ON FINITE GROUPS IN WHICH SS-SEMI-PERMUTABILITY IS A TRANSITIVE RELATION

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Abstract. Let $H$ be a subgroup of a finite group $G$. We say that $H$ is SS-semipermutable in $G$ if $H$ has a supplement $K$ in $G$ such that $H$ permutes with every Sylow subgroup $X$ of $K$ with $(|X|,|H|) = 1$. In this paper, the structure of SS-semi-permutable subgroups and the finite groups in which SS-semi-permutability is a transitive relation are described. It is shown that a finite solvable group $G$ is a PST-group if and only if whenever $H \triangleleft K$ are two $p$-subgroups of $G$, $H$ is SS-semipermutable in $N_G(K)$.

1. Introduction

Throughout this paper, all the groups are considered to be finite. Let $H$ be a subgroup of $G$. Then $\pi(G)$ denotes the set of prime divisors of $|G|$; $H^G$ is the normal closure of $H$ in $G$, i.e. the intersection of all normal subgroups of $G$ containing $H$; and $F^*(G)$ is the generalized fitting subgroup of $G$, i.e. the product of all normal quasinilpotent subgroup of $G$. $H$ is said to be permutable in $G$ if it permutes with all the subgroups of $G$. $H$ is said to be S-permutable (or $\pi$-quasinormal) in $G$ if it permutes with every Sylow subgroup of $G$. This concept has been introduced by Kegel [6], and has been widely studied by some authors; see, for example, [4] and [9].

MSC(2010): Primary: 20D10; Secondary: 20D20, 20D35
Keywords: SS-semipermutable subgroups, S-semipermutable subgroups, PST-groups.
Received: 21 October 2015, Revised: 18 June 2016.
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Permutability and $S$-permutability, like normality, are not, in general, transitive relations. This observation is the point of departure of the study of some relevant classes of groups such as $T$-groups, $PT$-groups, and $PST$-groups. Recall that a group $G$ is called a $T$-group (resp. $PT$-group, $PST$-group) if normality (resp. permutability, $S$-permutability) is a transitive relation. By [6], $PST$-groups are exactly those groups in which every subnormal subgroup of $G$ is $S$-permutable in $G$. Agrawal [1] have shown that a group $G$ is a solvable $PST$-group if and only if the nilpotent residual $L$ of $G$ is a normal abelian Hall subgroup of $G$ upon which $G$ acts by conjugation as power automorphisms. Solvable $PST$, $PT$, and $T$-groups have been studied and characterized by Agrawal [1], Gaschütz [5] and Zacher [11].

A subgroup $H$ of a group $G$ is said to be semipermutable (resp. $S$-semipermutable) in $G$ if $H$ permutes with every subgroup (resp. Sylow subgroup) $X$ of $G$ such that $(|X|, |H|) = 1$. A group $G$ is called a BT-group (resp. an SBT-group) [10] if semi-permutability (resp. $S$-semi-permutability) is a transitive relation. Wang et al. have shown that every subgroup of $G$ is semipermutable in $G$ if and only if every subgroup of $G$ is $S$-semipermutable in $G$.

In 2008, Li et al. [7], have introduced the concept of SS-permutability (or $SS$-quasinormality), which is a generalization of $S$-permutability. A subgroup $H$ of $G$ is said to be SS-permutable in $G$ if $H$ has a supplement $K$ in $G$ such that $H$ permutes with every Sylow subgroup of $K$. In this case, $K$ is called an SS-permutable supplement of $H$ in $G$. Also in 2014, Chen and Guo have introduced a new concept called NSS-permutable as follows: A subgroup $H$ of a group $G$ is said to be NSS-permutable [3] in $G$ if $H$ has a normal supplement $K$ in $G$ such that $H$ permutes with every Sylow subgroup of $K$. In this case, $K$ is called an NSS-permutable supplement of $H$ in $G$. Moreover, a group $G$ is called an SST-group (resp. an NSST-group) [3] if SS-permutability (resp. NSS-permutability) is a transitive relation. A subgroup $H$ of $G$ is said to be $\tau$-quasinormal in $G$ if $HG_p = G_pH$ for every $G_p \in Syl_p(G)$ such that $(|H|, p) = 1$ and $(|H|, |G_p^G|) \neq 1$.

In this paper, we introduce a new subgroup embedding property, namely, SS-semipermutable which may be viewed as a generalization of both SS-permutable and semipermutable concepts, as follows:

**Definition 1.1.** We say that a subgroup $H$ of a group $G$ is SS-semipermutable in $G$ if $H$ has a supplement $K$ in $G$ such that $H$ permutes with all Sylow subgroups $X$ of $K$ such that $(|X|, |H|) = 1$. In this case, $K$ is called an SS-semipermutable supplement of $H$ in $G$. 
We say that a group $G$ is an SSBT-group if SS-semi-permutability is a transitive relation.

2. Preliminaries

In this section, we give some results that are useful in the sequel. The following Lemma is easy to prove.

**Lemma 2.1.** Let $N_1$ and $N_2$ be the subgroups of a group $G$ and assume that $N_1N_2 \leq G$. If $P_1$ and $P_2$ are the Sylow $p$-subgroups of $N_1$ and $N_2$ respectively, where $p \in \pi(G)$, and $P_1P_2 \leq N_1N_2$, then $P_1P_2$ is a Sylow $p$-subgroup of $N_1N_2$.

**Lemma 2.2.** Suppose that a subgroup $H$ of a group $G$ is SS-semipermutable in $G$ with a SS-semipermutable supplement $K$ and $L \leq G$. Then

1. If $H \leq L$, then $H$ is SS-semipermutable in $L$.
2. Every conjugate of $K$ in $G$ is a SS-semipermutable supplement of $H$ in $G$.
3. If $H$ is a $p$-subgroup, where $p \in \pi(G)$ and $H \leq F(G)$, then $H$ is $S$-permutable in $G$.

**Proof.** (1) Since $HK = KH$, $L = (HK) \cap L = H(K \cap L)$, which means that $(K \cap L)$ is a supplement of $H$ in $L$. Now, suppose that $X \in Syl(K \cap L)$ with $(|X|, |H|) = 1$. Then there exists $Y \in Syl(K)$ such that $X \leq Y$. By hypothesis, $HY = YH$. Hence, $HY \cap L = H(Y \cap L) = HX$ and $L \cap (YH) = (L \cap Y)H = XH$. Therefore, $HX = XH$, and this shows that $H$ is SS-semipermutable in $L$.

(2) Let $g \in G$. Then it is easy to see that $K^gH = G$. Now, suppose that $X$ is a Sylow subgroup of $K^g$ such that $(|X|, |H|) = 1$, where $p \in \pi(G)$. Then $X^{g^{-1}}$ is a Sylow $p$-subgroup of $K$ with $(|X^{g^{-1}}|, |H|) = 1$. Hence, $X^{g^{-1}}H = HX^{g^{-1}}$. This shows that $XH = HX$.

(3) Let $Q$ be a Sylow $q$-subgroup of $K$, where $q \in \pi(G)$ and $q \neq p$. Then $HQ = QH$, and $HQ$ contains a Sylow $q$-subgroup $Q^*$ of $G$. As $H \leq O_p(G)$, it follows that $H = O_p(G) \cap HQ \leq HQ$, and thus $Q^*$ normalizes $H$. Since this holds for all primes $q \neq p$, we deduce that $O^p(G) \leq N_G(H)$. Now applying [9, Lemma A], we have that $H$ is $S$-permutable in $G$.

**Lemma 2.3.** Let $G$ be a group. Then every SS-semipermutable subgroup of $G$ is $\tau$-quasinormal in $G$.

**Proof.** Let $H$ be a SS-semipermutable subgroup of $G$, and $X$ be a Sylow subgroup of $G$ such that $(|X|, |H|) = 1$ and $(|H|, |G^p_H|) \neq 1$. Then there exists an element $h \in H$ such that $X^h \leq K$. It follows that
Suppose that a subgroup $H \trianglelefteq G$, and so $HX = XH$. Therefore, $H$ is $\tau$-quasinormal in $G$. \hfill \Box

**Lemma 2.4.** [8, Theorem 1.2] Let $G$ be a group. Then every subgroup of $F^*(G)$ is $\tau$-quasinormal in $G$ if and only if $G$ is a solvable PST-group.

**Lemma 2.5.** Let $T$ and $S$ be SS-semipermutable in a solvable group $G$ with $(|T|, |S|) = 1$. Then $(T, S)$ is SS-semipermutable in $G$.

**Proof.** Let $K_1$ and $K_2$ be SS-semipermutable supplements of $T$ and $S$, respectively. Note that $G$ is a solvable group. By Lemma 2.2(2), without less of generality, we may assume that $S \leq K_1$ and $T \leq K_2$. Then $TS(K_1 \cap K_2) = TK_1 = G$. This means that $K_1 \cap K_2$ is a supplement of $(T, S)$ in $G$. For any Sylow $p$-subgroup $X$ of $K_1 \cap K_2$ such that $p \in \pi(K_1 \cap K_2)$ and $(|X|, |(T, S)|) = 1$, there exist a Sylow $p$-subgroup $K_1p$ of $K_1$ and a Sylow $p$-subgroup $K_2p$ of $K_2$ such that $X = K_1p \cap K_2 = K_1 \cap K_2p$. Note that $p \nmid |T|$ and $p \nmid |S|$. Hence, $TK_1p = K_1p \cdot T$ and $SK_2p = K_2p \cdot S$. This shows that $T(K_1p \cap K_2) = (K_1p \cap K_2) \cdot T$ and $S(K_1 \cap K_2p) = (K_1 \cap K_2p) \cdot S$. Thus $(T, S)X = X(T, S)$, which implies that $K_1 \cap K_2$ is a SS-semipermutable supplement of $(T, S)$ in $G$. Therefore, $(T, S)$ is SS-semipermutable in $G$. \hfill \Box

**Proposition 2.6.** Suppose that a subgroup $H$ of a group $G$ is SS-semipermutable in $G$ with a SS-semipermutable supplement $K$, $L \leq G$ and $N \trianglelefteq G$. Then

1. If $H$ is a $p$-group, where $p \in \pi(G)$, then $(HN)/N$ is SS-semipermutable in $G/N$.
2. If $N \leq L$ and $L/N$ is SS-semipermutable in $G/N$, then $L$ is SS-semipermutable in $G$.
3. If $N$ is nilpotent, then $NK$ is a SS-semipermutable supplement of $H$ in $G$.

**Proof.** (1) It is clear that $KN/N$ is a supplement of $HN/N$. Let $A/N$ be a Sylow $q$-subgroup of $KN/N$ such that $(|A|, |HN/N|) = 1$, where $q \in \pi(G)$. Then there exists a Sylow $q$-subgroup $X$ of $KN$ such that $A = XN$. Further, there exist Sylow $q$-subgroups $K_q$ of $K$ and $N_q$ of $N$ such that $Y = K_qN_q$ is a Sylow $q$-subgroup of $KN$. Hence, $XN/N = (YN/N)^{kN} = (K_qN/N)^{kN} = K_q^kN/N$ for some $k \in K$.

Since $(|K_q^kN|/|N|, |HN/N|) = 1$, we have $(|K_q^kN|/|K_q^k \cap N|, |H|/|H \cap N|) = 1$. If $p \neq q$, it is clear that $(|K_q^k|, |H|) = 1$, and so $K_q^kH = HK_q^k$, which implies that $(A/N)(HN/N) = (HN/N)(A/N)$. If $p = q$ and $(|K_q^k|, |H|) = 1$, we have $(A/N)(HN/N) = (HN/N)(A/N)$. If $p = q$ and $(|K_q^k|, |H|) \neq 1$, we have the following two cases:
i. \( K^k_q = K^k_q \cap N \), which implies that \( K^k_q N/N = 1 \).

ii. \( H = H \cap N \), which implies that \( HN/N = 1 \).

Therefore, \( (A/N)(HN/N) = (HN/N)(A/N) \), and so \( HN/N \) is SS-semipermutable in \( G \).

(2) Let \( K/N \) be a SS-semipermutable supplement of \( L/N \) in \( G/N \). Then \( (K/N)(L/N) = G/N \), which means that \( KL = G \). If \( X \) is a Sylow \( p \)-subgroup of \( K \) such that \( (|X|, |L|) = 1 \), then \( XN/N \) is a Sylow \( p \)-subgroup of \( KN/N \) and \( (|XN/N|, |L/N|) = 1 \). Hence, \( (XN/N)(L/N) = (L/N)(XN/N) \). Therefore, \( XNL = LNX \) yields \( XL = LX \).

(3) Since \( N \) is nilpotent, for every \( p \in \pi(G) \) and every \( N_p \leq G \). By Lemma 2.1, for every \( K_p \in \text{syl}_p(K) \), \( N_pK_p \leq \text{syl}_p(NK) \). Now, suppose that \( X \) is a Sylow \( p \)-subgroup of \( NK \) such that \( (|X|, |H|) = 1 \), where \( p \in \pi(G) \). Then there exists an element \( g \in G \) such that \( X = (N_pK_p)^g \) for some \( N_p \leq \text{syl}_p(N) \) and \( K_p \leq \text{syl}_p(K) \). Hence, \( K_pH = HK_p \), which means that \( XH =HX \). Therefore, \( NK \) is a SS-semipermutable supplement of \( H \) in \( G \).

\[ \square \]

3. Main results

**Theorem 3.1.** Let \( G \) be a group. Then the following statements are equivalent:

(1) \( G \) is solvable, and every subnormal subgroup of \( G \) is SS-semipermutable in \( G \).

(2) Every subgroup of \( F^*(G) \) is SS-semipermutable in \( G \).

(3) \( G \) is a solvable PST-group.

Proof. Assume that \( G \) is a solvable PST-group. Then every subnormal subgroup of \( G \) is S-permutable in \( G \), and so SS-semipermutable in \( G \). Therefore, (3) implies (1).

Now, we show that (2) implies (3). Suppose that every subgroup of \( F^*(G) \) is SS-semipermutable in \( G \). Then by Lemma 2.3, every subgroup of \( F^*(G) \) is \( \tau \)-quasinormal in \( G \). Now, applying Lemma 2.4, we have that \( G \) is a solvable PST-group and thus (3) holds.

Finally, we prove that (1) implies (2). Assume that \( G \) is a solvable group and every subnormal subgroup of \( G \) is SS-semipermutable in \( G \). Then \( F^*(G) = F(G) \), and so every subgroup of \( F^*(G) \) is SS-semipermutable in \( G \).

\[ \square \]

**Theorem 3.2.** Let \( G \) be a group. Then the following statements are equivalent:

(1) Whenever \( H \leq K \) are two \( p \)-subgroups of \( G \) with \( p \in \pi(G) \), \( H \) is SS-semipermutable in \( N_G(K) \).

(2) \( G \) is a solvable PST-group.
Proof. Assume that (1) holds. By Lemma 2.2(3), whenever $H \leq K$ are two $p$-subgroups of $G$ with $p \in \pi(G)$, $H$ is $S$-permutable in $N_G(K)$. It follows from [2, Theorem 4] that $G$ is a solvable PST-group, and so (2) follows.

By [2, Theorem 4], again, we also see that (2) implies (1). □

Theorem 3.3. Let $G$ be a solvable group. Then the following statements are equivalent:

1. $G$ is a SSBT-group.
2. Every subgroup of $G$ is SS-semipermutable in $G$.
3. Every subgroup of $G$ of prime power order is SS-semipermutable in $G$.

Proof. Suppose that $G$ is a SSBT-group. Then every subnormal subgroup of $G$ is SS-semipermutable in $G$. By Theorem 3.1, $G$ is a PST-group. Let $L$ be the nilpotent residual of $G$. Since all subgroups of $L$ are normal in $G$, every subgroup $H$ of $G$ is SS-semipermutable in $HL$. As $HL$ is subnormal subgroup of $G$, it follows that $HL$ is SS-semipermutable in $G$. Hence, $H$ is SS-semipermutable in $G$. Since (2) implies (1), (1) and (2) are equivalent.

Now, assume that every subgroup of $G$ of prime power order is SS-semipermutable in $G$. By Lemma 2.5, it is easy to see that every subgroup of $G$ is SS-semipermutable in $G$, and it follows that (3) implies (2). This completes the proof. □

Corollary 3.4. Let $G$ be a solvable group. Then the following statements are equivalent:

1. $G$ is a SSBT group.
2. Every subgroup of $G$ is either SS-semipermutable or abnormal in $G$.

Proof. By Theorem 3.3, (1) implies (2). Suppose that every subgroup of $G$ is either SS-semipermutable or abnormal in $G$. According to proof of Lemma 2.3 and using [12, Lemma 1], $G$ is supersolvable. Let $H$ be a $p$-subgroup of $G$ with $p \in \pi(G)$. If $p$ is not the smallest prime divisor of $|G|$, then $H$ is not abnormal in $G$. Hence, $H$ is SS-semipermutable in $G$. Now assume that $p$ is the smallest prime divisor of $|G|$. If $H$ is not a Sylow $p$-subgroup of $G$, then $H$ is not abnormal in $G$, and by hypothesis, $H$ is SS-semipermutable in $G$. If $H \in Syl_p(G)$, $HG_q = G_qH$ for every $G_q \in Syl_q(G)$ with $q \in \pi(G)$ and $p \neq q$. Then every Hall $p'$-subgroup of $G$ is an SS-semipermutable supplement of $H$ in $G$, and so $H$ is SS-semipermutable in $G$. Hence, every subgroup of $G$ of prime power order is SS-semipermutable in $G$. Now, applying
Theorem 3.3, we have that $G$ is a SSBT-group, and this completes the proof. □

Corollary 3.5. The class of all solvable SSBT-groups is closed under taking subgroups and direct product.

Proof. Let $G$ be a solvable SSBT-group. If $H$ is a subgroup of $G$, then by Lemma 2.2(1) and Theorem 3.3, $H$ is an SSBT-group. Therefore, the class of all solvable SSBT-groups is closed under taking subgroups. Now, we prove that the class of all solvable SSBT-groups is closed under taking direct product. Let $G_1$ and $G_2$ be solvable SSBT-groups and $H_1 \times H_2$ be a subgroup of $G_1 \times G_2$. Then by Theorem 3.3, $H_1$ and $H_2$ are SS-semipermutable in $G_1$ and $G_2$, respectively. Let $K_1$ and $K_2$ be SS-semipermutable supplements of $H_1$ and $H_2$ in $G_1$ and $G_2$, respectively. Suppose further that $X_1 \times X_2$ is a Sylow subgroup of $K_1 \times K_2$ such that $(|X_1 \times X_2|, |H_1 \times H_2|) = 1$. Hence, $X_1 H_1 = H_1 X_1$ and $X_2 H_2 = H_2 X_2$, and so $(X_1 \times X_2)(H_1 \times H_2) = (H_1 \times H_2)(X_1 \times X_2)$. Therefore, $H_1 \times H_2$ is SS-semipermutable in $G_1 \times G_2$, and so by Theorem 3.3, $G_1 \times G_2$ is an SSBT-group. This shows that the class of all solvable SSBT-groups is closed under taking direct product. □

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ON FINITE GROUPS IN WHICH SS-SEMI-PERMUTABILITY IS A TRANSITIVE RELATION

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بررسی گروه‌های متناهی که SS-نیمه‌جابجایی‌پذیری یک خاصیت متعددی باشند

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فرض کنید $H$ یک زیرگروه از گروه متناهی $G$ باشد. زیرگروه $G$ را SS-نیمه‌جابجایی‌پذیر در نامیم هرگاه $H$ باشد به نحوی که $H$ با آن در ک منظم به ارتباط و از $K$ یک زیرگروه سیلوی $X$ باشد. زیرگروه $H$ نیمه‌جابجایی‌پذیر و گروه‌های متناهی $SS$-نیمه‌جابجایی‌پذیر، در این مقاله ساختار گروهرئی $SS$-نیمه‌جابجایی‌پذیری در آن یک خاصیت متعددی باشد بررسی شده است. به عنوان نمونه ثابت $PST$-گروه $G$ است. است $SS$-نیمه‌جابجایی‌پذیر در $N_G(H)$.

کلمات کلیدی: زیرگروه‌های SS-نیمه‌جابجایی‌پذیر، زیرگروه‌های PST-جوابجایی‌پذیر، گروه‌های $SS$. 