} \) of \( R \) is said to be a coassociated prime ideal of \( M \) if there is an Artinian quotient \( L \) of \( M \) such that \( \mathfrak{p} = (0 : R L) \). The set of all coassociated prime ideals of \( M \)
is denoted by $\text{Coass}_RM$. Also, $\text{Att}_RM$ is defined by

$$\text{Att}_RM := \{p \in \text{Spec } R \mid p = (0 :_R L) \text{ for some quotient } L \text{ of } M\}.$$  

Clearly, $\text{Coass}_RM \subseteq \text{Att}_RM$ and the equality holds if either $R$ or $M$ is Artinian. More generally, if $M$ is representable, then it is easy to check that $\text{Coass}_RM = \text{Att}_RM$. If $0 \to M \to N \to L \to 0$ is an exact sequence of $R$-modules and $R$-homomorphisms, then it is easy to check that:

$$\text{Coass}_RL \subseteq \text{Coass}_RN \subseteq \text{Coass}_RL \cup \text{Coass}_RM,$$

and:

$$\text{Att}_RL \subseteq \text{Att}_RN \subseteq \text{Att}_RL \cup \text{Att}_RM.$$  

Also if $R$ is local, then one can see that $\text{Coass}_RM = \text{Ass}_RM$.  

For an $R$-module $M$, set $\text{cd}_aM := \sup \{i | H_i^a(M) \neq 0\}$.  

By [9, Corollary 3.2], $H_i^a(M) = 0$ for all $i > \text{cd}_aR$.  

Next, we recall the definition of the notion $\text{mag}_RM$.  

**Def 2.1.** Let $M$ be an $R$-module.  

i) (See [20]) The magnitude of $M$ is defined by

$$\text{mag}_RM := \sup \{\dim R/p | p \in \text{Coass}_RM\}.$$  

If $M = 0$, then we put $\text{mag}_RM = -\infty$.  

ii) (See [15]) The Noetherian dimension of $M$ is defined inductively as follows: when $M = 0$, put $\text{Ndim}_RM = -1$. Then, by induction, for an integer $d \geq 0$, we put $\text{Ndim}_RM = d$ if $\text{Ndim}_RM < d$ is false, and for every ascending sequence $M_0 \subseteq M_1 \subseteq \ldots$ of submodules of $M$, there exists $n_0$ such that $\text{Ndim}_RM_{n+1}/M_n < d$ for all $n > n_0$.  

iii) (See [14]) The co-localization of $M$ at a prime ideal $p$ of $R$ is defined by

$$^pM := \text{Hom}_{R_p}((M^\vee)_p, E_{R_p}(R_p/pR_p)).$$  

Then $\text{Cosupp}_RM$ is defined by

$$\text{Cosupp}_RM := \{p \in \text{Spec } R | ^pM \neq 0\}.$$
iv) (See [3]) $M$ is said to be $N$-critical if $\text{Ndim}_N N < \text{Ndim}_N M$ for all proper submodules $N$ of $M$.

If $0 \to X \to Y \to Z \to 0$ is an exact sequence of $R$-modules and $R$-homomorphisms, then it is easy to verify that:

$$\text{mag}_R Y = \max\{\text{mag}_R X, \text{mag}_R Z\}.$$

Recall that an $R$-module $M$ is said to be Matlis reflexive if the natural homomorphism $M \to M^{\vee\vee}$ is an isomorphism.

Now we recall some definitions, which are required in the following statements.

We begin by recalling the definition of linearly-compact modules from [11]. Let $M$ be a topological $R$-module. Then $M$ is said to be linearly-topologized if $M$ has a base $\mathcal{M}$ consisting of sub-modules for the neighborhoods of its zero element. A Hausdorff linearly-topologized $R$-module $M$ is said to be linearly-compact if for any family $\mathcal{F}$ of cosets of closed submodules of $M$ which has the finite intersection property, the intersection of all cosets in $\mathcal{F}$ is non-empty. A Hausdorff linearly topologized $R$-module $M$ is called semi-discrete if every submodule of $M$ is closed. The class of semi-discrete linearly-compact modules is very large it contains many important classes of modules such as the class of Artinian modules or the class of finitely-generated modules over a complete local ring.

An $R$-module $M$ is called minimax if it has a finitely-generated submodule $N$ such that $M/N$ is Artinian. By [21, Lemma 1.1], over a complete local ring $R$, an $R$-module $M$ is minimax if and only if $M$ is semi-discrete linearly-compact and if and only if $M$ is Matlis reflexive. These definitions can be extended to complexes in obvious ways.

In the case $(R, m)$ is a local ring, by [20, Lemma 2.2], we have $\text{mag}_M = \dim M^{\vee}$ for any $R$-module $M$, where $(\cdot)^\vee := \text{Hom}_R(\cdot, E(R/m))$. Following this idea, one could expect $\text{mag}_RX = \dim_R X^{\vee}$.

Let $X \in D(R)$. We know that $\dim_R X^{\vee} = \sup\{\dim_R H_i(X^{\vee}) - i \mid i \in \mathbb{Z}\} = \sup\{\dim_R H_{-i}(X)^\vee - i \mid i \in \mathbb{Z}\} = \sup\{\dim_R H_j(X)^\vee + j \mid j \in \mathbb{Z}\}$.

Therefore, we define $\text{mag}_R X$ as follows:
Definition 2.2. Let $R$ be a commutative Noetherian ring, and $X \in \mathcal{D}(R)$. We define $\text{mag}_R X := \sup \{ \text{mag}_R H_i(X) + i \mid i \in \mathbb{Z} \}$.

Lemma 2.3. Let $(R, \mathfrak{m})$ be a local ring, and $X \in \mathcal{D}(R)$. Then:

$$\text{mag}_R X = \sup \{ \dim \frac{R}{p} + \sup \mathfrak{p} X \mid \mathfrak{p} \in \text{Cosupp}_R X \}.$$ 

Proof. By definition, we have $\text{mag}_R X = \dim_R X^\vee$. Now we know that $\dim_R X^\vee = \sup \{ \dim \frac{R}{p} - \inf(X^\vee)_p \mid p \in \text{Supp}_R X^\vee \}$. Let $p \in \text{Spec} R$. By definition, we have $\mathfrak{p} X = \text{Hom}_R((X^\vee)_p, E(\frac{R}{p}))$, and so:

$$\sup \mathfrak{p} X = - \inf(X^\vee)_p.$$ 

Also, by [14, Theorem 2.7], $\text{Cosupp}_R X = \text{Supp}_R X^\vee$. Hence,

$$\text{mag}_R X = \sup \{ \dim \frac{R}{p} + \sup \mathfrak{p} X \mid \mathfrak{p} \in \text{Cosupp}_R X \}.$$ 

Remark 2.4. Let $X$ and $Y$ be complexes in $\mathcal{D}(R)$. Observe that the isomorphism $\text{LA}^a(X) \simeq R \text{Hom}_R(C(\mathfrak{a}), X)$ immediately gives:

$$\text{LA}^a(R \text{Hom}_R(X, Y)) \simeq R \text{Hom}_R(X, \text{LA}^a(Y)) \simeq R \text{Hom}_R(R \Gamma_{\mathfrak{a}}(X), Y).$$

Definition 2.5. (See [6]) Let $X \in \mathcal{D}(R)$.

i) If $m$ is an integer, $\Sigma^m X$ denotes the complex $X$ shifted (or translated) $m$ degrees (to the left); it is given by

$$\Sigma^m X = X_{l-m}, \quad d_l^\Sigma^m X = (-1)^m d_l^{X_{l-m}},$$

for $l \in \mathbb{Z}$.

ii) If $m, n \in \mathbb{Z}$, the truncated complexes $\tau_{m \subset} X$ and $\tau_{n \supset} X$ are given by

$$\tau_{m \subset} X = 0 \longrightarrow C_m^X \xrightarrow{d_m^X} X_{m-1} \xrightarrow{d_{m-1}^X} X_{m-2} \xrightarrow{d_{m-2}^X} \cdots,$$

and

$$\tau_{n \supset} X = \cdots \xrightarrow{d_{n+3}^X} X_{n+2} \xrightarrow{d_{n+2}^X} X_{n+1} \xrightarrow{d_{n+1}^X} \xrightarrow{d_n^X} Z_n^X \longrightarrow 0,$$

where $d_m^X$ and $d_{n+1}^X$ are the induced maps.

Lemma 2.6. Let $R$ be a commutative Noetherian ring, and $X \in \mathcal{D}(R)$. 

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i) If $X \in D_{\square}(R)$, then $\sup L^a(X) \leq \sup \{ \sup L^a(H_i(X)) + i \mid i \in \mathbb{Z} \}$.

ii) If $(R, \mathfrak{m})$ is a complete local ring and $X \in D^\text{Art}_{\square}(R)$, then

$$\sup L^a(X) = \sup \{ \sup L^a(H_i(X)) + i \mid i \in \mathbb{Z} \}.$$

**Proof.** i) Let $s := \sup L^a(X)$, and assume that $s > \sup(\sup L^a(H_i(X))) + l$ for all $l$. By descending induction on $l$, we show that, $\sup(\sup L^a(\tau_{\subseteq}X)) = s$ for all $l \in \mathbb{Z}$. This gives the desired contradiction, since $\tau_{\subseteq}X \simeq 0$ for $l$ small enough. Since $\tau_{\subseteq}X \simeq X$ for $l$ large enough, the equality $\sup(\sup L^a(\tau_{\subseteq}X)) = s$ certainly holds for large $l$.

In the inductive step, note that the exact triple,

$$(\Sigma^l H_l(X), \tau_{\subseteq}X, \tau_{\subseteq-l}X)$$

(see [6, Corollary 1.43]) induces an exact sequence:

$$\ldots \rightarrow H^a_{m-l}(H_l(X)) \rightarrow H^a_m(\tau_{\subseteq}X) \rightarrow H^a_m(\tau_{\subseteq-l}X) \rightarrow H^a_{m-l-1}(H_l(X)) \rightarrow \ldots$$

from which the desired assertion, $\sup(\sup L^a(\tau_{\subseteq-l}X)) = s$, follows from the inductive assumption,

$\sup(\sup L^a(\tau_{\subseteq}X)) = s$, and the assumption $\sup(\sup L^a(H_l(X))) < s - l$ made earlier.

ii) As $X \in D^\text{Art}_{\square}(R)$, it follows that $X^\vee \simeq X$ in $D(R)$. Hence, we have:

$$\sup L^a(X) = \sup L^a(X^\vee)$$

$$= \sup L^a(\mathbf{R} \text{Hom}_R(X^\vee, E(\frac{R}{\mathfrak{m}})))$$

$$\overset{(a)}{=} \sup(\mathbf{R} \text{Hom}_R(X^\vee, L^a(E(\frac{R}{\mathfrak{m}}))))$$

$$\overset{(b)}{=} \sup \{ \sup \mathbf{R} \text{Hom}_R(H_i(X^\vee), L^a(E(\frac{R}{\mathfrak{m}}))) - i \mid i \in \mathbb{Z} \}$$

$$\overset{(c)}{=} \sup \{ \sup L^a(\mathbf{R} \text{Hom}_R(H_i(X^\vee), E(\frac{R}{\mathfrak{m}}))) - i \mid i \in \mathbb{Z} \}$$

$$= \sup \{ \sup L^a(H_i(X^\vee)^\vee) - i \mid i \in \mathbb{Z} \}$$

$$= \sup \{ \sup L^a((H_{-i}(X))^\vee) - i \mid i \in \mathbb{Z} \}$$

$$\overset{(d)}{=} \sup \{ \sup L^a(H_j(X)) + j \mid j \in \mathbb{Z} \}$$
The equalities (a) and (c) follow by Remark 2.4, and since $X \in \mathcal{D}_A^f(R)$, (b) follows from [6, Lemma 16.26]. Since $X \in \mathcal{D}^{\text{Art}}_A(R)$, the equality (d) holds.

Now we recall the following definition of Zoschinger:

**Definition 2.7.** Let $M$ be an $R$-module. Then $\text{Coass}(M)$ has a finite final subset, when the set of minimal elements of $\text{Coass}(M)$ is finite, or equivalently, there exists a finite subset, $\{p_1, \ldots, p_n\}$ of $\text{Coass}(M)$ such that $\bigcap \text{Coass}(M) = \bigcap_{i=1}^n p_i$.

**Theorem 2.8.** Let $\mathfrak{a}$ be an ideal of $R$, and $X \in \mathcal{D}_A(R)$. Assume that for all $i \in \mathbb{Z}$, either:

i) $\text{Coass}_R H_i(X) = \mathcal{A}tt_R H_i(X)$,

ii) $H_i(X)$ is $N$-critical,

iii) $H_i(X)$ is finitely-generated or Matlis reflexive,

iv) $H_i(X)$ is linearly-compact,

v) $R$ is complete local and $H_i(X)$ has finitely many minimal coassociated prime ideals; or:

vi) $R$ is complete local with the maximal ideal $\mathfrak{m}$, and $\mathfrak{m}^n H_i(X)$ is minimax for some integer $n \geq 0$.

Then $\sup L \Lambda^a(X) \leq \text{mag}_R X$ and equality holds if $R$ is complete local with the maximal ideal $\mathfrak{a}$ and $X \in \mathcal{D}^{\text{Art}}_A(R)$.

**Proof.** From Lemma 2.6 i), we have:

$$\sup L \Lambda^a(X) \leq \sup \{\sup L \Lambda^a(H_i(X)) + i \mid i \in \mathbb{Z}\}.$$ 

On the other hand, by [10, Theorem 2.8], in each of these cases, $\sup L \Lambda^a(H_i(X)) \leq \text{mag}_R H_i(X)$ for all $i \in \mathbb{Z}$. Now the result follows by the definition of $\text{mag}_R X$.

Now let $(R, \mathfrak{m})$ be a complete local ring. From [5, Theorem 4.8, 4.10], for each Artinian module $M$, $\text{Ndim}_R M = \max \{i \mid H_i^\mathfrak{m}(M) \neq 0\}$. Hence, the last part follows from Lemma 2.6 ii) and [20, Theorem 2.10].
Corollary 2.9. Let \((R, \mathfrak{m})\) be a complete local ring, and \(X \in \mathcal{D}^{\text{ht}}(R)\). Then \(\text{mag}_R X \leq \text{Gfd}_R X\).

Proof. From [13, Theorem 2.5], \(\sup(\mathcal{L}a(X)) \leq \text{Gfd}_R X\) for any ideal \(a\) of \(R\). Now, the result follows by Theorem 2.8. \(\square\)

Definition 2.10. (See [6])

i) Let \(X \in \mathcal{D}(R)\), and \(Y \in \mathcal{D}(R)\). The module:

\[ H_{-i}(R \text{Hom}_R(X, Y)) \]

is often denoted by \(\text{Ext}^i_R(X, Y)\), and called the \(i\)-th hyper \(\text{Ext}\) module of the complexes \(X\) and \(Y\).

ii) Let \(X, Y \in \mathcal{D}(R)\). The module: \(H_i(X \otimes^L_R Y)\) is sometimes denoted by \(\text{Tor}^R_i(X, Y)\), and called the \(i\)-th hyper \(\text{Tor}\) module of the complexes \(X\) and \(Y\).

Assume that \(M\) and \(N\) are two \(R\)-modules, and \(X\) and \(Y\) are two complexes. The following result is deduced from Theorem 2.8.

Corollary 2.11. Assume that \(M\) is a linearly-compact \(R\)-module, \(N\) an \(R\)-module, and \(X, Y \in \mathcal{D}(R)\).

i) If \(R \text{Hom}_R(N, M) \in \mathcal{D}(R)\), then \(H^a_i(R \text{Hom}_R(N, M)) = 0\) for all \(i > \text{mag}(R \text{Hom}_R(N, M))\).

ii) If \(N\) is finitely-generated, and \(N \otimes^L_R M \in \mathcal{D}(R)\), then:

\[ H^a_i(N \otimes^L_R M) = 0 \]

for all \(i > \text{mag}(N \otimes^L_R M)\).

Proof. By [4, Lemma 2.5 and 2.6], the \(R\)-modules \(\text{Ext}^i_R(N, M)\) and \(\text{Tor}^R_i(N, M)\) are linearly-compact for all non-negative integers \(i\). Hence, the result follows by Theorem 2.8. \(\square\)

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صفر شدن مدول‌های همولوژی موضعی مشتق شده

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شناخته شده کارگری مشتق شده $D(R)$ و $R$ ایدهآلی از $R$ به جای آن باشد. همچنین مدول ار ار alleged، به عنوان یک مولفه برای هموگواری کردن X، یک گره هموگواری کردن X را به ایدهآل $a$ را باشد و در هر حال مختلف آن را ثابت می‌کنیم.

کلمات کلیدی: کوهمولوژی موضعی، مدول‌های همولوژی موضعی، یک گره مدول، بعد نوتری.