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ON FINITE GROUPS IN WHICH SS-SEMI-PERMUTABILITY IS A TRANSITIVE RELATION

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ABSTRACT. Let H be a subgroup of a finite group G. We say that H is SS-semipermutable in G if H has a supplement K in G such that H permutes with every Sylow subgroup X of K with $(|X|, |H|) = 1$. In this paper, the structure of SS-semipermutable subgroups and the finite groups in which SS-semi-permutability is a transitive relation are described. It is shown that a finite solvable group G is a PST-group if and only if whenever $H \leq K$ are two p-subgroups of G, H is SS-semipermutable in $N_G(K)$.

1. Introduction

Throughout this paper, all the groups are considered to be finite. Let H be a subgroup of G. Then $\pi(G)$ denotes the set of prime divisors of $|G|$; H^G is the normal closure of H in G, i.e. the intersection of all normal subgroups of G containing H; and $F^*(G)$ is the generalized fitting subgroup of G, i.e. the product of all normal quasinilpotent subgroup of G. H is said to be permutable in G if it permutes with all the subgroups of G. H is said to be S-permutable (or π-quasinormal) in G if it permutes with every Sylow subgroup of G. This concept has been introduced by Kegel [6], and has been widely studied by some authors; see, for example, [4] and [9].

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Permutability and S-permutability, like normality, are not, in general, transitive relations. This observation is the point of departure of the study of some relevant classes of groups such as T-groups, PT-groups, and PST-groups. Recall that a group G is called a T-group (resp. PT-group, PST-group) if normality (resp. permutability, S-permutability) is a transitive relation. By [6], PST-groups are exactly those groups in which every subnormal subgroup of G is S-permutable in G. Agrawal [1] have shown that a group G is a solvable PST-group if and only if the nilpotent residual L of G is a normal abelian Hall subgroup of G upon which G acts by conjugation as power automorphisms. Solvable PST, PT, and T-groups have been studied and characterized by Agrawal [1], Gaschütz [5] and Zacher [11].

A subgroup H of a group G is said to be semipermutable (resp. S-semipermutable) in G if H permutes with every subgroup (resp. Sylow subgroup) X of G such that $(|X|, |H|) = 1$. A group G is called a BT-group (resp. an SBT-group) [10] if semi-permutability (resp. S-semi-permutability) is a transitive relation. Wang et al. have shown that every subgroup of G is semipermutable in G if and only if every subgroup of G is S-semipermutable in G.

In 2008, Li et al. [7], have introduced the concept of SS-permutability (or SS-quasinormality), which is a generalization of S-permutability. A subgroup H of G is said to be SS-permutable in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K. In this case, K is called an SS-permutable supplement of H in G. Also in 2014, Chen and Guo have introduced a new concept called NSS-permutable as follows: A subgroup H of a group G is said to be NSS-permutable [3] in G if H has a normal supplement K in G such that H permutes with every Sylow subgroup of K. In this case, K is called an NSS-permutable supplement of H in G. Moreover, a group G is called an SST-group (resp. an NSST-group) [3] if SS-permutability (resp. NSS-permutability) is a transitive relation. A subgroup H of G is said to be τ-quasinormal in G if $HG_p = G_pH$ for every $G_p \in Syl_p(G)$ such that $(|H|, p) = 1$ and $(|H|, |G_p^G|) \neq 1$.

In this paper, we introduce a new subgroup embedding property, namely, SS-semipermutable which may be viewed as a generalization of both SS-permutable and semipermutable concepts, as follows:

Definition 1.1. We say that a subgroup H of a group G is SS-semipermutable in G if H has a supplement K in G such that H permutes with all Sylow subgroups X of K such that $(|X|, |H|) = 1$. In this case, K is called an SS-semipermutable supplement of H in G.

We say that a group G is an SSBT-group if SS-semi-permutability is a transitive relation.

2. Preliminaries

In this section, we give some results that are useful in the sequel. The following Lemma is easy to prove.

Lemma 2.1. *Let N_1 and N_2 be the subgroups of a group G and assume that $N_1N_2 \leq G$. If P_1 and P_2 are the Sylow p-subgroups of N_1 and N_2 respectively, where $p \in \pi(G)$, and $P_1P_2 \leq N_1N_2$, then P_1P_2 is a Sylow p-subgroup of N_1N_2.*

Lemma 2.2. *Suppose that a subgroup H of a group G is SS-semipermutable in G with a SS-semipermutable supplement K and $L \leq G$. Then*

1. *If $H \leq L$, then H is SS-semipermutable in L.*
2. *Every conjugate of K in G is a SS-semipermutable supplement of H in G.*
3. *If H is a p-subgroup, where $p \in \pi(G)$ and $H \leq F(G)$, then H is S-permutable in G.*

Proof. (1) Since $HK = KH$, $L = (HK) \cap L = H(K \cap L)$, which means that $(K \cap L)$ is a supplement of H in L. Now, suppose that $X \in Syl(K \cap L)$ with $(|X|, |H|) = 1$. Then there exists $Y \in Syl(K)$ such that $X \leq Y$. By hypothesis, $HY = YH$. Hence, $HY \cap L = H(Y \cap L) = HX$ and $L \cap (YH) = (L \cap Y)H = XH$. Therefore, $HX = XH$, and this shows that H is SS-semipermutable in L.

(2) Let $g \in G$. Then it is easy to see that $K^gH = G$. Now, suppose that X is a Sylow subgroup of K^g such that $(|X|, |H|) = 1$, where $p \in \pi(G)$. Then $X^{g^{-1}}$ is a Sylow p-subgroup of K with $(|X^{g^{-1}}|, |H|) = 1$. Hence, $X^{g^{-1}}H = HX^{g^{-1}}$. This shows that $XH = HX$.

(3) Let Q be a Sylow q-subgroup of K, where $q \in \pi(G)$ and $q \neq p$. Then $HQ = QH$, and HQ contains a Sylow q-subgroup Q^* of G. As $H \leq O_p(G)$, it follows that $H = O_p(G) \cap HQ \trianglelefteq HQ$, and thus Q^* normalizes H. Since this holds for all primes $q \neq p$, we deduce that $O^p(G) \leq N_G(H)$. Now applying [9, Lemma A], we have that H is S-permutable in G. □

Lemma 2.3. *Let G be a group. Then every SS-semipermutable subgroup of G is τ-quasinormal in G.*

Proof. Let H be a SS-semipermutable subgroup of G, and X be a Sylow subgroup of G such that $(|X|, |H|) = 1$ and $(|H|, |G_p^G|) \neq 1$. Then there exists an element $h \in H$ such that $X^h \leq K$. It follows that

$HX^h = X^hH$, and so $HX = XH$. Therefore, H is τ-quasinormal in G. □

Lemma 2.4. [8, Theorem 1.2] *Let G be a group. Then every subgroup of $F^*(G)$ is τ-quasinormal in G if and only if G is a solvable PST-group.*

Lemma 2.5. *Let T and S be SS-semipermutable in a solvable group G with $(|T|, |S|) = 1$. Then $\langle T, S\rangle$ is SS-semipermutable in G.*

Proof. Let K_1 and K_2 be SS-semipermutable supplements of T and S, respectively. Note that G is a solvable group. By Lemma 2.2(2), without less of generality, we may assume that $S \leq K_1$ and $T \leq K_2$. Then $TS(K_1 \cap K_2) = TK_1 = G$. This means that $K_1 \cap K_2$ is a supplement of $\langle T, S\rangle$ in G. For any Sylow p-subgroup X of $K_1 \cap K_2$ such that $p \in \pi(K_1 \cap K_2)$ and $(|X|, |\langle T, S\rangle|) = 1$, there exist a Sylow p-subgroup K_{1p} of K_1 and a Sylow p-subgroup K_{2p} of K_2 such that $X = K_{1p} \cap K_2 = K_1 \cap K_{2p}$. Note that $p \nmid |T|$ and $p \nmid |S|$. Hence, $TK_{1p} = K_{1p}T$ and $SK_{2p} = K_{2p}S$. This shows that $T(K_{1p} \cap K_2) = (K_{1p} \cap K_2)T$ and $S(K_1 \cap K_{2p}) = (K_1 \cap K_{2p})S$. Thus $\langle T, S\rangle X = X\langle T, S\rangle$, which implies that $K_1 \cap K_2$ is a SS-semipermutable supplement of $\langle T, S\rangle$ in G. Therefore, $\langle T, S\rangle$ is SS-semipermutable in G. □

Proposition 2.6. *Suppose that a subgroup H of a group G is SS-semipermutable in G with a SS-semipermutable supplement K, $L \leq G$ and $N \trianglelefteq G$. Then*

(1) *If H is a p-group, where $p \in \pi(G)$, then $(HN)/N$ is SS-semipermutable in G/N.*
(2) *If $N \leq L$ and L/N is SS-semipermutable in G/N, then L is SS-semipermutable in G.*
(3) *If N is nilpotent, then NK is a SS-semipermutable supplement of H in G.*

Proof. (1) It is clear that KN/N is a supplement of HN/N. Let A/N be a Sylow q-subgroup of KN/N such that $(|A/N|, |HN/N|) = 1$, where $q \in \pi(G)$. Then there exists a Sylow q-subgroup X of KN such that $A = XN$. Further, there exist Sylow q-subgroups K_q of K and N_q of N such that $Y = K_qN_q$ is a Sylow q-subgroup of KN. Hence, $XN/N = (YN/N)^{kN} = (K_qN/N)^{kN} = K_q^kN/N$ for some $k \in K$.

Since $(|K_q^kN|/|N|, |HN/N|) = 1$, we have $(|K_q^k|/|K_q^k \cap N|, |H| /|H \cap N|) = 1$. If $p \neq q$, it is clear that $(|K_q^k|, |H|) = 1$, and so $K_q^kH = HK_q^k$, which implies that $(A/N)(HN/N) = (HN/N)(A/N)$. If $p = q$ and $(|K_q^k|, |H|) = 1$, we have $(A/N)(HN/N) = (HN/N)(A/N)$. If $p = q$ and $(|K_q^k|, |H|) \neq 1$, we have the following two cases:

i. $K_q^k = K_q^k \cap N$, which implies that $K_q^k N/N = 1$.

ii. $H = H \cap N$, which implies that $HN/N = 1$.

Therefore, $(A/N)(HN/N) = (HN/N)(A/N)$, and so HN/N is SS-semipermutable in G.

(2) Let K/N be a SS-semipermutable supplement of L/N in G/N. Then $(K/N)(L/N) = G/N$, which means that $KL = G$. If X is a Sylow p-subgroup of K such that $(|X|, |L|) = 1$, then XN/N is a Sylow p-subgroup of K/N and $(|XN/N|, |L/N|) = 1$. Hence, $(XN/N)(L/N) = (L/N)(XN/N)$. Therefore, $XNL = LXN$ yields $XL = LX$.

(3) Since N is nilpotent, for every $p \in \pi(G)$ and every $N_p \in Syl_p(N)$, $N_p \trianglelefteq G$. By Lemma 2.1, for every $K_p \in syl_p(K)$, $N_pK_p \in Syl_p(NK)$. Now, suppose that X be a Sylow p-subgroup of NK such that $(|X|, |H|) = 1$, where $p \in \pi(G)$. Then there exists an element $g \in G$ such that $X = (N_pK_p)^g$ for some $N_p \in Syl_p(N)$ and $K_p \in Syl_p(K)$. Hence, $K_pH = HK_p$, which means that $XH = HX$. Therefore, NK is a SS-semipermutable supplement of H in G. □

3. Main results

Theorem 3.1. *Let G be a group. Then the following statements are equivalent:*

(1) *G is solvable, and every subnormal subgroup of G is SS-semipermutable in G.*

(2) *Every subgroup of $F^*(G)$ is SS-semipermutable in G.*

(3) *G is a solvable PST-group.*

Proof. Assume that G is a solvable PST-group. Then every subnormal subgroup of G is S-permutable in G, and so SS-semipermutable in G. Therefore, (3) implies (1).

Now, we show that (2) implies (3). Suppose that every subgroup of $F^*(G)$ is SS-semipermutable in G. Then by Lemma 2.3, every subgroup of $F^*(G)$ is τ-quasinormal in G. Now, applying Lemma 2.4, we have that G is a solvable PST-group and thus (3) holds.

Finally, we prove that (1) implies (2). Assume that G is a solvable group and every subnormal subgroup of G is SS-semipermutable in G. Then $F^*(G) = F(G)$, and so every subgroup of $F^*(G)$ is SS-semipermutable in G. □

Theorem 3.2. *Let G be a group. Then the following statements are equivalent:*

(1) *Whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is SS-semipermutable in $N_G(K)$.*

(2) *G is a solvable PST-group.*

Proof. Assume that (1) holds. By Lemma 2.2(3), whenever $H \leq K$ are two p-subgroups of G with $p \in \pi(G)$, H is S-permutable in $N_G(K)$. It follows from [2, Theorem 4] that G is a solvable PST-group, and so (2) follows.

By [2, Theorem 4], again, we also see that (2) implies (1). □

Theorem 3.3. *Let G be a solvable group. Then the following statements are equivalent:*

(1) *G is a SSBT-group.*
(2) *Every subgroup of G is SS-semipermutable in G.*
(3) *Every subgroup of G of prime power order is SS-semipermutable in G.*

Proof. Suppose that G is a SSBT-group. Then every subnormal subgroup of G is SS-semipermutable in G. By Theorem 3.1, G is a PST-group. Let L be the nilpotent residual of G. Since all subgroups of L are normal in G, every subgroup H of G is SS-semipermutable in HL. As HL is subnormal subgroup of G, it follows that HL is SS-semipermutable in G. Hence, H is SS-semipermutable in G. Since (2) implies (1), (1) and (2) are equivalent.

Now, assume that every subgroup of G of prime power order is SS-semipermutable in G. By Lemma 2.5, it is easy to see that every subgroup of G is SS-semipermutable in G, and it follows that (3) implies (2). This completes the proof. □

Corollary 3.4. *Let G be a solvable group. Then the following statements are equivalent:*

(1) *G is a SSBT group.*
(2) *Every sugroup of G is either SS-semipermutable or abnormal in G.*

Proof. By Theorem 3.3, (1) implies (2). Suppose that every subgroup of G is either SS-semipermutable or abnormal in G. According to proof of Lemma 2.3 and using [12, Lemma 1], G is supersolvable. Let H be a p-subgroup of G with $p \in \pi(G)$. If p is not the smallest prime divisor of $|G|$, then H is not abnormal in G. Hence, H is SS-semipermutable in G. Now assume that p is the smallest prime divisor of $|G|$. If H is not a Sylow p-subgroup of G, then H is not abnormal in G, and by hypothesis, H is SS-semipermutable in G. If $H \in Syl_p(G)$, $HG_q = G_qH$ for every $G_q \in Syl_q(G)$ with $q \in \pi(G)$ and $p \neq q$. Then every Hall p'-subgroup of G is an SS-semipermutable supplement of H in G, and so H is SS-semipermutable in G. Hence, every subgroup of G of prime power order is SS-semipermutable in G. Now, applying

Theorem 3.3, we have that G is a SSBT-group, and this completes the proof. □

Corollary 3.5. *The class of all solvable SSBT-groups is closed under taking subgroups and direct product.*

Proof. Let G be a solvable SSBT-group. If H is a subgroup of G, then by Lemma 2.2(1) and Theorem 3.3, H is an SSBT-group. Therefore, the class of all solvable SSBT-groups is closed under taking subgroups. Now, we prove that the class of all solvable SSBT-groups is closed under taking direct product. Let G_1 and G_2 be solvable SSBT-groups and $H_1 \times H_2$ be a subgroup of $G_1 \times G_2$. Then by Theorem 3.3, H_1 and H_2 are SS-semipermutable in G_1 and G_2, respectively. Let K_1 and K_2 be SS-semipermutable supplements of H_1 and H_2 in G_1 and G_2, respectively. Suppose further that $X_1 \times X_2$ is a Sylow subgroup of $K_1 \times K_2$ such that $(|X_1 \times X_2|, |H_1 \times H_2|) = 1$. Hence, $X_1H_1 = H_1X_1$ and $X_2H_2 = H_2X_2$, and so $(X_1 \times X_2)(H_1 \times H_2) = (H_1 \times H_2)(X_1 \times X_2)$. Therefore, $H_1 \times H_2$ is SS-semipermutable in $G_1 \times G_2$, and so by Theorem 3.3, $G_1 \times G_2$ is an SSBT-group. This shows that the class of all solvable SSBT-groups is closed under taking direct product. □

References

1. R. K. Agrawal, Finite groups whose Subnormal subgroups permutes with all Sylow subgroups, *Proc. Amer. Math. soc.* **47** (1975), 77–83.
2. A. Ballester-Bolinches and R. Esteban-Romero, On finite soluble groups in which Sylow permutability is a transitive relation, *Acta Math. Hungar.* **101** (2003), 193–202.
3. X. Chen and W. Guo, Finite groups in which SS-permutability is a transitive relation, *Acta Math. Hungar.* **143** (2014), 466–479.
4. W. E. Deskins, On quasinormal subgroups of finite groups, *Math. Z.* **82** (1963), 125–132.
5. W. Gaschütz, Gruppen, in denen das normalteilersein transitiv ist, *J. Reine Angew. Math.* **198** (1957), 87–92.
6. O. H. Kegel, Sylow-gruppen und subnormalteiler endlicher gruppen, *Math. Z.* **78** (1962), 205–221.
7. S. Li, Z. Shen, J. Liu and X. Liu, The influence of SS-quasinormality of some subgroups on the structure of finite groups, *J. Algebra.* **319** (2008), 4275–4287.
8. V. O. Lukyanenko and A. N. Skiba, Finite groups in which τ-quasinormality is a transitive relation, *Rend. Semin. Math. Univ. Podova.* **124** (2010), 231–246.
9. P. Schmid, Subgroup permutable with all Sylow subgroups, *J. Algebra.* **207** (1998), 285–293.
10. L. Wang, Y. Li and Y. Wang, Finite groups in which (S-)semi-permutability is a transitive relation, *Int. J. Algebra.* **2** (2008), 143–152.
11. G. Zacher, Sui gruppi risolubili finiti col reticolo dei sottogruppi di composizione dotato di duale, *Rend. Sem. Mat. Univ. Padova.* **32** (1962), 325–327.

12. Q. Zhang, S-semi-permutability and abnormality in finite groups, *Comm. Algebra* **27** (1999), 4515–4524.

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ON FINITE GROUPS IN WHICH SS-SEMI-PERMUTABILITY IS A TRANSITIVE RELATION

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بررسی گروه‌های متناهی که SS- نیمه‌جابجاپذیری یک خاصیت متعدی باشد

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فرض کنید H یک زیرگروه از گروه متناهی G باشد. زیرگروه H را SS- نیمه‌جابجاپذیر در G نامیم هرگاه H دارای مکمل K در G باشد به نحوی که H با هر زیرگروه سیلوی X از K با شرط $(|X|,|H|)=۱$ جابجا گردد. در این مقاله ساختار گروه‌های SS- نیمه‌جابجاپذیر و گروه‌های متناهی که SS- نیمه‌جابجاپذیری در آن یک خاصیت متعدی باشد، بررسی شده است. به عنوان نمونه ثابت شده است گروه حل‌پذیر متناهی G یک PST- گروه است اگر و تنها اگر به ازای هر دو $P-$زیرگروه $K \leq H$ از G، H SS- نیمه‌جابجاپذیر در $N_G(H)$ است.

کلمات کلیدی: زیرگروه‌های SS- نیمه‌جابجاپذیر، زیرگروه‌های S- جابجاپذیر، گروه‌های PST.