**Journal of Algebraic Systems**
**Vol. 4, No. 1, (2016), pp 15-27**
**DOI: 10.22044/jas.2016.725**

# SOME REMARKS ON GENERALIZATIONS OF MULTIPLICATIVELY-CLOSED SUBSETS

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Abstract. Let $R$ be a commutative ring with an identity, and $M$ be a unitary $R$-module. In this paper, the concept of multiplicatively-closed subset of $R$ is generalized, and some properties of the generalized subsets of $M$ are studied. Some well-known theorems about multiplicatively-closed subsets of $R$ are also generalized, and it is shown that some other well-known results are not valid for $M$.

## 1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$ and all modules are unital. Let $R$ be a ring, $M$ be an $R-$module, and $N$ be a submodule of $M$. We know that $(N : M) = \{r \in R | rM \subseteq N\}$ is an ideal of $R$. The $R$-module $M$ is multiplicative if for every submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. It is easy to show that $N = (N : M)M$.

Let $R$ be a ring. The subset $S$ of $R$ is called a multiplicatively-closed subset if $a, b \in S$ implies $ab \in S$, where $a, b \in R$. Let $P$ be a prime ideal of $R$, i.e., a proper ideal with the property that $ab \in P$ implies $a \in P$ or $b \in P$, where $a, b \in R$. A proper submodule $P$ of $M$ is called a prime submodule if $r \in R$ and $x \in M$ together with $rx \in P$ implies

MSC(2010): Primary: 13C05; Secondary:13C13.
Keywords: Multiplicative module, multiplicatively-closed subset of $R$, $(n-1,n)$-multiplicatively $S$-closed subset of $M$, $(n-1,n)$-weakly multiplicatively $S$-closed subset of $M$, $(n-1,n)-\phi_m$-multiplicatively $S$-closed subset of $M$.
Received: 2 September 2015, Revised: 27 April 2016.
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$r \in (P : M)$ or $x \in P$. It is easy to show that if $P$ is a prime submodule of $M$, then $(P : M)$ is a prime ideal of $R$.

In [6], Badawi has introduced the concept of 2-absorbing ideal, and has generalized this concept to $n$-absorbing ideal, i.e. a proper ideal $P$ of $R$ with the property that $a_1 \dots a_{n+1} \in P$ implies $a_1 \dots a_{i-1}a_{i+1} \dots a_{n+1} \in P$, for some $i \in \{1, \dots, n+1\}$, where $a_1, \dots, a_{n+1} \in R$. In [2], Anderson and Badawi have studied $n$-absorbing ideals for $n \geq 2$.

In [9] and [11], an $(n-1)$-absorbing ideal $P$ of $R$ has been denoted by $(n-1, n)$-prime ideal. Thus a $(1, 2)$-prime ideal is just a prime ideal. In [4], this concept has been studied with respect to non-unique factorization for principal ideals in an integral domain.

Also in [10], Ebrahimpour and Nekooei have established the concept of $(n-1, n)$-prime submodule, i.e., a proper submodule $P$ of $M$ with the property that $a_1...a_{n-1}x \in P$ implies $a_1...a_{n-1} \in (P : M)$ or $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$ for some $i \in \{1, ..., n-1\}$, where $a_1, ..., a_{n-1} \in R$ and $x \in M$. Note that a $(1, 2)$-prime submodule is just a prime submodule.

In [5], Anderson and Smith have defined a weakly prime ideal, i.e. a proper ideal $P$ of $R$ with the property that $0 \neq ab \in P$ implies $a \in P$ or $b \in P$, where $a, b \in R$. The notion of a weakly prime element (i.e. an element $p \in R$ such that $(p)$ is a weakly prime ideal) has been introduced by Galovich [14], while the subject of unique factorization rings with zero divisors has been studied. In [19], Nekooei has extended this concept to weakly prime submodule, i.e. a proper submodule $P$ of $M$ with the property that $0 \neq rx \in P$ implies $x \in P$ or $r \in (P : M)$, where $r \in R$ and $x \in M$.

In [9], Ebrahimpour and Nekooei have defined a proper ideal $P$ of $R$ to be $(n-1, n)$-weakly prime if $a_1, \dots, a_n \in R$ together with $0 \neq a_1 \dots a_n \in P$ imply $a_1 \dots a_{i-1}a_{i+1} \dots a_n \in P$, for some $i \in \{1, \dots, n\}$. In [10], it has been established that a proper submodule $P$ of $M$ is $(n-1, n)$-weakly prime if $0 \neq a_1...a_{n-1}x \in P$ implies $a_1...a_{n-1} \in (P : M)$ or $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in P$, for some $i \in \{1, ..., n-1\}$, where $a_1, ..., a_{n-1} \in R$ and $x \in M$. Thus a $(1, 2)$-weakly prime submodule is just a weakly prime submodule.

In studying unique factorization domains, Bhatwadekar and Sharma [8] have defined the notion of almost prime ideals, i.e. a proper ideal $P$ of $R$ with the property that $ab \in P \backslash P^2$ implies $a \in P$ or $b \in P$, where $a, b \in R$. Thus a weakly prime ideal is almost prime, and any proper idempotent ideal is also almost prime.

In [3], Anderson and Bataineh have extended the concept of prime ideals to $\Phi$-prime ideals, as follows: Let $R$ be a commutative ring, and $S(R)$ be the set of ideals of $R$. Let $\Phi : S(R) \to S(R) \cup \{\emptyset\}$ be a function.

A proper ideal $P$ of $R$ is called $\Phi$-prime if $ab \in P\backslash\Phi(P)$ implies $a \in P$ or $b \in P$, where $a, b \in R$. They defined $\Phi_m : S(R) \to S(R) \cup \{\emptyset\}$ with $\Phi_m(J) = J^m$ for all $J \in S(R)$; $(m \geq 2)$.

In [9], Ebrahimpour and Nekooei have introduced the concept of $(n-1,n)$-$\Phi_m$-prime ideal, i.e., a proper ideal $P$ of $R$ with the property that $a_1 \dots a_n \in P\backslash P^m$ implies $a_1 \dots a_{i-1}a_{i+1} \dots a_n \in P$ for some $i \in \{1, \dots, n\}$, where $a_1, \dots, a_n \in R$ ;$(m, n \geq 2)$.

In [20], Zamani has extended the concept of $\phi$-prime ideal to $\phi$-prime submodule. Let $S(M)$ be the set of all submodules of $M$ and $\phi : S(M) \to S(M) \cup \{\emptyset\}$ be a function. A proper submodule $P$ of $M$ is called $\phi$-prime submodule if $rx \in P\backslash\phi(P)$ implies $r \in (P : M)$ or $x \in P$, where $r \in R$ and $x \in M$. Zamani defined $\Phi_m : S(M) \to S(M) \cup \{\emptyset\}$ together with $\Phi_m(N) = (N : M)^{m-1}N$ for all $N \in S(M)$; $(m \geq 2)$.

In [10], Ebrahimpour and Nekooei have introduced the concept of $(n-1,n)-\phi_m$-prime submodule, i.e. a proper submodule $P$ of $M$ with the property that $a_1 \dots a_{n-1}x \in P\backslash(P : M)^{m-1}P$ implies

$$a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P,$$

for some $i \in \{1, \dots, n-1\}$ or $a_1 \dots a_{n-1} \in (P : M)$, where $a_1, \dots, a_{n-1} \in R$ and $x \in M$. Thus a $(1,2)$-$\Phi_2$-prime submodule is just almost prime. The $(1,2)$-$\Phi_m$-prime submodules is called "$\Phi_m$-prime"

In [12], Ebrahimpour has established the concept of $(n-1,n)$-weakly multiplicatively-closed subset of $R$, denoted by $(n-1,n)$-W. M. closed, i.e. a subset $S$ of $R$ with the property that $a_1, \dots, a_n \in R$, and

$$a_1 \dots a_{i-1}a_{i+1} \dots a_n \in S,$$

imply $a_1 \dots a_n \in S \cup \{0\}$ for all $i \in \{1, \dots, n\}$. Moreover, it has been said that $S$ is an $(n-1,n)$-$\Phi_m$-multiplicatively-closed subset of $R$, denoted by $(n-1,n)-\phi_m$-M. closed if $a_1, \dots, a_n \in R$, and

$$a_1 \dots a_{i-1}a_{i+1} \dots a_n \in S,$$

imply $a_1 \dots a_n \in S \cup (R\backslash S)^m$ for all $i \in \{1, \dots, n\}$; $(n, m \geq 2)$.

Let $R$ be a ring, $M$ be an $R-$module, and $S, S^*$ be non-empty subsets of $R$ and $M$, respectively. In this paper, we introduce the concepts of $(n-1,n)$-multiplicatively $S$-closed, $(n-1,n)$-weakly multiplicatively $S$-closed and $(n-1,n)-\phi_m$- multiplicatively $S$-closed subsets $S^*$ of $M$, and prove some basic properties of these subsets. Also we prove the generalized version of some well-known theorems about multiplicatively-closed subsets of $R$.

## 2. $(n-1,n)$-MULTIPLICATIVELY $S$-CLOSED SUBSETS

We say that a non-empty subset $S$ of $R$ is $(n-1,n)$-multiplicatively-closed, denoted $(n-1,n)$-M. closed, if $a_1,...,a_n \in R$, and

$$a_1...a_{i-1}a_{i+1}...a_n \in S,$$

for all $i \in \{1,...,n\}$ implies $a_1...a_n \in S$, $(n \geq 2)$. Also we say that an $(n-1,n)$-multiplicatively-closed subset $S$ of $R$ is saturated if $a_1,...,a_n \in R$ together with $a_1...a_n \in S$ imply $a_1...a_{i-1}a_{i+1}...a_n \in S$ for all $i \in \{1,...n\}$. The $(1,2)$-M. closed and saturated $(1,2)$-M. closed subsets of $R$ are denoted by M. closed and saturated M. closed, respectively.

Let $R$ be a ring, $M$ be an $R-$module, and $S, S^*$ be non-empty subsets of $R$ and $M$, respectively. We say that $S^*$ is $(n-1,n)$-multiplicatively $S$-closed, denoted by $(n-1,n)$-M. $S$-closed if $a_1,...,a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^*$ for all $i \in \{1,...,n-1\}$, imply $a_1...a_{n-1}x \in S^*$ and $a_1...a_{n-1} \in S$. Furthermore, we say that $S^*$ is a saturated $(n-1,n)$-M. $S$-closed subset of $M$ if $a_1,...,a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{n-1}x \in S^*$ imply $a_1...a_{n-1} \in S$ and $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^*$ for all $i \in \{1,...n-1\}$ $(n \geq 2)$. The $(1,2)$-M. $S$-closed and saturated $(1,2)$-M. $S$-closed subsets of $M$ are denoted by M. $S$-closed and saturated M. $S$-closed, respectively.

**Proposition 2.1.** *Let $R$ be a ring, $M$ be an $R-$module, and $P$ be a proper submodule of $M$. Then $P$ is an $(n-1,n)$-prime submodule of $M$ if and only if $M \setminus P$ is an $(n-1,n)$-M. $S$-closed subset of $M$, where $S = R \setminus (P : M)$; $(n \geq 2)$.*

*Proof.* $(\Rightarrow)$ Let $S^* = M \setminus P$, and $P$ be an $(n-1,n)$-prime submodule of $M$. Let $a_1,\dots,a_{n-1} \in R$, and $x \in M$ with $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in S^*$, for all $i \in \{1,\dots,n-1\}$, and $a_1...a_{n-1} \in S$. Let $a_1 \dots a_{n-1}x \notin S^*$. Thus $a_1...a_{n-1}x \in P$. Since $P$ is $(n-1,n)$-prime, we have

$$a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P,$$

for some $i \in \{1,\dots,n-1\}$ or $a_1...a_{n-1} \in (P : M)$, which are contradictions. Thus $a_1 \dots a_{n-1}x \in S^*$. Therefore, $M \setminus P$ is an $(n-1,n)$-M. $S$-closed subset of $M$.

$(\Leftarrow)$ Let $M \setminus P$ be an $(n-1,n)$-M. $S$-closed subset of $M$, $a_1,\dots,a_{n-1} \in R$, and $x \in M$ with $a_1 \dots a_{n-1}x \in P$. If $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \notin P$, for all $i \in \{1,\dots,n-1\}$, and $a_1...a_{n-1} \notin (P : M)$, then $a_1 \dots a_{n-1}x \in (M \setminus P)$ because $S^*$ is $(n-1,n)$-multiplicatively $S$-closed, which is a contradiction. Thus there exists $i \in \{1,\dots,n-1\}$ such that

$$a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P$$

or $a_1...a_{n-1} \in (P : M)$. Therefore, $P$ is an $(n-1, n)$-prime submodule of $M$. □

Let $R$ be a ring, and $S$ be a multiplicatively-closed subset of $R$. It is well-known that if an ideal $I$ is maximal with respect to $I \cap S = \emptyset$, then $I$ is a prime ideal of $R$ [16, Theorem 2.2]. In Example 2.2, we show that a similar result does not hold for $(n-1, n)$-M. $S$-closed subsets of an $R-$module $M$, where $S$ is a non-empty subset of $R$.

**Example 2.2.** Let $R = \frac{\mathbb{Z}}{16\mathbb{Z}}$, $M = R$, and $N = \frac{8\mathbb{Z}}{16\mathbb{Z}}$ and $S^* = S = \{\overline{1}, \overline{3}, \overline{9}, \overline{11}\}$. Then $S^*$ is an M. $S$-closed subset of $M$, and $N$ is maximal with respect to $N \cap S^* = \emptyset$. But $N$ is not prime because $\overline{2}.\overline{4} \in N$ but $\overline{2} \notin (N : M)$ and $\overline{4} \notin N$.

**Theorem 2.3.** *Let $R$ be an Artinian ring, $M$ be a multiplicative $R-$module, and $S^*$ be a saturated M. $S$-closed subset of $M$, where $S$ is a non-empty subset of $R$. Let $N$ be a submodule of $M$ that is maximal with respect to $N \cap S^* = \emptyset$. Then $N$ is a prime submodule of $M$.*

*Proof.* We show that $(N : M) \cap S = \emptyset$. If $(N : M) \cap S \neq \emptyset$, then there exists an $s \in (N : M) \cap S$. Thus for every $x \in S^*$, we have $sx \in S^* \cap N$ which is a contradiction. Therefore, $(N : M) \cap S = \emptyset$.

Now, we show that $N$ is maximal with $(N : M) \cap S = \emptyset$. If not, then there exists an ideal $I$ of $R$ such that $(N : M) \subset I$ and $I \cap S = \emptyset$. Notice that $M$ is cyclic, by [13, Page 764]. Let $M = Rm$. If $IM = (N : M)M = N$, then for every $a \in I$, there exists a $b \in (N : M)$ such that $am = bm$. Thus $(a - b)M = \{0\} \subseteq N$. Thus $(a - b) \in (N : M)$, and hence $a \in (N : M)$. Thus $I = (N : M)$, which is a contradiction. Thus $N = (N : M)N \subset IM$. Thus $S^* \cap IM \neq \emptyset$. Thus there exists an $r \in I$ such that $rm \in S^*$. Since $S^*$ is saturated we have $r \in S$. Thus $r \in I \cap S$, which is a contradiction. Therefore, $(N : M)$ is maximal with $(N : M) \cap S = \emptyset$.

We show that $(N : M)_S$ is a maximal ideal of $R_S$. Let $Q_S$ be a maximal ideal of $R_S$ over $(N : M)_S$. Thus $Q$ is a prime ideal of $R$ with $Q \cap S = \emptyset$. Let $r \in (N : M)$. Thus $\frac{r}{1} \in Q_S$. Thus there exists an $s \in S$ such that $sr \in Q$. Since $Q \cap S = \emptyset$, we have $r \in Q$. Thus $(N : M) \subseteq Q$, and so $(N : M) = Q$, because, $Q \cap S = \emptyset$. Therefore, $(N : M)_S$ is a maximal ideal of $R_S$.

Since $M$ is finitely generated and $(N : M) \cap S = \emptyset$, we have $N_S \neq M_S$. Also $(N : M)_S \subseteq (N_S : M_S) \neq R$. Thus $(N : M)_S = (N_S : M_S)$. Thus $N_S$ is a prime submodule of $M_S$, by [18, Proposition 1 ], and so $N$ is a prime submodule of $M$, by [17, Proposition 2]. □

**Theorem 2.4.** *Let $R$ be a ring, $M$ be an $R-$module, $S$ be a non-empty subset of $R$, and $S^*$ be an $(n, n+1)$-M. $S$-closed subset of $M$. If for* *every* $x_1, ..., x_n \in S^*$ *there exists* $r_i \in (Rx_i : M)$*, for* $i = 1, ...n$*, such that* $r_1...r_n \in S$ *and* $r_1...r_{j-1}r_{j+1}...r_nS^* \subseteq S^*$*, for all* $j \in \{1, ..., n\}$*, and* $N$ *is a submodule of* $M$ *with the property that* $N \cap S^* = \emptyset$*. Then there exists an* $(n-1, n)$*-prime submodule* $P$ *of* $M$ *such that* $N \subseteq P$ *and* $P \cap S^* = \emptyset$*,* $(n \geq 2)$*.*

*Proof.* Put $\mu = \{T \leq M | N \subseteq T, T \cap S^* = \emptyset\}$. By Zorn's lemma, $\mu$ has a maximal element $P$. We show that $P$ is an $(n-1, n)$-prime submodule of $M$. Let $a_1, ..., a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{n-1}x \in P$. Assume that $a_1...a_{n-1} \notin (P : M)$ and $a_1...a_{i-1}a_{i+1}...a_{n-1}x \notin P$ for all $i \in \{1, ..., n-1\}$. Then $(P + (a_1...a_{n-1})M) \cap S^* \neq \emptyset$ and $(P + (a_1...a_{i-1}a_{i+1}...a_{n-1}x)) \cap S^* \neq \emptyset$, for all $i \in \{1, ..., n-1\}$. Hence, there exists $x_1, ..., x_n \in S^*$ such that $x_i \in P + (a_1..a_{i-1}a_{i+1}...a_{n-1}x)$ for all $i \in \{1, ..., n-1\}$ and $x_n \in P + (a_1...a_{n-1}M)$. By assumption, there exists an $r_i \in (Rx_i : M)$ such that $r_1...r_n \in S$ and $r_1...r_{i-1}r_{i+1}...r_nS^* \subseteq S^*$ for all $i \in \{1, ..., n\}$. But $r_1...r_{n-1}(a_1...a_{n-1})M \subseteq P + (a_1...a_{n-1}x) \subseteq P$. Therefore, $r_1...r_nM \subseteq (r_1...r_{n-1}x_n) \subseteq P + (r_1...r_{n-1})(a_1...a_{n-1})M \subseteq P$. Since $S^*$ is $(n, n+1)$-M. $S$-closed, we have $r_1...r_nS^* \subseteq P \cap S^*$, which is a contradiction. Thus $P$ is an $(n-1, n)$-prime submodule of $M$. □

## 3. $(n-1, n)$-WEAKLY MULTIPLICATIVELY $S$-CLOSED SUBSETS

Let $R$ be a ring. A non-empty subset $S$ of $R$ is weakly multiplicatively-closed, denoted by W. M. closed, if $a, b \in S$ implies $ab \in S \cup \{0\}$.

We say that an $(n-1, n)$-weakly multiplicatively-closed subset $S$ of $R$ is saturated if $a_1, ..., a_n \in R$ together with $a_1...a_n \in S \cup \{0\}$ imply

$$a_1...a_{i-1}a_{i+1}...a_n \in S \cup \{0\},$$

for all $i \in \{1, ...n\}$, $(n \geq 2)$.

Remark that the $(1, 2)$-W. M. closed and saturated $(1, 2)$-W. M. closed subsets of $R$ are W. M. closed and saturated W. M. closed, respectively.

Now, we generalize these concepts to modules.

Let $R$ be a ring, $M$ be an $R-$module, and $S, S^*$ be non-empty subsets of $R$ and $M$, respectively. We say that $S^*$ is $(n-1, n)$-weakly multiplicatively $S$-closed, denoted by $(n-1, n)$-W. M. $S$-closed if $a_1, ..., a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^*$ for all $i \in \{1, ..., n-1\}$ and $a_1...a_{n-1} \in S$ imply $a_1...a_{n-1}x \in S^* \cup \{0\}$. Furthermore, we say that $S^*$ is a saturated $(n-1, n)$-W. M. $S$-closed subset of $M$ if $a_1, ..., a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{n-1}x \in S^* \cup \{0\}$ imply $a_1...a_{n-1} \in S \cup \{0\}$ and $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^* \cup \{0\}$, for all $i \in \{1, ...n-1\}$, $(n \geq 2)$.

The $(1,2)$-W. M. $S$-closed and saturated $(1,2)$-W. M. $S$-closed subsets of $M$ are denoted by W. M. $S$-closed and saturated W. M. $S$-closed, respectively.

It is clear that every $(n-1,n)$-M.$S$-closed subset of $M$ is $(n-1,n)$-W. M. $S$-closed. But the converse is not true, in general. For example, let $M = R = \mathbf{Z}_6$ and $S = S^* = \{\overline{3}, \overline{4}\}$. Since $0 = \overline{3}.\overline{4} \notin S^*$, $S^*$ is not M. $S$-closed but it is clear that $S^*$ is W. M. $S$-closed.

**Proposition 3.1.** *Let $R$ be a ring, $M$ an $R-$module and $S$ be a W. M. closed subset of $R$. If $S^*$ is a saturated W. M. $S$-closed subset of $M$, then $S$ is a saturated W. M. closed subset of $R$.*

*Proof.* Let $a, b \in R$ and $ab \in S \cup \{0\}$. Since $S^* \neq \emptyset$, there exists $x \in S^*$. Thus $abx \in S^* \cup \{0\}$. Since $S^*$ is saturated, we have $a \in S \cup \{0\}$ and $bx \in S^* \cup \{0\}$. Thus $a \in S \cup \{0\}$ and $b \in S \cup \{0\}$. Therefore, $S$ is saturated. □

It is clear that every $(n-1,n)$-W. M. closed subset $S$ of $R$ is an $(n-1,n)$-W. M. $S$-closed subset of $R$ as an $R-$module. But every $(n-1,n)$-W. M. $S$-closed subset of $R$ as $R-$module is not an $(n-1,n)$-W. M. closed subset of $R$ as a ring in general.

**Example 3.2.** Let $R = M = \mathbf{Z}_6$ and $S = \{\overline{3}\}$ and $S^* = \{\overline{0}, \overline{2}\}$. It is clear that $S^*$ is a W. M. $S$-closed subset of $M$. But $S^*$ is not a W. M. closed subset of $R$. Because $\overline{2}.\overline{2} = \overline{4} \notin S^* \cup \{0\}$.

**Theorem 3.3.** *Let $R$ be a ring, and $S, S^*$ be non-empty subsets of $R$. Then $S^*$ is a saturated W. M. $S$-closed subset of $R$ as $R-$module if and only if $S^* \cup \{0\} = S \cup \{0\}$, and $S^*$ be a saturated W. M. closed subset of $R$.*

*Proof.* ($\Rightarrow$) Let $S^*$ be a saturated W. M. $S$-closed subset of $R$ as $R-$module. Then $S$ is a saturated W. M. closed, by Proposition 2.1. Moreover, for every $x \in S^* \setminus \{0\}$ and $a \in S \setminus \{0\}$, $xa = ax \in S^* \cup \{0\}$. Since $S^*$ is saturated, then $x \in S$ and $a \in S^*$. Thus $S^* \cup \{0\} = S \cup \{0\}$.

($\Leftarrow$) This is clear. □

**Lemma 3.4.** *Let $R$ be a ring and $\{q_i\}_{i\in I}$, $(n-1.n)$-weakly prime ideals of $R$. Then $S = R \setminus \bigcup_{i\in I} q_i$ is a $(n-1,n)$-W. M. closed subset of $R$.*

*Proof.* Let $a_1, ..., a_n \in R$ with $a_1...a_{j-1}a_{j+1}...a_n \in S$ for all $j \in \{1, ..., n\}$. Then $a_1...a_{j-1}a_{j+1}...a_n \notin q_i$ for every $i \in I$. If $a_1...a_n \notin S \cup \{0\}$, then $0 \neq a_1...a_n \notin S$. Thus there exists $i \in I$ such that $0 \neq a_1...a_n \in q_i$. Since $q_i$ is $(n-1,n)$-weakly prime, we have $a_1...a_{j-1}a_{j+1}...a_n \in q_i$ for some $j \in \{1, ..., n\}$. Therefore, $a_1...a_{j-1}a_{j+1}...a_n \notin S$, which is a contradiction. Thus $a_1...a_n \in S \cup \{0\}$ and $S$ is an $(n-1,n)$- W. M. closed subset of $R$. □

**Lemma 3.5.** *Let $R$ be a ring, $M$ be an $R-$module, and $\{P_j\}_{j\in J}$ be $(n-1,n)$-weakly prime submodules of $M$ such that $(P_j : M) = q_j$ for all $j \in J$. Then $S^* = M \setminus \bigcup_{j\in J} P_j$ is an $(n-1,n)$-W. M. $S$-closed subset of $M$, where $S = R \setminus \bigcup_{j\in J} q_j$, $(n \geq 2)$.*

*Proof.* Let $S^* = M \setminus \cup_{j\in J} P_j$, $P_j$ be an $(n-1,n)$-weakly prime submodule of $M$ for all $j \in J$, $a_1, \dots, a_{n-1} \in R$ and $x \in M$ together with $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in S^*$ for all $i \in \{1, \dots, n-1\}$ and $a_1...a_{n-1} \in S$. Let $a_1 \dots a_{n-1}x \notin S^* \cup \{0\}$. Thus $0 \neq a_1...a_{n-1}x \in P_j$ for some $j \in J$. Since $P_j$ is $(n-1,n)$-weakly prime, we have $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P_j$ for some $i \in \{1, \dots, n-1\}$ or $a_1...a_{n-1} \in q_j$, which are contradictions. Thus $a_1 \dots a_{n-1}x \in S^* \cup \{0\}$. Therefore, $M \setminus \cup_{j\in_J} P_j$ is an $(n-1,n)$-W. M. $S$-closed subset of $M$. □

**Lemma 3.6.** *Let $R$ be a ring, $M$ be an $R-$module, and $P$ be a proper submodule of $M$.Then, $P$ is an $(n-1,n)$-weakly prime submodule of $M$ if and only if $M \setminus P$ is an $(n-1,n)$-W. M. $S$-closed subset of $M$, where $S = R \setminus (P : M)$; $(n \geq 2)$.*

*Proof.* (⇒) Let $P$ be an $(n-1,n)$-weakly prime submodule of $M$. We have $M \setminus P$ is an $(n-1,n)$-W. M. $S$-closed subset of $M$, by Lemma 3.5.

(⇐) Let $M \setminus P$ be an $(n-1,n)$-W. M. $S$-closed subset of $M$ and $a_1, \dots, a_{n-1} \in R$ and $x \in M$ together with $a_1 \dots a_{n-1}x \in P \setminus \{0\}$. If

$$a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \notin P,$$

for all $i \in \{1, \dots, n-1\}$ and $a_1...a_{n-1} \notin (P : M)$, then $a_1 \dots a_{n-1}x \in (M \setminus P) \cup \{0\}$ because $S^*$ is $(n-1,n)$-weakly multiplicatively $S$-closed, which is a contradiction. So there exists $i \in \{1, \dots, n-1\}$ such that $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P$ or $a_1...a_{n-1} \in (P : M)$. Therefore, $P$ is an $(n-1,n)$-weakly prime submodule of $M$. □

Let $R$ be a ring, $M$ be an $R-$module, and $Z(M) = \{r \in R | \exists 0 \neq m \in M; rm = 0\}$. Let $S = R \setminus Z(M)$, and $S^*$ be the set of torsion-free elements of $M$. If $S^* \neq \emptyset$, then $S^*$ is a W. M. $S$-closed subset of $M$. Now, we show that $S^*$ is not saturated.

**Example 3.7.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\{p_n | n \in \mathbb{N}\}$ be the set of all prime numbers. Let $E(p_n) = \{a \in \frac{\mathbb{Q}}{\mathbb{Z}} | a = \frac{r}{p_n^t} + \mathbb{Z}; r \in \mathbb{Z}, t \in \mathbb{N}_0\}$. Then $E(p_n)$ is a non-zero submodule of $\frac{\mathbb{Q}}{\mathbb{Z}}$ as $\mathbb{Z}-$module for all $n \in \mathbb{N}$.

Set $M = \prod_{n\in\mathbb{N}} E(p_n)$. It is clear that $M$ is a $\mathbb{Z}-$module. Let $\alpha = (\alpha_n)_{n\in\mathbb{Z}} \in M$ be such that $\alpha_n = \frac{r_n}{p_n^{t_n}} + \mathbb{Z} \neq 0$, for infinite $n$.

We claim that $\alpha$ is a torsion-free element of $M$, because if there exists $m \in \mathbb{Z}$ with $m\alpha = 0$, then $m(\frac{r_n}{p_n^{t_n}} + \mathbb{Z}) = \mathbb{Z}$ and so $p_n^{t_n} | mr_n$. Since the greatest common divisor of $p_n$ and $r_n$ is 1, we have $p_n|m$ for all $n \in \mathbb{Z}$. Thus $m = 0$ . Therefore, $\alpha$ and so $p_k\alpha$ are torsion-free for all $k \in \mathbb{Z}$. Thus $p_k\alpha \in S^* \cup \{0\}$. If $\beta = (\beta_n)_{n\in\mathbb{Z}}$, and $\beta_n = 0$, for $n \neq k$ and $\beta_k = \frac{1}{p^k} + \mathbb{Z}$, then $p_k\beta = 0$. Therefore, $p_k \notin S \cup \{0\}$.

**Theorem 3.8.** *Let $R$ be a ring, $M$ be a torsion-free $R-$module, and $P$ be a weakly prime submodule of $M$. Let $S^* = M \setminus P$ and $S = R \setminus (P : M)$. Also let $N$ be a submodule of $M$ together with $N \cap S^* = \emptyset$. Then*

*(i) $(N : M) \cap S = \emptyset$.*

*(ii) If $N$ is maximal with respect to $N \cap S^* = \emptyset$, then $N = \{m \in M|sm \in N; \exists s \in S\}$.*

*Proof.* (i) Let $(N : M) \cap S \neq \emptyset$. Thus there exists $s \in (N : M) \cap S$. Let $x \in S^*$. We have $sx \in S^* \cap N$ or $sx = 0$, which are contradictions, Because $ann(x) = 0$ and $0 \notin S, S^*$.

(ii) Set $T = \{m \in M|sm \in N; \exists s \in S\}$, and assume that $N \subset T$. Thus $T \cap S^* \neq \emptyset$, and so there exists $x \in S^*$ such that $sx \in N$ for some $s \in S$. Since $S^*$ is W. M. $S$-closed, we have $sx \in S^* \cup \{0\}$. Thus $sx \in N \cap S^*$ or $sx = 0$, which are contradictions, because $ann(x) = 0$ and $0 \notin S, S^*$. □

Unlike the case of rings, we show that for a W. M. $S$-closed subset $S^*$ of an $R-$module $M$ and a submodule $N$ of $M$ that is maximal with respect to $N \cap S^* = \emptyset$, it is not necessary that $N$ be weakly prime in general, where $S$ is a non-empty subset of $R$.

**Example 3.9.** Let $R = \frac{\mathbb{Z}}{16\mathbb{Z}}$, $M = R$, $N = \frac{8\mathbb{Z}}{16\mathbb{Z}}$, and $S^* = S = \{\overline{1}, \overline{4}\}$. Then $S^*$ is a W. M. $S$-closed subset of $M$, and $N$ is maximal with respect to $N \cap S^* = \emptyset$. But $N$ is not weakly prime, because $0 \neq \overline{2}.\overline{4} \in N$ but $\overline{2}, \overline{4} \notin N$.

## 4. $(n-1, n) - \phi_m$- MULTIPLICATIVELY $S$-CLOSED SUBSETS

We say that an $(n-1, n) - \phi_m$-multiplicatively-closed subset $S$ of $R$ is saturated if $a_1, ..., a_n \in R$ together with $a_1...a_n \in S \cup (R \setminus S)^m$ implying $a_1...a_{i-1}a_{i+1}...a_n \in S \cup (R \setminus S)^m$ for all $i \in \{1, ...n\}$. For $n = 2$, $(1, 2) - \phi_m$-multiplicatively-closed and saturated $(1, 2) - \phi_m$-multiplicatively-closed subsets of $R$ are denoted by $\phi_m$-multiplicatively-closed and saturated $\phi_m$-multiplicatively-closed, respectively; $(n, m \geq 2)$.

Let $R$ be a ring, $M$ be an $R-$module, and $S, S^*$ be non-empty subsets of $R$ and $M$, respectively. We say that $S^*$ is $(n-1,n)-\phi_m$-multiplicatively $S$-closed, denoted by $(n-1,n)-\phi_m$-M. $S$-closed, if $a_1,...,a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^*$, for all $i \in \{1,...,n-1\}$ and $a_1...a_{n-1} \in S$ imply $a_1...a_{n-1}x \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. Furthermore, we say that $S^*$ is a saturated $(n-1,n)-\phi_m$-M. $S$-closed subset of $M$ if $a_1,...,a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{n-1}x \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$ imply $a_1...a_{n-1} \in S \cup (R \setminus S)^m$ and $a_1...a_{i-1}a_{i+1}...a_{n-1}x \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$, for all $i \in \{1,...n-1\}$, $(n,m \geq 2)$. For $n=2$, $(1,2)-\phi_m$-M. $S$-closed and saturated $(1,2)-\phi_m$-M. $S$-closed subsets of $M$ are denoted by $\phi_m$-M. $S$-closed and saturated $\phi_m$-M. $S$-closed, respectively.

It is clear that every $(n-1,n)$-W. M. $S$-closed subset of $M$ is $(n-1.n)-\phi_m$-M. $S$-closed. But the converse is not true, in general.

**Example 4.1.** Let $M = R = \mathbb{Z}_6$ and $S = S^* = \{\overline{0}, \overline{2}\}$. Since $\overline{2}.\overline{2} \notin S^* \cup \{0\}$, $S^*$ is not a W. M. $S$-closed subset of $M$. But $SS^* \subseteq S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. So $S^*$ is $\phi_m$-M. $S$-closed subset of $M$, $(m \geq 2)$.

**Proposition 4.2.** *Let $R$ be a ring, $M$ an $R$-module and $S, S^*$ non-empty subsets of $R$ and $M$, respectively. If $S^*$ be an $(n-1,n)$-W. M. $S$-closed subset of $M$, then it is $(n-1,n)-\phi_m$-M. $S$-closed, $(n,m \geq 2)$.*

*Proof.* Let $a_1,...,a_{n-1} \in R$ and $x \in M$ together with $a_1...a_{n-1} \in S$ and

$$a_1....a_{i-1}a_{i+1}...a_{n-1}x \in S^*,$$

for all $i \in \{1,...,n-1\}$. Since $S^*$ is $(n-1,n)$-W. M. $S$-closed, we have $a_1...a_{n-1}x \in S^* \cup \{0\}$.

If $a_1...a_{n-1}x \in S^*$, then we are done. Now assume that $a_1...a_{n-1}x = 0$. If $0 \in S^*$, then we are done. Thus we assume that $0 \notin S^*$. Thus $0 \in M \setminus S^*$. So $a_1...a_{n-1}x \in (R \setminus S)^{m-1}(M \setminus S^*)$ . Therefore, $S^*$ is $(n-1,n)-\phi_m$-M. $S$-closed. □

**Lemma 4.3.** *Let $R$ be a ring and $\{q_i\}_{i \in I}$, $(n-1.n)-\phi_m$-prime ideals of $R$. Then $S = R \setminus \bigcup_{i \in I} q_i$ is a $(n-1,n)-\phi_m$-M. closed subset of $R$.*

*Proof.* Let $a_1,...,a_n \in R$ together with $a_1...a_{j-1}a_{j+1}...a_n \in S$, for all $j \in \{1,...,n\}$. If $a_1...a_n \notin S \cup (R \setminus S)^m$, then $a_1...a_n \in (\cup q_i) \setminus (\cup q_i)^m$. Thus there exists $i \in I$ such that $a_1...a_n \in q_i \setminus q_i^m$. Since $q_i$ is $(n-1,n)-\phi_m$- prime, we have $a_1...a_{j-1}a_{j+1}...a_n \in q_i$, for some $j \in \{1,...,n\}$. Therefore, $a_1...a_{j-1}a_{j+1}...a_n \notin S$, a contradiction. Thus $a_1...a_n \in S \cup (R \setminus S)^m$ and $S$ is an $(n-1,n)-\phi_m$-M. closed subset of $R$. □

**Proposition 4.4.** *Let $R$ be a ring, $M$ a multiplicative $R-$module and $P$ a $\phi_m$-prime submodule of $M$ . If $S^* = M \setminus P$ is a saturated $\phi_m$-M.*

*$S$-closed subset of $M$, then $S$ is a saturated $\phi_m$-M. closed subset of $R$, where $S = R \setminus (P : M)$.*

*Proof.* We have $P = (P : M)M$, and $(P : M)$ is a $\phi_m$-prime ideal of $R$, by [10, Lemma 4.3.(i)]. Thus $S$ is a $\phi_m$-M. closed subset of $R$, by Lemma 4.3 . Let $a, b \in R$ and $ab \in S \cup (R \setminus S)^m$. Since $S^* \neq \emptyset$, there exists $x \in S^*$.

We show that $abx \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. Let $ab \in S$. Since $S^*$ is $\phi_m$-M. $S$-closed, we have $abx \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. Now, let $ab \in (R \setminus S)^m = (P : M)^m$. Thus $abx \in (P : M)^{m-1}((P : M)M) = (P : M)^{m-1}P$. Thus $abx \in (R \setminus S)^{m-1}(M \setminus S^*)$.

Since $S^*$ is saturated, we have $a \in S \cup (R \setminus S)^m$ and $bx \in S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. Thus $a \in S \cup (R \setminus S)^m$ and $b \in S \cup (R \setminus S)^m$. Therefore, $S$ is saturated. □

**Lemma 4.5.** *Let $R$ be a ring, $M$ an $R-$module and $\{P_j\}_{j\in J}$, $(n-1, n) - \phi_m$-prime submodules of $M$ such that $(P_j : M) = q_j$ for all $j \in J$. Then $S^* = M \setminus \bigcup_{j\in J} P_j$ is an $(n-1, n) - \phi_m$-M. $S$-closed subset of $M$, where $S = R \setminus \bigcup_{j\in J} q_j$, $(n, m \geq 2)$.*

*Proof.* Let $S^* = M \setminus \cup_{j\in J} P_j$ and $P_j$ be an $(n-1, n) - \phi_m$-prime submodule of $M$ for all $j \in J$ and $a_1, \dots, a_{n-1} \in R$ and $x \in M$ together with $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in S^*$ for all $i \in \{1, \dots, n-1\}$ and $a_1 ... a_{n-1} \in S$. Let $a_1 \dots a_{n-1}x \notin S^* \cup (R \setminus S)^{m-1}(M \setminus S^*)$. Thus $a_1 ... a_{n-1}x \in P_j \setminus q_j^{m-1}P_j$ for some $j \in J$. Since $P_j$ is $(n-1, n) - \phi_m$-prime we have $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P_j$ for some $i \in \{1, \dots, n-1\}$ or $a_1 ... a_{n-1} \in q_j$, which are contradictions. Thus $a_1 \dots a_{n-1}x \in S^* \cup (R \setminus S)^{m-1}(M \setminus P)$. Therefore, $M \setminus \cup_{j\in J} P_j$ is an $(n-1, n) - \phi_m$-M. $S$-closed subset of $M$. □

**Proposition 4.6.** *Let $R$ be a ring, $M$ an $R-$module and $P$ a proper submodule of $M$. Then $P$ is an $(n-1, n) - \phi_m$- prime submodule of $M$ if and only if $M \setminus P$ is an $(n-1, n) - \phi_m$-M. $S$-closed subset of $M$, where $S = R \setminus (P : M)$; $(n, m \geq 2)$.*

*Proof.* ($\Rightarrow$) Let $S^* = M \setminus P$ and $P$ be an $(n-1, n) - \phi_m$-prime submodule of $M$. Then $M \setminus P$ is an $(n-1, n) - \phi_m$-M. $S$-closed subset of $M$, by Lemma 3.5.

($\Leftarrow$) Let $M \setminus P$ be an $(n-1, n) - \phi_m$-M. $S$-closed subset of $M$ and $a_1, \dots, a_{n-1} \in R$ and $x \in M$ together with $a_1 \dots a_{n-1}x \in P \setminus (P : M)^{m-1}P$. If $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \notin P$, for all $i \in \{1, \dots, n-1\}$ and $a_1 ... a_{n-1} \notin (P : M)$, then $a_1 \dots a_{n-1}x \in (M \setminus P) \cup (P : M)^{m-1}P$ because $S^*$ is $(n-1, n) - \phi_m$-M. $S$-closed, which is a contradiction. Thus there exists an $i \in \{1, \dots, n-1\}$ such that $a_1 \dots a_{i-1}a_{i+1} \dots a_{n-1}x \in P$

or $a_1 ... a_{n-1} \in (P : M)$. Therefore, $P$ is an $(n-1, n) - \phi_m$-prime submodule of $M$. □

Let $R$ be a ring, $M$ an $R-$module and $S, S^*$ non-empty subsets of $R$ and $M$, respectively. In Example 3.7, we show that if $S^*$ is a $\phi_m$-M. $S$-closed subset of $M$ and $N$ is a submodule of $M$ that is maximal with respect to $N \cap S^* = \emptyset$, then it is not necessary that $N$ be a $\phi_m$-prime submodule of $M$, $(m \geq 2)$.

**Example 4.7.** Let $R, M, N, S$ and $S^*$ be as Example 3.9. Then $S^*$ is a $\phi_m$-M. $S$-closed subset of $M$ and $N$ is maximal with respect to $N \cap S^* = \emptyset$. But $N$ is not $\phi_m$-prime, because $\overline{2}.\overline{4} \in N \setminus (N : M)^{m-1}N$ but $\overline{2} \notin (N : M)$ and $\overline{4} \notin N$, $(m \geq 2)$.

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**Journal of Algebraic Systems**

# SOME REMARKS ON GENERALIZATIONS OF MULTIPLICATIVELY-CLOSED SUBSETS

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# چند نکته در تعمیم‌هایی از زیرمجموعه‌های بسته ضربی

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فرض کنید $R$ حلقه‌ای جابجایی و یکدار و $M$ یک $R$-مدول یکانی باشد. در این مقاله ما مفهوم زیرمجموعه‌های بسته ضربی $R$ را به مدول $M$ تعمیم داده و برخی از خواص این زیرمجموعه‌های تعمیم داده شده را مطالعه می‌کنیم. در بین نتایج زیادی که در این مقاله آمده می توان به تعمیم برخی از قضایای معروف در مورد زیرمجموعه‌های بسته ضربی در حلقه $R$ به مدول $M$ اشاره کرد. همچنین نشان می‌دهیم که برخی دیگر از قضایای معروف درباره زیرمجموعه‌های بسته ضربی در حلقه $R$ در تعمیم این زیرمجموعه‌ها برای مدول $M$ برقرار نیستند.

کلمات کلیدی: مدول‌های ضربی، زیرمجموعه‌های بسته ضربی، زیر مجموعه‌های $S-$بسته $(n-1,n)$-ضربی از $M$، زیر مجموعه‌های $S-$بسته $(n-1,n)$-ضربی ضعیف از $M$، زیر مجموعه‌های $S-$بسته $(n-1,n)$-$\phi_m-$ضربی از $M$.