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ON COMPOSITION FACTORS OF A GROUP WITH THE SAME PRIME GRAPH AS $\mathbf{L_n}(5)$

A. MAHMOUDIFAR*

Abstract. The prime graph of a finite group G is denoted by $\Gamma(G)$. A nonabelian simple group G is called quasirecognizable by prime graph if, for every finite group H, where $\Gamma(H) = \Gamma(G)$, there exists a nonabelian composition factor of H that is isomorphic to G. Until now, it has been proved that some finite linear simple groups are quasirecognizable by prime graph, for instance, the linear groups $L_n(2)$ and $L_n(3)$ are quasirecognizable by prime graph. In this paper, we consider the quasirecognition by prime graph of the simple group $L_n(5)$.

1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers. If $n \in \mathbb{N}$, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. Also the set of element orders of G is denoted by $\pi_e(G)$. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and the two distinct primes p and q are joined by an edge (and we write $p \sim q$), whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. A subset of vertices of $\Gamma(G)$ is called an independent subset of $\Gamma(G)$ if its vertices are pairwise nonadjacent. The maximal size of an independent subset of $\Gamma(G)$ is denoted by $t(G)$. For every $p \in \pi(G)$, we denote, by $t(p, G)$, the maximal size of an independent subset of $\Gamma(G)$, which contains p.

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*Corresponding author.

In addition, we denote by $\rho(G)$ (by $\rho(p, G)$) some independent sets in $\Gamma(G)$ (containing p) with maximal number of vertices.

A finite nonabelian simple group P is called *quasirecognizable by prime graph* (*by element orders*) if every finite group G with $\Gamma(G) = \Gamma(P)$ ($\pi_e(G) = \pi_e(P)$) has a unique composition factor isomorphic to P. Also P is called *recognizable by prime graph* (*by element orders*) if $\Gamma(G) = \Gamma(P)$ ($\pi_e(G) = \pi_e(P)$) implies that $G \cong P$. Also we note that quasirecognition (recognition) by prime graph implies quasirecognition (recognition) by element orders, but the converse is not true, in general.

Hagie in [10] has determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. It has been proved that if $q = 3^{2n+1}$ ($n > 0$). Then the simple group ${}^2G_2(q)$ is recognizable by its prime graph [12] and [32]. A group G is called a CIT group if G is of even order and the centralizer in G of any involution is a 2-group. In [19] finite groups with the same prime graph as a CIT simple group are determined. Also in [20] it has been proved that if $p > 11$ is a prime number and $p \not\equiv 1 (\mathrm{mod}\ 12)$, then $L_2(p)$ is recognizable by prime graph. In [13] and [17], finite groups with the same prime graph as $L_2(q)$, where q is not prime, have been determined. In [1] and [16], finite groups with the same prime graph as ${}^2F_4(q)$, where $q = 2^{2n+1} > 2$ and $F_4(q)$, where $q = 2^n > 2$ are determined. Also in [11], it has been proved that if p is a prime number that is not a Mersenne or Fermat prime and $p \neq 11$, 13, 19 and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition factor that is isomorphic to $L_2(p)$, and if $p = 13$, then G has a unique nonabelian composition factor that is isomorphic to $L_2(13)$ or $L_2(27)$. Then it has been proved that if p and $k > 1$ are odd and $q = p^k$ is a prime power, then $\mathrm{PGL}(2, q)$ is recognizable by its prime graph [2] (see also [5]). In [14], [15], and [18], finite groups with the same prime graph as $L_n(2)$, where $n = 9$, 10, 16 have been obtained. Also for more results, see [4], [9], [21], and [24].

In this paper, as the main result, at first, we prove some theorems that are useful for considering quasirecognition by prime graph. Also as an application of these theorems, we show the simple group $L_n(5)$, where $n \geq 11$ is quasirecognizable by prime graph. In [7], it has been proved that if $\pi_e(G) = \pi_e(L_5(5))$, then $G \cong L_5(5)$ or $L_5(5) \cdot 2$, the extension of $L_5(5)$ by a graph automorphism. Also in [8], it has been proved that if $\pi_e(G) = \pi_e(L_4(5))$, then $G/N \cong L_4(5)$, where N is a normal 5-subgroup of G. As a consequence of our result, we prove that if $\pi_e(G) = \pi_e(L_n(5))$, where $n \geq 11$, then $G \cong L_n(5)$ or $G \cong L_n(5) \cdot 2$, the extension of $L_n(5)$ by graph automorphism.

Throughout this paper, all groups are finite, and by simple groups, we mean nonabelian simple groups. All further unexplained notations are standard and refer to [6]. Let m be a positive integer and p be a prime number. Then m_p denotes the p-part of m. In other words, $m_p = p^k$ if $p^k \big| m$ but $p^{k+1} \not\big| m$.

2. Preliminary Results

Lemma 2.1. [26, Theorem 1] *Let G be a finite group satisfying the following two conditions:*

(a) *there exist three primes in $\pi(G)$ pairwise nonadjacent in $\Gamma(G)$; i.e., $t(G) \geq 3$;*
(b) *there exists an odd prime in $\pi(G)$ nonadjacent in $\Gamma(G)$ to the prime 2; i.e. $t(2, G) \geq 2$.*

Then there is a finite nonabelian simple group S such that $S \leq \overline{G} = G/K \leq \mathrm{Aut}(S)$ for the maximal normal solvable subgroup K of G. Furthermore, $t(S) \geq t(G) - 1$, and one of the following statements holds:

(1) *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, S) = 3$.*
(2) *For every prime $p \in \pi(G)$ nonadjacent to 2 in $\Gamma(G)$, a Sylow p-subgroup of G is isomorphic to a Sylow p-subgroup of S. In particular, $t(2, S) \geq t(2, G)$.*

Lemma 2.2. [22, Lemma 1] *Let G be a finite group and $N \trianglelefteq G$ such that G/N is a Frobenius group with kernel F and cyclic complement C. If $(|F|, |N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$, for some prime divisor p of $|N|$.*

Lemma 2.3. [28, Lemma 5] *If $L \cong L_n(q)$ and $d = (q-1, n)$, then L includes a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/d$.*

Lemma 2.4. [33, Zsigmondy's Theorem] *Let p be a prime, and let n be a positive integer. Then one of the following holds:*

(1) *there is a primitive prime p' for $p^n - 1$, i.e. $p' \big| (p^n - 1)$ but $p' \not\big| (p^m - 1)$, for every $1 \leq m < n$,*
(2) *$p = 2$, $n = 1$ or 6,*
(3) *p is a Mersenne prime and $n = 2$.*

If q is a natural number, r is an odd prime, and $(q, r) = 1$, then by $e(r, q)$, we denote the smallest natural number m such that $q^m \equiv 1 \pmod{r}$. Given an odd q, put $e(2, q) = 1$ if $q \equiv 1 \pmod{4}$, and put $e(2, q) = 2$ if $q \equiv 3 \pmod{4}$. Obviously, we can see that if r is an odd prime such that $r \mid (q^n - 1)$, then $e(r, q) \mid n$.

Lemma 2.5. [29, Proposition 1.1] *Let $G = A_n$ be an alternating group of degree n.*

(1) *Let* $r,\ s \in \pi(G)$ *be odd primes. Then* r *and* s *are nonadjacent if and only if* $r + s > n$.
(2) *Let* $r \in \pi(G)$ *be an odd prime. Then* 2 *and* r *are nonadjacent if and only if* $r + 4 > n$.

The next lemma determines the structure of the maximal tori of finite simple groups of Lie type.

Lemma 2.6. [29, Lemma 1.2] *Let* $\overline{G}$ *be a connected simple classical algebraic group of adjoint type, and let* $G = O^{p'}(\overline{G}_\sigma)$ *be the finite simple classical group:*

(1) *Every maximal torus* T *of* $G = A^{\varepsilon}_{n-1}(q)$ $(\varepsilon \in \{+,-\})$ *has the order*
$$\frac{1}{(n, q-\varepsilon 1)(q-\varepsilon 1)}(q^{n_1} - (\varepsilon 1)^{n_1})(q^{n_2} - (\varepsilon 1)^{n_2}) \dots (q^{n_k} - (\varepsilon 1)^{n_k})$$
for an appropriate partition $n_1 + n_2 + \cdots + n_k = n$ *of* n. *Moreover, for every partition, there exists a torus of corresponding order.*
(2) *Every maximal torus* T *of* G, *where* $G = B_n(q)$ *or* $G = C_n(q)$, *has the order*
$$\frac{1}{(2, q-1)}(q^{n_1} - 1)(q^{n_2} - 1)...(q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots$$
$$(q^{l_m} + 1)$$
for an appropriate partition $n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_m = n$ *of* n. *Moreover, for every partition, there exists a torus of corresponding order.*
(3) *Every maximal torus* T *of* $G = D^{\varepsilon}_n(q)$ *has the order*
$$\frac{1}{(4, q^n-\varepsilon 1)}(q^{n_1} - 1)(q^{n_2} - 1) \dots (q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots$$
$$(q^{l_m} + 1)$$
for an appropriate partition $n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_m = n$ *of* n, *where* m *is even if* $\varepsilon = +$, *and* m *is odd if* $\varepsilon = -$. *Moreover, for every partition, there exists a torus of corresponding order.*

Lemma 2.7. [29, Proposition 2.1] *Let* $G = A_{n-1}(q)$ *be a finite simple group of Lie type over a field of characteristic* p. *Let* r,s *be odd primes, and* $r,s \in \pi_e(G) \setminus \{p\}$. *Denote* $k = e(r, q)$, $l = e(s, q)$, *and suppose that* $2 \leq k \leq l$. *Then* r *and* s *are nonadjacent if and only if* $k + l > n$ *and* k *does not divide* l.

3. Main Results

Throughout this section, let L be a nonabelian simple group such that $t(2, L) \geq 2$, $t(L) \geq 3$ and G be a finite group such that $\Gamma(G) = \Gamma(L)$. Thus by Lemma 2.1, there exists a nonabelian simple group S, such that $S \leq \overline{G} := G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. Also if $L = L_n(q)$, then we assume that r_m is a primitive prime divisor of $q^m - 1$. It is obvious that $e(r_m, q) = m$ if r_m is odd. In this section, we use these notations.

Theorem 3.1. *Using the above-mentioned and assumptions, let ρ be an independent subset of $\Gamma(G)$ such that $|\rho| \geq (p+3)/2$ for some $p \in \rho$. If $7 \leq p < \max \rho$, then $S \ncong A_m$, where $m \geq 5$.*

Proof. Let $t = \max \rho$ and $S \cong A_m$ for some $m \in \mathbb{N}$. Since ρ is an independent subset of $\Gamma(G)$ and $|\rho| \geq (p+3)/2 \geq (7+3)/2 > 3$, thus by Lemma 2.1, at most, one element of ρ may not belong to the subset $\rho' := \rho \cap \pi(S)$, i.e. $|\rho| - 1 \leq |\rho'| \leq |\rho|$. We claim that $p \in \rho'$. For this purpose, we consider the following cases. If $t \in \rho' \subseteq \pi(S)$, then every prime less than t is contained in $\pi(S)$ since $S \cong A_m$ and $\pi(S) = \pi(m!)$. But by our hypothesis, $p < t$. Thus $p \in \pi(S) \cap \rho = \rho'$. If $t \notin \rho'$, then ρ' contains every element of $\rho \setminus \{t\}$ since, at most, one element of ρ is not contained in ρ'. Again, we have $p \in \rho'$ since $p \in \rho \setminus \{t\}$. Therefore, in each case $p \in \rho'$ and so ρ' is an independent subset of $\Gamma(S)$ that contains p.

Now we show that $2 \notin \rho'$. If $2 \in \rho'$, then by Lemma 2.5, for each $r \in \rho' \setminus \{2\}$, we have $m-3 \leq r \leq m$. On the other hand, $|\rho'|-1 \geq |\rho| - 2 \geq (p-1)/2 \geq 3$, and so there exist at least 3 prime numbers between $m-3$ and m, which implies that $m = 5$, and this is a contradiction, since $p \in \rho'$ and $p \geq 7$.

Therefore, $2 \notin \rho'$ and ρ' is an independent subset of $\Gamma(S)$, which contains p. Since, for each $r \in \rho' \setminus \{p\}$, $r \not\sim p$ in $\Gamma(S)$, using Lemma 2.5, we conclude that if $r \in \rho'$, then $m - p + 1 \leq r \leq m$. Thus there exist at least $(p-1)/2$ odd prime numbers between p numbers $m - p + 1$, $m - p + 2$, $\ldots$, m. Since $p \geq 7$, an easy calculation shows that this is impossible. Therefore, S is not isomorphic to any alternating group. $\square$

Theorem 3.2. *Let p be an odd prime number and $L = L_n(p)$, where $n \geq 11$. If S is a simple group of Lie type over $\mathrm{GF}(p^\alpha)$, where $\alpha \geq 1$, then $S \cong L$.*

Proof. Our main tools to prove this theorem are the following facts:

(a) By Lemma 2.1, $\rho(2,G) \subseteq \pi(S)$. Therefore, $\{r_{n-1}, r_n\} \cap \pi(S) \neq \emptyset$. Also, by Table 8 in [29] and Lemma 2.1, we have:

$$t(S) \geq t(G) - 1 = [(n+1)/2] - 1 \geq 5.$$

(b) Since $n \geq 11$, by Table 8 in [29], $\{r_{n-3}, r_{n-2}, r_{n-1}, r_n\}$ is an independent subset of $\Gamma(G)$. Also by Lemma 2.1, at most one element of this set may not belong to $\pi(S)$. Therefore, $|\{r_{n-3}, r_{n-2}, r_{n-1}, r_n\} \cap \pi(S)| \geq 3$.

(c) Since $t(S) \geq 5$, by Table 9 in [29], $S \ncong {}^3D_4(p^\alpha)$. Thus the order of the finite simple group S is of the form:

$$|S| = p^x(p^{x_1} - 1)(p^{x_2} - 1) \cdots (p^{x_k} - 1)(p^{y_1} + 1)(p^{y_2} + 1) \cdots (p^{y_l} + 1)/d$$

where $x_i, y_j \in \mathbb{N}$, $x_1 < x_2 < \cdots < x_k$, $y_1 < y_2 < \cdots < y_l$ and x, d are natural numbers. Let $t = \max\{x_k, 2y_l\}$. We know that $\{r_{n-1}, r_n\} \cap \pi(S) \neq \emptyset$. According to the order of S, and using Lemma 2.4, it follows that $n - 1 \leq t \leq n$. Moreover, if $r_n \in \pi(S)$, then $t = n$, and if $r_n \notin \pi(S)$, then $r_{n-1} \in \pi(S)$, and so $t = n - 1$.

(d) By Table 6 in [29], we know that if n is odd, then $r_n \in \rho(2,G) \subseteq \pi(S)$. Let $t = \max\{x_k, 2y_l\}$. Then by (c), $t = n - 1$ or $t = n$. As an application of this result, we prove that if t is even, then $t = n$. If t is even and $t = n-1$, then n is odd, and so $r_n \in \pi(S)$, by the above discussion, which implies that $t = n$, by (c), and this is a contradiction.

Now, using the classification of finite simple groups, we consider each possibility for S such that $t(S) \geq 5$ and $t(2,S) \geq 2$.

Case1. Let $S \cong {}^2A_{m-1}(p^\alpha)$. Since $t(S) \geq 5$, by Table 8 in [29], we have $t(S) = [(m+1)/2]$ and $m \geq 9$.

If m is an odd number, then $2\alpha m \leq n$, by Lemma 2.4. On the other hand, by (a), $[(m+1)/2] = t(S) \geq t(G) - 1 = [(n+1)/2] - 1 \geq [(2\alpha m + 1)/2] - 1$, which shows that $4 > (2\alpha - 1)m$. But this is impossible since $\alpha \geq 1$ and $m \geq 9$.

Therefore, m is an even number. Then the order of ${}^2A_{m-1}(p^\alpha)$ and (c) imply that $t = \max\{2\alpha(m-1), \alpha m\} = 2\alpha(m-1)$, and so by (d), $2\alpha(m-1) = n$. Now, by (a), $[(m+1)/2] = t(S) \geq t(G) - 1 = [(n+1)/2]-1 = [(2\alpha(m-1)+1)/2]-1$, which shows that $(m-1)(2\alpha-1) < 5$. But this is impossible, since $m \geq 9$ and $\alpha \geq 1$. Therefore, S is not isomorphic to ${}^2A_{m-1}(p^\alpha)$.

Case2. Let $S \cong B_m(p^\alpha)$ or $C_m(p^\alpha)$. Since $t(S) \geq 5$, by Table 8 in [29], we have $t(S) = [(3m+5)/4]$ and $m \geq 5$. Also by (c) and the orders of $B_m(p^\alpha)$ and $C_m(p^\alpha)$, we have $t = 2\alpha m \leq n$. Since $2\alpha m$ is even by (d), $2\alpha m = n$. By (b), $\pi(S)$ consists of a primitive prime divisor of $p^{n-1} - 1 = p^{2\alpha m-1} - 1$ or $p^{n-3} - 1 = p^{2\alpha m-3} - 1$. But the orders of $B_m(p^\alpha)$ and $C_m(p^\alpha)$ show that this is impossible. Therefore, S is not isomorphic to $B_m(p^\alpha)$ and $C_m(p^\alpha)$.

Case3. Let $S \cong D_m(p^\alpha)$ or ${}^2D_m(p^\alpha)$. Since $t(S) \geq 5$, by Table 8 in [29], $t(S) \leq (3m+4)/4$ and $m \geq 6$. By (c) and the orders of $D_m(p^\alpha)$ and ${}^2D_m(p^\alpha)$, $2\alpha(m-1) \leq n$. Now, by (a), $(3m+4)/4 \geq t(S) \geq t(G) - 1 = [(n+1)/2] - 1 \geq [(2\alpha(m-1)+1)/2] - 1$, which implies that $(4\alpha - 3)(m-1) < 13$. Since $m \geq 6$, we conclude that $\alpha = 1$, and so $S \cong D_m(p)$ or ${}^2D_m(p)$.

Now, using (c) and the orders of $D_m(p)$ and ${}^2D_m(p)$, we have $2(m-1) \leq n$ or $2m \leq n$, respectively. Also since $2(m-1)$ and $2m$ are even, by (d), $n = 2(m-1)$ or $n = 2m$. On the other hand, by (b), $\pi(S)$ consists of a primitive prime divisor of $p^{n-1} - 1$ or $p^{n-3} - 1$. But by the orders of $D_m(p)$ and ${}^2D_m(p)$, we get a contradiction, since $m \geq 6$. Therefore, S is not isomorphic to $D_m(p^\alpha)$ and ${}^2D_m(p^\alpha)$.

Case4. Let $S \cong F_4(p^\alpha)$, $E_7(p^\alpha)$ or $E_8(p^\alpha)$. In this case, the order of S is of the form:

$$|S| = p^x(p^{2x_1} - 1)(p^{2x_2} - 1)\cdots(p^{2x_k} - 1)/d$$

where $x_1 < x_2 < \cdots < x_k$, and x and d are natural numbers. Thus by (d), $2x_k = n$, and so by (b), $\pi(S)$ consists of primitive prime divisors of $p^{n-1} - 1$ or $p^{n-3} - 1$. But the order of S shows that this is impossible.

Case5. Let $S \cong E_6(q)$, where $q = p^\alpha$. Thus

$$|S| =$$
$$q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1)/(3, q-1).$$

In this case, by Table 4 in [30], $t(S) = 5$, and so by (a), $n \in \{11, 12\}$. Also (c) and the order of S show that $12\alpha \leq n$, and so $\alpha = 1$ and $n = 12$. Now, by (b), $\pi(S)$ consists of primitive prime divisors of $p^{n-1} - 1 = p^{11} - 1$ or $p^{n-2} - 1 = p^{10} - 1$, which is impossible.

Case6. Let $S \cong {}^2E_6(p^\alpha)$ or ${}^2G_2(3^{2\alpha+1})$, where $\alpha \geq 1$. The order of S and (c) show that $n \geq 18$. But by Table 9 in [29] and (a), we have $5 = t(S) \geq t(G) - 1 \geq [(18+1)/2] - 1 = 8$, which is a contradiction.

Case7. Let $S \cong L_m(p^\alpha)$. The order of S and (c) show that $n - 1 \leq \alpha m \leq n$. Since $t(S) \geq 5$, by Table 8 in [29], we have $t(S) = [(m+1)/2]$ and $m \geq 9$. Now, the relation $[(m+1)/2] = t(S) \geq t(G) - 1 = [(n+1)/2] - 1 \geq [(\alpha m+1)/2] - 1$ implies that $m(\alpha - 1) < 4$. Since $m \geq 9$, we conclude that $\alpha = 1$, and so $S \cong L_m(p)$, where $n - 1 \leq m \leq n$.

If $m = n-1$, then by (c), $r_n \notin \pi(S)$, and so $r_n \in \pi(\overline{G}/S)$ or $\pi(K)$. Since $\pi(\overline{G}/S) \subseteq \pi(\mathrm{Out}(S))$ and $|\mathrm{Out}(S)| = |\mathrm{Out}(L_m(p))| = 2(m, p-1) = 2(n-1, p-1)$, so $r_n \notin \pi(\overline{G}/S)$ and $r_n \in \pi(K)$. By Lemma 2.3, S

contains a Frobenius subgroup of the form $p^{n-2} : (p^{n-2}-1)/(n-1, p-1)$. By Table 4 in [29], $r_n \nsim p$ in $\Gamma(G)$. Thus by Lemma 2.2, $r_n \sim r_{n-2}$, in $\Gamma(G)$, which by (b), this is a contradiction. Therefore, $m = n$ and $S \cong L_n(p)$. □

Theorem 3.3. *With the notations and assumptions at the beginning of this section, we have:*

(a) If S is a simple group of Lie type over $\mathrm{GF}(r^\beta)$, where $r \in \pi(G)$, then $t(r, G) \leq 6$. In particular, if $L = L_n(q)$, where $n \geq 13$, then S is not isomorphic to any simple group of Lie type over $\mathrm{GF}(r^\beta)$, where $e(r, q) \geq 7$.

(b) Let $L = L_n(p^\alpha)$ and S be a classical simple group of Lie type over $\mathrm{GF}(r^\beta)$. If there exists $r' \in \pi(S)$ and $e(r', r) = 2k$, then $e(r', p^\alpha) \leq 4k + 6$.

(c) If $t(G) \geq 14$, then S is not isomorphic to any exceptional simple group of Lie type. In particular, if $L = L_n(q)$, $r \in \pi(L)$ and $e(r, q) \geq 27$, then S is not isomorphic to any exceptional simple group of Lie type.

Proof. a) To the contrary, suppose that S is a simple group of Lie type over $\mathrm{GF}(r^\beta)$, where $\beta \geq 1$ and $t(r, G) \geq 7$. Then there exists an independent subset ρ of $\pi(G)$ such that $r \in \rho$ and $|\rho| \geq 7$. If $\rho' := \rho \cap \pi(S)$, then, by Lemma 2.1, $|\rho'| \geq 6$. Therefore, $t(r, S) \geq |\rho'| \geq 6$. But Tables 4 and 5 in [29] show that for such simple groups, $t(r, S) \leq 5$, which is a contradiction. Therefore, $t(r, G) \leq 6$.

For the second part of this result, let $L = L_n(q)$, where $n \geq 13$ and $t = e(r, q) \geq 7$. Let $t \geq (n+3)/2$. Then, by Table 8 in [29], we get that the subset $\rho_0 := \{r_{[n/2]+1}, r_{[n/2]+2}, \ldots, r_t(= r), \ldots, r_{n-1}, r_n\}$ is an independent subset of $\Gamma(G)$ including r such that $|\rho_0| = n - [n/2]$. Since $n \geq 13$, by Lemma 2.7, we conclude that $t(r, G) \geq |\rho_0| \geq 7$, which is a contradiction since $t(r, G) \leq 6$. Now, let $t < (n + 3)/2$. Then, by Lemma 2.7, we get that $\rho(r, L) \subseteq \{r, r_{n-t+1}, r_{n-t+2}, \ldots, r_{n-1}, r_n\}$. On the other hand, Lemma 2.7 shows that r is adjacent to exactly one element in $r_{n-t+1}, r_{n-t+2}, \ldots, r_{n-1}, r_n$, and these numbers are mutually nonadjacent by Lemma 2.7 and also by Table 8 in [29]. Therefore, $t(r, L) \geq 7$, and so $t(r, G) \geq 7$, since $\Gamma(G) = \Gamma(L)$, and this is impossible by the above discussion.

b) To the contrary, suppose that $e(r', p^\alpha) \geq 4k + 7$. Therefore, $n \geq 4k+7$. Thus $t(G) = [(n+1)/2] \geq [(4k+7+1)/2] = 2k+4$. Therefore, by Table 8 in [29], the subset $\rho = \{r_{n-2k-3}, r_{n-2k-2}, \ldots, r_{n-1}, r_n\}$ is an independent subset of $\Gamma(G)$. Also, by Lemma 2.7, at most, one element of ρ is adjacent to r'. Thus, by Lemma 2.1, $\pi(S) \cap \rho$ has, at least, $2k+2$ elements of $\pi(S)$, which are nonadjacent to r' in $\Gamma(S)$. Therefore, $t(r', S) \geq 2k+3$.

We claim that if $e(r', r) = 2k$. Then $t(r', S) \leq 2k+2$, which is impossible. By the assumption, $e(r', r) = 2k$, then $e(r', r^{\beta})\big|2k$.

Let $S \cong L_m(r^{\beta})$ or $S \cong U_m(r^{\beta})$ and the prime numbers r'_i satisfy the equation $e(r'_i, r^{\beta}) = i$, where $m-2k+1 \leq i \leq m$. Then, by Lemma 2.6, r' may be nonadjacent to r, and, at most, the prime numbers r'_i. Therefore, in this case, $t(r', S) \leq 2k+2$, which is a contradiction.

Now, let $S \cong B_m(r^{\beta})$, $C_m(r^{\beta})$, $D_m(r^{\beta})$ or ${}^2D_m(r^{\beta})$ and the prime numbers r'_i satisfy the equation $e(r'_i, r^{\beta}) = i$ or $2i$, where $m-k+1 \leq i \leq m$. Then by Lemma 2.6, r' may be nonadjacent to r and at most the prime numbers r'_i. Therefore in each case we have $t(r', S) \leq 2k+2$, which is a contradiction.

c) If $t(G) \geq 14$, then there exists an independent subset ρ of $\Gamma(G)$, such that $|\rho| \geq 14$. Thus if $\rho' := \pi(S) \cap \rho$, then, by Lemma 2.1, $|\rho'| \geq 13$. Thus $t(S) \geq |\rho'| \geq 13$, and, by Table 9 in [29], S is not isomorphic to any exceptional simple group of Lie type. In particular, if $L = L_n(q)$ and $e(r, q) \geq 27$, then $t(G) = [(n+1)/2] \geq [(27+1)/2] = 14$. □

4. Applications in quasirecognition by prime graph of $\mathbf{L_n(5)}$

Now, by the results proved in the earlier section, we consider quasirecognition of finite simple group $L_n(5)$, where $n \geq 11$. Throughout this section, we assume that r_i is a primitive prime divisor of $5^i - 1$, where $i \geq 3$.

Theorem 4.1. *Let G be a finite group such that $\Gamma(G) = \Gamma(L_n(5))$, where $n \geq 11$. Then $L_n(5) \leq G/K \leq \mathrm{Aut}(L_n(5))$, where K is a normal $\{2,3,5\}$-subgroup of G. In particular, the finite simple group $L_n(5)$, where $n \geq 11$, is quasirecognizable by prime graph.*

Proof. By Table 4 in [29], $\{5, r_n, r_{n-1}\}$ is an independent subset of $\Gamma(G)$. Also, by Table 6 in [29], r_n or r_{n-1} is nonadjacent to 2 in $\Gamma(G)$. Thus, by Lemma 2.1, there exists a nonabelian simple group S, such that $S \leq \overline{G} := G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G. Now, we show that S can not be isomorphic to any simple group, except $L_n(5)$. By classification theorem, S may be isomorphic to an alternating group, a sporadic simple group or a simple group of Lie type.

Step 1. Since $e(7, 5) = 6$, there exist 5 elements of the independent subset $\{r_{n-5}, r_{n-4}, \ldots, r_n\}$ of $\Gamma(G)$, which are nonadjacent to 7 in $\Gamma(G)$. Thus there exists an independent subset ρ of $\Gamma(G)$ such that $7 \in \rho$,

$|\rho| > 5 = (7+3)/2$, and, obviously, $\max \rho \neq 7$. Now, by Theorem 3.1, we conclude that S is not isomorphic to any alternating simple group.

Step 2. Let S be isomorphic to a sporadic simple group. The orders of the sporadic simple groups show that all prime divisors of $|S|$ are less than 100. Thus $r_n < 100$ or $r_{n-1} < 100$, since $r_n \big| |S|$ or $r_{n-1} \big| |S|$. Also, by Table 2 in [29], $11 \geq t(S) \geq t(G) - 1 = [(n+1)/2] - 1$, which shows that $11 \leq n \leq 24$. But by an easy calculation, we see that if $11 \leq n \leq 24$, then there exist some primitive prime r_{n-1} and r_n greater than 100, which is a contradiction.

Step 3. Let S be a simple group of Lie type over $\mathrm{GF}(r^\beta)$, where $\beta \geq 1$ and $\rho := \{2, 3, 5, 7, 11, 13, 31, 71\}$.

3.1. Let $r \notin \rho$. Since ρ contains all prime numbers $x \in \pi(G)$ such that $e(x, 5) \leq 6$, it follows that $e(r, 5) \geq 7$. Since $n \geq 11$, Theorem 3.3(a) implies that $11 \leq n \leq 12$. Then $L = L_{11}(5)$ or $L = L_{12}(5)$, and so $\pi(G) \subseteq \rho \cup \{19, 313, 521, 601, 829, 19531, 12207031\}$.

Now, by the orders of simple groups of Lie type over $\mathrm{GF}(r^\beta)$ and Tables 8 and 9 in [29], we conclude that if $t(S) \geq 5$, then $|S|$ is divisible by $r^2 - 1$ and $r^3 - 1$. But if $r \in \{19, 313, 521, 601, 829, 19531, 12207031\}$. Then $\pi((r^2-1)(r^3-1)) \nsubseteq \pi(G)$, which is a contradiction.

3.2. Let $r \in \rho$.

• Let $r \in \rho \setminus \{2, 3, 5\}$. Since $t(S) \geq t(G) - 1 \geq 5$, by Tables 8 and 9 in [29] and the order of S, we conclude that $(r^6 - 1) \big| |S|$. If $r = 7$, then $e(43, 7) = 6$ and $e(43, 5) = 42$. Now, using Theorem 3.3(b) and (c), we get a contradiction. Similarly, since $e(37, 11) = 6$ and $e(37, 5) = 36$; $e(61, 13) = 3$ and $e(61, 5) = 30$; $e(331, 31) = 3$ and $e(331, 5) = 165$; $e(5113, 71) = 3$ and $e(5113, 5) = 1704$, by Theorem 3.3(b) and (c), we get a contradiction.

• Let $r = 3$.

If S is isomorphic to an exceptional simple group of Lie type with $t(S) \geq 5$, except ${}^2G_2(3^{2m+1})$, then the order of S shows that $(3^{12} - 1) \big| |S|$. But $73 \in \pi(3^{12} - 1) \subseteq \pi(S) \subseteq \pi(G)$ and $e(73, 5) = 72$, which, by Theorem 3.3(c), is a contradiction.

Let $S \cong {}^2G_2(3^{2m+1})$, where $m \geq 1$. Then, by Table 9 in [29], $t(S) = 5$ and $t(G) = 6$. Since $6 = t(G) = [(n+1)/2]$, we conclude that $\Gamma(G) = \Gamma(L_{11}(5))$ or $\Gamma(G) = \Gamma(L_{12}(5))$. Thus, by Table 6 in [29], $12207031 = r_{11} \in \rho(2, G) \subseteq \pi(S)$. Also we have $|S| = 3^{3(2m+1)}(3^{3(2m+1)}+1)(3^{2m+1} - 1)$. Since $e(12207031, 3) = 2 \cdot 5 \cdot 71 \cdot 521$, the order of S shows that $5 \big| (2m+1)$, and so $\pi(3^5 + 1) \subseteq \pi(S)$. Since $61 \in \pi(3^5 + 1)$ and $e(61, 5) = 30 \leq n \leq 12$, we get a contradiction.

Let S be isomorphic to a classical simple group over $\mathrm{GF}(3^\beta)$, where $\beta \geq 1$. Since $t(S) \geq t(G) - 1 \geq 5$, the order of S and Table 8 in [29] show that $(3^8 - 1)\big||S|$. Since $41 \in \pi(3^8 - 1) \subseteq \pi(S) \subseteq \pi(G)$ and $n \geq e(41, 5) = 20$, we have $t(S) \geq t(G) - 1 \geq [(n+1)/2] - 1 \geq 9$. Now, since $t(S) \geq 9$, by the order of S and Table 8 in [29], $(3^{10} - 1)\big||S|$. But $61 \in \pi(3^{10} - 1) \subseteq \pi(S) \subseteq \pi(G)$ and $e(61, 3) = 10$ and $e(61, 5) = 30$, which, by Theorem 3.3(b), is a contradiction. Therefore, S is not isomorphic to any simple group of Lie type over $\mathrm{GF}(3^\beta)$.

• Let $r = 2$. If S is isomorphic to an exceptional simple group of Lie type with $t(S) \geq 5$, except ${}^2E_6(2^\beta)$ and ${}^2F_4(2^{2\beta+1})$, then the order of S shows that $(2^9 - 1)\big||S|$. Since $73 \in \pi(2^9 - 1)$ and $e(73, 5) = 72$, by Theorem 3.3(c), we get a contradiction.

Let $S \cong {}^2E_6(2^\beta)$. Then, by Table 9 in [29], $t(S) = 5$, and so $t(G) = 6$, which implies that $11 \leq n \leq 12$. If β is even, then the order of ${}^2E_6(2^\beta)$ shows that $241 \in \pi(2^{24} - 1) \subseteq \pi(S) \subseteq \pi(G)$. Thus $n \geq e(241, 5) = 40$, which is impossible since $11 \leq n \leq 12$.

Therefore, β is odd. Since $11 \leq n \leq 12$, by Table 8 in [29], $\{r_8, r_9, r_{10}\}$ is an independent subset of $\Gamma(G)$. Then by Lemma 2.1, $313 = r_8 \in \pi(S)$ or $829 = r_9 \in \pi(S)$. On the other hand, $|S| = 2^{36\beta}(2^{12\beta} - 1)(2^{9\beta} + 1)(2^{8\beta} - 1)(2^{6\beta} - 1)(2^{5\beta} + 1)(2^{2\beta} - 1)/(3, 2^\beta + 1)$. If $829 \in \pi(S)$, since $e(829, 2) = 2^2 \cdot 3^2 \cdot 23$, then the order of S shows that $23\big|\beta$. Therefore, $47 \in \pi(2^{2\cdot 23} - 1) \subseteq \pi(S) \subseteq \pi(G)$, and so $n \geq e(47, 5) = 46$, which is a contradiction since $11 \leq n \leq 12$. If $313 \in \pi(S)$, since $e(313, 2) = 2^2 \cdot 3 \cdot 13$, then by the order of S, $13\big|\beta$. Therefore, $8191 \in \pi(2^{26} - 1) \subseteq \pi(S)$, and so $n \geq e(8191, 5) = 1365$, which is impossible since $11 \leq n \leq 12$.

Similarly, if $S \cong {}^2F_4(2^{2\beta+1})$, then we can show that $2\beta+1$ is divisible by 13 or 23, and we get a contradiction. Therefore, S is not isomorphic to any exceptional simple group of Lie type.

Now, let S be a classical simple group of Lie type over $\mathrm{GF}(2^\beta)$, where $\beta \geq 1$. Since $t(S) \geq t(G) - 1 \geq 5$, by the order of S and Table 8 in [29], we conclude that, $(2^8 - 1)\big||S|$. Since $17 \in \pi(2^8 - 1) \subseteq \pi(S) \subseteq \pi(G)$ and $e(17, 5) = 16$, we have $t(S) \geq t(G) - 1 = [(n+1)/2] - 1 \geq [(16+1)/2] - 1 = 7$. Now, since $t(S) \geq 8$, by the order of these groups and Table 8 in [29], we have $8191 \in \pi(2^{13} - 1) \subseteq \pi(S)$ or $43 \in \pi(2^{14} - 1) \subseteq \pi(S)$. Then $e(8191, 2) = 13$ and $e(8191, 5) = 1365$, $e(43, 2) = 14$, and $e(43, 5) = 42$, which, by Theorem 3.3(b), is a contradiction. Therefore, S is not isomorphic to any simple group of Lie type over $\mathrm{GF}(2^\beta)$.

By the classification of finite simple groups, we see that S can only be isomorphic to a simple group of Lie type over $\mathrm{GF}(5^\beta)$. Therefore, by Theorem 3.2, we conclude that $S \cong L_n(5)$.

Step 4. Finally, in this step, we will prove that K is a $\{2,3,5\}$-group.

Up to now, we show that if G is a finite group such that $\Gamma(G) = \Gamma(L_n(5))$, where $n \geq 11$. Then there exists a normal subgroup of G, say H, such that $L_n(5) \cong H/K \leq \overline{G} := G/K \leq \mathrm{Aut}(L_n(5))$, where K is the maximal normal solvable subgroup of G. First, we claim that either $K = 1$ or $C_G(K) \leq K$.

Since H/K is a simple group, $H/K \cap C_G(K)K/K$ is equal to a trivial group or is equal to H/K. If $H/K \cap C_G(K)K/K = H/K$, then $H \leq C_G(K)K$, and so for all $r \in \pi(H) \setminus \pi(K)$, we have $r \in \pi(C_G(K))$. Thus if $r \in \pi(H/K) = \pi(L_n(5))$ and $r' \in \pi(K)$, then $r' \sim r$ in $\Gamma(G)$. But by Table 4 in [29] and Lemma 2.6, if r' is any prime number in $\pi(L_n(5))$, then $r' \nsim r_n$ or $r' \nsim r_{n-1}$ in $\Gamma(L_n(5))$, which implies that $K = 1$.

Therefore, $H/K \cap C_G(K)K/K$ is a trivial group. If $C_G(K)K/K$ is not a trivial group, then the product $(H/K)(C_G(K)K/K)$ is a direct product in G/K. Thus if $r \in \pi(H/K) = \pi(L_n(5)) = \pi(G)$ and $r' \in \pi(C_G(K)K/K)$, then $r' \sim r$ in $\Gamma(G)$. Again, we get that $r' \sim r_n$ or $r' \sim r_{n-1}$. Thus similar to the previous paragraph, we get that $C_G(K)K/K$ is a trivial group. Therefore, by the previous argument, we deduce that either $K = 1$ or $C_G(K) \leq K$, which satisfies the above claim.

Let $t \in \pi(K)$. In this part, we use some familiar results about the solvable groups (for example see [26]). We know that we may assume that $O^t(K) \neq K$ since K is solvable. Then $K/O^t(K)$ is a nontrivial t-group. Put $\widehat{K} = K/O^t(K)$ and $\widehat{G} = G/O^t(K)$ since $O^t(K)$ is a characteristic subgroup of K and $K \trianglelefteq G$. If the Frattini subgroup of $\widehat{K}$ is denoted by $\Phi(\widehat{K})$, then $\widehat{K}/\Phi(\widehat{K})$ is an elementary abelian t-group, and we have $G/K \cong \widehat{G}/\widehat{K} \cong (\widehat{G}/\Phi(\widehat{K}))/(\widehat{K}/\Phi(\widehat{K}))$. Therefore, without loss of generality, we can assume that K is an elementary abelian t-group.

Now, we show that K is a $\{2,3,5\}$-group. For proving this result, let $t \in \pi(K)$. We know that $L_{n-1}(5)$ is isomorphic to a subgroup of $L_n(5)$. Now, by Lemma 2.3, $L_n(5)$ consists of some Frobenius subgroups of the form $5^{n-1} : (5^{n-1}-1)/(n,4)$ and $5^{n-2} : (5^{n-2}-1)/(n-1,4)$. Since $C_G(K)K/K$ is a trivial group, if $t \neq 5$, then Lemma 2.2 implies that $t \sim r_{n-1}$ or $t \sim r_{n-2}$ in $\Gamma(G)$. But Lemma 2.6, implies that $e(t,5) \leq 2$, and so $t \big| (5^2-1)$. Therefore, K is a $\{2,3,5\}$-group. $\square$

Corollary 4.2. *If G is a finite group such that $\Gamma(G) = \Gamma(L_n(5))$ (or $\pi_e(G) = \pi_e(L_n(5))$) and $|G| = |L_n(5)|$, where $n \geq 11$, then $G \cong L_n(5)$.*

Proof. By Theorem 4.1, $L_n(5) \leq G/K \leq \mathrm{Aut}(L_n(5))$, where K is a normal $\{2, 3, 5\}$-subgroup of G. Since $|G| = |L_n(5)|$, we have $|K| = 1$. Therefore, $G \cong L_n(5)$. □

There is a conjecture due to W. Shi and H. Bi [25], which states that if G is a finite group and M a finite simple group, then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$. We mention that this conjecture has been proved. However, Corollary 4.3 is a new proof of the conjecture of Shi and Bi for the simple group $L_n(5)$, where $n \geq 11$.

Remark 4.3. In [8], it has been proved that if $\pi_e(G) = \pi_e(L_4(5))$. Then $G/N \cong L_4(5)$. Now, using Theorem 1 in [31], it follows that $N = 1$. Therefore, $L_4(5)$ is recognizable by element orders.

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Ali Mahmoudifar
Department of Mathematics, Tehran North Branch, Islamic Azad University, Tehran, Iran.

Email: alimahmoudifar@gmail.com

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ON COMPOSITION FACTORS OF A GROUP WITH THE SAME PRIME GRAPH AS $\mathbf{L_n(5)}$

A. MAHMOUDIFAR

پیرامون سری‌های ترکیبی گروهی با گراف اول یکسان با $\mathbf{L_n(5)}$

علی محمودی فر
دانشکده علوم دانشگاه آزاد اسلامی واحد تهران شمال

گراف اول گروه متناهی G را با نماد $\Gamma(G)$ نمایش می‌دهند. گروه ساده متناهی G را شبه شناسایی پذیر توسط گراف اول می‌نامیم هرگاه هر گروه متناهی H که $\Gamma(H) = \Gamma(G)$ دارای یک سری ترکیبی یکریخت با G باشد. تا کنون ثابت شده است که برخی از گروه‌های ساده خطی متناهی توسط گراف اول خود شبه‌شناسایی‌پذیر می‌باشند، برای نمونه گروه‌های خطی $L_n(۲)$ و $L_n(۳)$ شبه‌شناسایی‌پذیر توسط گراف خطی خود می‌باشند. در این مقاله به بررسی شبه شناسایی‌پذیر بودن گروه‌های ساده خطی $L_n(۵)$ می‌پردازیم.

کلمات کلیدی: گروه خطی خاص تصویری، گراف اول، مرتبه عنصر.