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# STRONGLY DUO AND CO-MULTIPLICATION MODULES

S. SAFAEEYAN*

Abstract. Let $R$ be a commutative ring. An $R$-module $M$ is called co-multiplication, provided that for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = (0 :_M I)$. In this paper, we show that co-multiplication modules are a generalization of strongly duo modules. Uniserial modules of finite length, and hence, valuation Artinian rings are some distinguished classes of co-multiplication modules. In addition, if $R$ is a Noetherian quasi-injective ring, then $R$ is strongly duo if and only if $R$ is co-multiplication. We also show that J-semisimple strongly duo rings are precisely semisimple rings. Moreover, if $R$ is a perfect ring, then uniserial $R$-modules are co-multiplication of finite length modules. Finally, we show that Abelian co-multiplication groups are all reduced, and co-multiplication $\mathbb{Z}$-modules (Abelian groups) are characterized as well.

## 1. Introduction

Throughout this paper, all rings are commutative with identity, and all modules are unitary. Let $M$ be an $R$-module. For each subset $X$ of $M$, set $\operatorname{ann}_R(X) = \{r \in R \mid Xr = 0\}$. The submodule $N$ of $M$ is called fully invariant, provided that for each $f \in \operatorname{End}(M_R)$, $f(N) \subseteq N$. A ring $R$ is said to be right duo if right ideals of $R$ are two-sided. It is easy to check that $R$ is a right duo ring if and only if $\operatorname{Rej}(R, R/I) = I$ for every right ideal $I$ of $R$. In [14], the authors have studied the dual of

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∗Corresponding author.

this concept, i.e. $\mathrm{Tr}(I, R) = I$ for all right ideal $I$ of $R$. Generalizing to modules, we call an $R$-module $M$ strongly duo if $\mathrm{Tr}(N, M) = N$ for all submodule, $N$ of $M$. Clearly, every strongly duo module $M_R$ is a duo module (i.e. all submodules of $M_R$ are fully invariant). Therefore, the concept of right strongly duo rings is the dual of duo rings and, simultaneously, its generalization. In [14, Theorem 2.1], it has been shown that a right $R$-module $M$ is right strongly duo if and only if for each $m, n \in M$, $\mathrm{ann}_r(m) \subseteq \mathrm{ann}_r(n)$ implies that $n \in mR$. This observation persuade us to investigate right $R$-modules $M$ such that for every two submodules $N$ and $K$ of $M$, $\mathrm{ann}_r(N) \subseteq \mathrm{ann}_r(K)$ implies that $K \subseteq N$. We show that over a commutative ring $R$, $R$-modules with aforementioned property are precisely co-multiplication $R$-modules (Theorem 2.3). Co-multiplication modules and some of their properties have been investigated in [2], [3], and [4]. An $R$-module $M$ is called multiplication, provided that for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = MI$; see [6]. Duo modules and multiplication modules have been investigated by several authors. Some of their recent works (not all) are cited in the references. Multiplication modules are duo but the converse is not true. In [6], [7], [8], and [9], conditions have been found, under which duo modules are multiplication modules. Inasmuch as co-multiplication modules are strongly duo (Corollary 2.4(1)). In this paper, we will find some conditions, under which strongly duo modules are co-multiplication. In Section 2, we investigate co-multiplication modules, and show that uniserial modules of finite length, and hence, uniserial Artinian rings are classes of co-multiplication modules. Theorem 2.10 shows that J-semisimple co-multiplication rings are precisely semisimple rings. In Section 3, some important properties of co-multiplication modules are studied.

Recall that an $R$-module $M$ is said to be *quasi-injective* (*PQ-injective*) if, for any (cyclic) submodule $N$ of $M$, any $f \in \mathrm{Hom}_R(N, M)$ can be extended to an endomorphism of $M$. A nonzero $R$-module $M$ is called *prime*, provided that for each nonzero submodule $N$ of $M$, $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ (see[16, Lemma 3.54]). Let $M$ be an $R$-module. An ideal $P$ of $R$ is called an *associated prime* of $M$ if there exists a prime submodule $N \subseteq M$ such that $P = \mathrm{ann}(N)$. The set of all associated primes of $M$ is denoted by $\mathrm{Ass}(M)$.

Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [1], [5], [15], and [16].

## 2. Main results

Let $M$ be an $R$-module. For each ideal $I$ of $R$ and each submodule $N$ of $M$, define $(N :_M I) = \{m \in M|\ ma \in N\ for\ each\ a \in I\}$ and $\mathrm{ann}_M(I) = \{m \in M|\ ma = 0\ for\ each\ a \in I\}$. It is clear that $(N :_M I)$ and $\mathrm{ann}_M(I)$ are two submodules of $M$.

**Definition 2.1.** An $R$-module $M$ is called co-multiplication, provided that for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = (0 :_M I) = \mathrm{ann}_M(I)$.

It is clear that $M$ is a co-multiplication $R$-module if and only if for each submodule $N$ of $M$, $N = (0 :_M \mathrm{ann}_R(N))$.

The following Lemma is well-known.

**Lemma 2.2.** *Let $M$ be an $R$-module and $N$ be a submodule of $M$. Then*

$$\mathrm{ann}_R(\mathrm{ann}_M(\mathrm{ann}_R(N))) = \mathrm{ann}_R(N).$$

The following result shows that co-multiplication modules are a generalization of strongly duo modules.

**Theorem 2.3.** *Let $M$ be an $R$-module. The following assertions are equivalent.*

1. *$M$ is a co-multiplication module;*
2. *For any two submodules $N$ and $K$ of $M$, $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(K)$ implies that $K \subseteq N$;*
3. *For each submodule $N$ of $M$, and each $m \in M$, $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(m)$ implies that $m \in N$.*

*Proof.* $(1 \Rightarrow 2)$. Let $N$ and $K$ be two submodules of $M$ such that $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(K)$. Then $(0 :_M \mathrm{ann}_R(K)) \subseteq (0 :_M \mathrm{ann}_R(N))$. Since $M$ is a co-multiplication $R$-module, $K = (0 :_M \mathrm{ann}_R(K)) \subseteq (0 :_M \mathrm{ann}_R(N)) = N$.

$(2 \Rightarrow 1)$. Assume that $N$ is a submodule of $M$. By Lemma 2.2,

$$\mathrm{ann}_R(\mathrm{ann}_M(\mathrm{ann}_R(N))) = \mathrm{ann}_R(N).$$

Therefore, $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(\mathrm{ann}_M(\mathrm{ann}_R(N)))$, and by assumption,

$$\mathrm{ann}_M(\mathrm{ann}_R(N)) = (0 :_M \mathrm{ann}_R(N)) \subseteq N.$$

Hence, the equality holds.

$(1 \Rightarrow 3)$. Since $R$ is a commutative ring, for each $m \in M$, $\mathrm{ann}_R(mR) = \mathrm{ann}_R(m)$. Hence, by (1), the verification is immediate.

$(3 \Rightarrow 2)$. Assume that $N$ and $K$ are submodules of $M$ such that $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(K)$. For each $x \in K$, $\mathrm{ann}_R(K) \subseteq \mathrm{ann}_R(x)$. Thus by hypotheses, for each $x \in K$, we have $x \in N$. □

**Corollary 2.4.** *Let $M$ be a co-multiplication $R$-module. The following statements hold.*

1. *$M$ is a strongly duo $R$-module. In particular, $M$ is a PQ-injective duo module;*
2. *$M$ is a prime $R$-module if and only if $M$ is a simple $R$-module;*
3. *$P \in \mathrm{Ass}(M)$ if and only if $P = \mathrm{ann}_R(m)$ is a maximal ideal of $R$ for some $m \in M$.*

*Proof.* (1). Let $m, m' \in M$ such that $\mathrm{ann}_R(m) \subseteq \mathrm{ann}_R(m')$. By Theorem 2.3, $m' \in mR$. Therefore, by [14, Theorem 2.1], $M_R$ is a strongly duo module. Now, by [14, Theorem 3.5] and the fact that our rings are commutative, $M$ is a PQ-injective duo module.
(2). The verification is immediate.
(3). Let $P \in \mathrm{Ass}(M)$. There exists a prime submodule $N$ of $M$ such that $P = \mathrm{ann}_R(N)$. Since $N$ is a prime co-multiplication module, by part (2), $N$ is a simple $R$-module. Hence, $P = \mathrm{ann}_R(N)$ is a maximal ideal of $R$. Conversely, assume that $m \in M$ such that $P = \mathrm{ann}_R(m)$ is a maximal ideal of $R$. Thus $mR$ is simple, and hence, a prime $R$-module. □

**Corollary 2.5.** *Let $R$ be a ring. The following hold.*

1. *If $R$ is Noetherian, then each co-multiplication $R$-module is Artinian.*
2. *If $R$ is an Artinian ring, then each co-multiplication $R$-module is of finite length.*

*Proof.* (1). Let $M$ be a co-multiplication $R$-module and

$$N_1 \supseteq N_2 \supseteq \cdots$$

be a descending chain of submodules of $M$. Then

$$\mathrm{ann}_R(N_1) \subseteq \mathrm{ann}_R(N_2) \subseteq \cdots$$

is an ascending chain of ideals of $R$. Since $R$ is a Noetherian ring, there exists $k \in \mathbb{N}$ such that, for each $n \geq k$, $\mathrm{ann}_R(N_k) = \mathrm{ann}_R(N_n)$. Since $M$ is a co-multiplication $R$-module, by Theorem 2.3, for each $n \geq k$, $N_k = N_n$.
(2). Let $M$ be a co-multiplication $R$-module. Since Artinian rings are Noetherian, $R$ is Noetherian. Now, by part (1), any co-multiplication $R$-module is Artinian, and hence, $M_R$ is Artinian. Now, by Hopkins-Levitzki Theorem [16, Theorem 4.15], we have over Artinian rings, Noetherian modules are precisely Artinian modules. □

In the following, our main concern is the class of conditions under which strongly duo modules are co-multiplication.

**Lemma 2.6.** *Let $M$ be a strongly duo $R$-module. Then $S = \mathrm{End}(M_R)$ is a commutative ring.*

*Proof.* Assume that $f \in \mathrm{End}(M_R)$ and $m \in M$. Since $\mathrm{ann}_R(m) \subseteq \mathrm{ann}_R(f(m))$, $f(m) \in mR$, and hence, $f(m) = mr$ for some $r \in R$. Then for every $m \in M$ and $g, f \in \mathrm{End}(M_R)$, there exist $r, s \in R$ such that $f(m) = mr$ and $g(m) = ms$. Therefore, $fg(m) = mrs = msr = gf(m)$. □

**Theorem 2.7.** *Let $M$ be a quasi-injective strongly duo $R$-module with endomorphism ring $S$. Then for every finitely generated $S$-submodule $K$ of $M$, we have $K = \mathrm{ann}_M\mathrm{ann}_S(K)$.*

*Proof.* First, we show that for each $x \in M$, $\mathrm{ann}_M\mathrm{ann}_S(Sx) = Sx$. It is clear that $Sx \subseteq \mathrm{ann}_M\mathrm{ann}_S(Sx)$. Assume that $y \in \mathrm{ann}_M\mathrm{ann}_S(Sx)$. Define the map $f : xR \longrightarrow M$ by $f(xr) = yr$. If for some $r \in R$, $xr = 0$, then $g_r \in \mathrm{ann}_S(Sx)$, and hence, $g_r(y) = yr = 0$, which is implies that $f$ is an $R$-homomorphism. By [14, Theorem 3.5], $M$ is a PQ-injective module, and hence, there exists $\overline{f} \in \mathrm{End}(M_R)$ such that $\overline{f}|_{xR} = f$. Therefore, $y = f(x) = \overline{f}(x) \in Sx$. Now, suppose that $K = Sx_1 + Sx_2$. Obviously,

$$Sx_1 + Sx_2 \subseteq \mathrm{ann}_M\mathrm{ann}_S(Sx_1 + Sx_2) = \mathrm{ann}_M(\mathrm{ann}_S(Sx_1) \cap \mathrm{ann}_S(Sx_2)).$$

Suppose that $y \in \mathrm{ann}_M\mathrm{ann}_S(Sx_1 + Sx_2)$. Define the map

$$\phi : \mathrm{ann}_S(Sx_2)x_1 \to M$$

by $\phi(f(x_1)) = f(y)$ for each $f \in \mathrm{ann}_S(Sx_2)$. For each $f \in \mathrm{ann}_S(Sx_2)$ and $r \in R$, we have

$$f(x_1)r = f(x_1r) = fg_r(x_1) = g_rf(x_1) \in \mathrm{ann}_S(Sx_2)(x_1).$$

Therefore, $\mathrm{ann}_S(Sx_2)x_1$ is an $R$-submodule of $M$. For each $r \in R$ and $f \in \mathrm{ann}_S(Sx_2)$, $\phi(f(x_1)r) = \phi(g_rf(x_1)) = g_rf(y) = f(y)r = \phi(f(x_1))r$. Hence, $\phi$ is an $R$-homomorphism. Since $M$ is a quasi-injective module, there exists $\overline{\phi} \in \mathrm{End}(M_R)$ such that $\overline{\phi}|_{\mathrm{ann}_S(Sx_2)x_1} = \phi$. By Lemma 2.6, $S$ is a commutative ring, and hence, $\mathrm{ann}_S(Sx_2)(\overline{\phi}(x_1) - y) = 0$. Then $\overline{\phi}(x_1) - y \in \mathrm{ann}_M\mathrm{ann}_S(Sx_2) = Sx_2$. Therefore, for some $g \in S$, $\overline{\phi}(x_1) - y = g(x_2)$, and hence, $y \in Sx_1 + Sx_2$. □

**Corollary 2.8.** *Let $M$ be a quasi-injective Noetherian strongly duo $R$-module with endomorphism ring $S$. Then $M$ is a co-multiplication $S$-module.*

*Proof.* First, we show that every $S$-submodule of $M$ is finitely generated. Let $K$ be a $S$-submodule of $M$. For each $k \in K$ and $r \in R$, we have $kr = g_r(k) \in SK = K$. Thus $K$ is an $R$-submodule of $M$, and hence, $K$ is finitely generated as an $R$-submodule. Assume that $K = x_1R+x_2R+\cdots+x_nR$. It is clear that $K = Sx_1+Sx_2+\cdots+Sx_n$. Therefore, every $S$-submodule $K$ of $M$ is finitely generated, and hence, by Theorem 2.7, $K = \text{ann}_M\text{ann}_S(K) = (0 :_M \text{ann}_S(K))$. □

**Corollary 2.9.** *Let $R$ be a quasi-injective Noetherian ring. Then $R$ is a co-multiplication ring if and only if $R$ is a strongly duo ring.*

*Proof.* Since $R \cong \text{End}(R_R)$, by Corollary 2.4(1)and Corollary 2.8, the proof is clear. □

Let $R$ be a ring. The Jacobson radical of $R$ is denoted by J(R). A ring $R$ is said to be J-semisimple if, J(R) = 0. The next result shows that for J-semisimple strongly duo rings are co-multiplication.

**Theorem 2.10.** *Let $R$ be a ring, the following statements are equivalent.*

(1) *$R$ is a* J*-semisimple co-multiplication ring;*
(2) *$R$ is a* J*-semisimple strongly duo ring;*
(3) *$R$ is a semisimple ring.*

*Proof.* $(1 \Rightarrow 2)$. It is clear by Corollary 2.4(1).
$(2 \Rightarrow 3)$. Assume that $\text{J} = \text{J}(R)$. By assumption, $T = \bigcap_{k=1}^{\infty} \text{J}^k = 0$. Therefore, by [11, Theorem 5.4], $R$ is a Notherian ring. Then $R$ is a Notherian strongly duo ring, and hence, by [14, Proposition 4.6], $R$ is an Artinian ring. Now, by [15, Theorem 4.14], $R$ is a semisimple ring.
$(3 \Rightarrow 1)$. Let $I$ be an ideal of $R$. There exists an idempotent $e \in R$ such that $I = eR$. Therefore, we have $\text{ann}_R(eR) = (1-e)R$ and $\text{ann}_R((1-e)R) = eR$. Hence,

$$\text{ann}_R(\text{ann}_R(I)) = \text{ann}_R((1-e)R) = eR = I,$$

as desired. □

An $R$-module $M$ is called uniserial if its submodules are linearly, ordered by inclusion. If $R_R$ is uniserial, we call $R$ right uniserial. Note that right uniserial rings are, in particular, local rings. Commutative uniserial rings are also known as valuation rings.

**Proposition 2.11.** *Uniserial modules of finite length are co- multiplication. In particular, uniserial Artinian rings are co-multiplication.*

*Proof.* Let $M$ be a uniserial $R$-module of finite length and $N$ a submodule of $M$. Since $M$ is a Noetherian $R$-module, $N$ is finitely generated, and hence, $N = n_1R + n_2R + \cdots + n_kR$, where $n_i \in M$ and $k \in \mathbb{N}$. Since $M$ is a uniserial $R$-module, $N = n_iR$ for some $1 \leq i \leq k$. For each $m \in M$, $\text{ann}_R(N) \subseteq \text{ann}_R(m)$ implies that

$\text{ann}_R(N) = \text{ann}_R(n_i) \subseteq \text{ann}_R(m)$. Since $M$ is a uniserial Artinian module, $M$ is a strongly duo module, [14, Proposition 2.5]. Hence, $\text{ann}_R(n_i) \subseteq \text{ann}_R(m)$ implies that $m \in n_iR = N$. Thus, by Theorem 2.2, $M$ is a co-multiplication module. Let $R$ be a uniserial Artinian ring. By [15, Theorem 4.15], $R_R$ is a uniserial module of finite length. Thus, $R_R$ is co-multiplication. □

**Lemma 2.12.** *Every uniserial module with ascending chain condition on its cyclic submodules is Noetherian.*

*Proof.* Let $M$ be an $R$-module with ascending chain condition on its cyclic submodules. We show that any submodule of $M$ is cyclic. Let $N$ be a non-cyclic submodule of $M$ and $0 \neq n_1 \in N$. There exists $0 \neq n_2 \in N \setminus n_1R$. Since $M_R$ is a uniserial module and $n_2R \nsubseteq n_1R$, $n_1R \subset n_2R$. Again, there exists $0 \neq n_3 \in N \setminus n_2R$. Since $M_R$ is a uniserial module and $n_3R \nsubseteq n_2R$, $n_2R \subset n_3R$. Repeat this argument to obtain the proper ascending chain $n_1R \subset n_2R \subset \ldots \subset n_kR \subseteq \cdots$ of cyclic submodules. It is a contradiction. □

*Remark* 2.13. It is a surprising result of D. Jonah (see [13]) that a ring is left perfect if and only if every right $R$-module has ascending chain condition on its cyclic submodules. As a direct consequence of D. Jonah's Theorem, we have, if $R$ is a left perfect ring and $M \in \text{Mod}-R$, the following are equivalent:

(1) $M_R$ is a Noetherian module;
(2) $M_R$ is an Artinian module;
(3) $M_R$ has a finite length.

In regard to Lemma 2.12 and D. Jonah's Theorem, we have the following Theorem.

**Theorem 2.14.** *Let $R$ be a perfect ring, and $M$ be a uniserial $R$-module. The following hold:*

(1) *$M$ is an $R$-module of finite length.*
(2) *$M$ is a co-multiplication $R$-module.*

*Proof.* (1). Since $R$ is a perfect ring, by D. Jonah's Theorem, $M_R$ satisfies in ascending chain condition on its cyclic submodules. By Lemma 2.12, $M_R$ is Noetherian, and hence, has a finite length.
(2). By part (1), $M_R$ is a uniserial module of finite length. Therefore, by Proposition 2.11, $M$ is a co-multiplication $R$-module. □

## 3. Some properties of co-multiplication modules

In this section, some of the interesting properties of co-multiplication modules are investigated. First, we show that in a co-multiplication $R$-module, every two isomorphic submodules are equal.

**Proposition 3.1.** *Let $M$ be a co-multiplication $R$-module, $K$ and $L$ be submodules of $M$, and $f \in \mathrm{Hom}_R(K, L)$. The following statements hold:*

1. *If $f$ is a monomorphism, then $K \subseteq L$.*
2. *If $f$ is an epimorphism, then $L \subseteq K$.*
3. *If $f$ is an isomorphism, then $K = L$.*

*Proof.* (1). Assume that $f$ is a monomorphism. For each $x \in K$, $\mathrm{ann}_R(x) = \mathrm{ann}_R(f(x))$. Thus by Corollary 2.4(1), $x \in f(x)R \subseteq \mathrm{Im}\, f$. Hence, $K \subseteq L$.
(2). Let $f$ be an epimorphism. For each $y \in L$, there exists an element $x \in K$ such that $f(x) = y$. On the other hand, $\mathrm{ann}_R(x) \subseteq \mathrm{ann}_R(y)$. Therefore, by Corollary 2.4(1), $y = f(x) \in xR \subseteq K$.
(3) By part (1) and (2), it is clear. □

An $R$-module $M$ is called *compressible*, provided that for each nonzero submodule $N$ of $M$, there exists a monomorphism $f \in \mathrm{Hom}_R(M, N)$. Now, we introduce some properties of co-multiplication modules.

**Corollary 3.2.** *Let $R$ be a ring.*

1. *Co-multiplication $R$-modules are co-hopfian.*
2. *Compressible co-multiplication modules are precisely simple modules.*
3. *Let $M$ be a co-multiplication and $X$ be an arbitrary $R$-module. If there exists a monomorphism $f \in \mathrm{Hom}_R(X \oplus X, M)$, then $X = 0$.*
4. *For each nonzero $R$-module $M$, $M \oplus M$ is not a co-multiplication $R$-module.*

*Proof.* (1). Let $f \in \mathrm{End}(M_R)$ be a monomorphism. Therefore, $M \cong Im\, f$. By Proposition 3.1, $M = \mathrm{Im}\, f$, and hence, $f$ is an isomorphism.
(2). Simple modules are compressible and co-multiplication. Conversely, assume that $M$ is a compressible co-multiplication module, and $N$ be a nonzero submodule of $M$. There exists a monomorphism $f \in \mathrm{Hom}_R(M, N)$. By Proposition 3.1, $M \subseteq N$, and hence, $M = N$.
(3). It is clear that $f(X \oplus 0) \cong f(0 \oplus X)$. Then by Proposition 3.1, $f(X \oplus 0) = f(0 \oplus X)$. Since $f$ is a monomorphism, $X \oplus 0 = 0 \oplus X$, and hence, $X = 0$.

(4). It is clear by part 3 because $\mathrm{Hom}_R(M \oplus M, M \oplus M)$ contains identity map. □

**Proposition 3.3.** *Let $R$ be a ring. The following hold:*

1. *If $M$ is a nonzero free co-multiplication $R$-module, then $M \cong R$.*
2. *Let $M$ be a co-multiplication $R$-module, $X$ be a faithful $R$-module, and $f$ be a monomorphism in $\mathrm{Hom}_R(X, M)$. Then $f$ is an isomorphism.*

*Proof.* (1). Since $M$ is a free $R$-module, $M \cong \sum_X R$ for some index set $X$. If $|X| \geq 2$, then there exists a monomorphism $f \in Hom_R(R \oplus R, M)$. Since $M_R$ is a co-multiplication module, by Corollary 3.2(3), $R = 0$. It is a contradiction. Thus $|X| = 1$, and hence, $M \cong R$.
(2). We know that $\mathrm{Im}\ f$ is a submodule of $M$. Since $M$ is a co-multiplication $R$-module, there exists an ideal $I$ of $R$ such that $\mathrm{Im}\ f = (0 :_M I)$. Therefore, $f(XI) = f(X)I = 0$. Since $f$ is a monomorphism, $XI = 0$, and hence, $I = 0$ since $X$ is a faithful $R$-module. Thus $\mathrm{Im}\ f = (0 :_M 0) = M$. □

**Corollary 3.4.** *Let $M$ be a co-multiplication $R$-module. The following hold:*

1. *If $M$ has a non-zero free submodule, then $M \cong R$.*
2. *If $M$ is non-zero, then $R \oplus M$ is not a co-multiplication $R$-module.*

*Proof.* (1). Let $N$ be a non-zero free $R$-submodule of $M$. There exists a monomorphism $f \in \mathrm{Hom}_R(R, N) \subseteq \mathrm{Hom}_R(R, M)$. Since $R$ is a faithful $R$-module, by Proposition 3.3(2), $M \cong R$.
(2). To the contrary, assume that $R \oplus M$ is a co-multiplication $R$-module. By part (1), $R \oplus M \cong R \cong R \oplus 0$. By Proposition 3.1, since $R \oplus M$ is a co-multiplication module, $R \oplus M = R \oplus 0$. Hence, $M = 0$, a contradiction. □

Corollary 3.2 implies that semisimple $R$-modules are not generally co-multiplication. In the following, we characterize an important class of semisimple modules which are co-multiplications. Let $M$ be a semisimple $R$-module and $S$ be a simple submodule of $M$. The sum of all simple submodules of $M$ that are isomorphic to $S$ is called a *homogeneous component* of $M$.

**Proposition 3.5.** *Let $M$ be a finitely generated semisimple $R$-module. Then $M$ is a co-multiplication module if and only if homogeneous components of $M$ are simple.*

*Proof.* If $M$ is a co-multiplication $R$-module, then, by Proposition 3.1(3), isomorphic submodules are equal. Hence, homogeneous components of $M$ are simple. Conversely, assume that $\{S_i\}_{i=1}^n$ is a family of simple submodules of $M$ such that $M = \oplus_{i=1}^n S_i$, and $K, N$ are submodules of $M$ such that $\text{ann}_R(N) \subseteq \text{ann}_R(K)$. There exist finite subsets $F_1$ and $F_2$ of $\{1, 2, \cdots, n\}$ such that $N = \oplus_{j \in F_1} S_j$ and $K = \oplus_{t \in F_2} S_t$. Therefore, $\text{ann}_R(N) = \bigcap_{j \in F_1} \text{ann}_R(S_j)$ and $\text{ann}_R(K) = \bigcap_{t \in F_2} \text{ann}_R(S_t)$. Thus $\text{ann}_R(N) \subseteq \text{ann}_R(K)$ implies that for each $t \in F_2$, $\bigcap_{j \in F_1} \text{ann}_R(S_j) \subseteq \text{ann}_R(S_t)$. Since $\text{ann}_R(S_t)$ is a maximal ideal of $R$, there exists a $j_t \in F_1$ such that $\text{ann}_R(S_{j_t}) \subseteq \text{ann}_R(S_t)$. Inasmuch as $\text{ann}_R(S_{j_t})$ is a maximal ideal of $R$. Then $\text{ann}_R(S_{j_t}) = \text{ann}_R(S_t)$, and hence, $S_{j_t} \cong S_t$. Since homogeneous components of $M$ are simple, $S_{j_t} = S_t$. Hence, $K = \oplus_{t \in F_2} S_t = \oplus_{t \in F_2} S_{j_t} \subseteq \oplus_{j \in F_1} S_j = N$. Now, by Theorem 2.3, the proof is complete. □

It is suitable to answer this question that " When is an Abelian group co-multiplication as a $\mathbb{Z}$-module?" Proposition 3.3 and Corollary 3.4, answer this question in some special cases. In the following, we study this subject.

An Abelian group $D$ is called *divisible*, provided that for each $x \in D$ and each positive number $n$, there exists $y \in D$ such that $ny = x$ (i.e. $nD = D$). An Abelian group is called *reduced* if it has no non-zero divisible subgroup. By [10, Theorem 21.3], every Abelian group $M$ is the direct sum of divisible group $D$ and reduced group $C$, $M = D \oplus C$.

**Lemma 3.6.** *Co-multiplication Abelian groups are reduced.*

*Proof.* Let $M$ be an Abelian group. Then $M = D \oplus C$, where $D$ is a divisible, and $C$ is a reduced subgroups of $M$. Since $M$ is a co-multiplication $\mathbb{Z}$-module, there exists a positive integer $n$ such that $D = (0 :_M n\mathbb{Z})$. Therefore, $(n\mathbb{Z})D = nD = 0$. On the other hand, $nD = D$ since $D$ is a divisible group. Hence, $D = 0$. □

**Proposition 3.7.** (1) *For any positive integer $n \geq 2$, $\mathbb{Z}_n$ is a co-multiplication $\mathbb{Z}$-module.*

(2) *If $M$ is a finitely generated Abelian group with* rank $M \geq 1$, *then $M$ is not a co-multiplication $\mathbb{Z}$-module.*

(3) *If $M$ is a finite non-cyclic Abelian group, then $M$ is not a co-multiplication $\mathbb{Z}$-module.*

*Proof.* (1). Assume that $K$ be a $\mathbb{Z}$-submodule of $\mathbb{Z}_n$. There exist $k, a \in \mathbb{Z}$ such that $n = ka$ and $K =< \bar{k} >= \bar{k}\mathbb{Z}$. It is clear that $\text{ann}_{\mathbb{Z}}(K) = a\mathbb{Z}$. Hence,

$$(0 :_{\mathbb{Z}_n} \text{ann}_{\mathbb{Z}}(K)) = (0 :_{\mathbb{Z}_n} a\mathbb{Z}) = \bar{k}\mathbb{Z} = K.$$

(2). By [10, Theorem 15.5], there exist integers $n_1, n_2, \cdots, n_k$ such that

$$M \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \mathbb{Z}_{n_k} \oplus \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k-times}.$$

Since $k \geq 1$, by Corollary 3.4, $M$ can not be a co-multiplication $\mathbb{Z}$-module since $\mathbb{Z}$ is not clearly a co-multiplication $\mathbb{Z}$-module.
(3). By [12, Theorem 2.6], there exist integers $n_1, n_2, \cdots, n_t$ such that $M \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus Z_{n_t}$, where $t \geq 2$ and $n_1|n_2|\cdots|n_t$. If $M$ is a co-multiplication $\mathbb{Z}$-module, then for $N = (0) \oplus (0) \oplus \cdots \oplus Z_{n_t}$, we have $\mathrm{ann}_{\mathbb{Z}}(N) = n_t\mathbb{Z}$ and $N = (0 :_M n_t\mathbb{Z})$. On the other hand, since $n_1|n_2|\cdots|n_t$,

$$(0 :_M n_t\mathbb{Z}) = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus Z_{n_t},$$

which is a contradiction. □

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**Saeed Safaeeyan**
Department of Mathematics, University of Yasouj , P.O.Box 75914, Yasouj, Iran.
Email: safaeeyan@yu.ac.ir

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# STRONGLY DUO AND CO-MULTIPLICATION MODULES

S. SAFAEEYAN

# مدول‌های قویا دئو و هم-ضربی

سعید صفاییان
دانشگاه یاسوج

فرض کنید $R$ یک حلقه تعویض‌پذیر باشد. $R$-مدول $M$ را هم-ضربی نامند، هرگاه برای هر زیر مدول $N$ از $M$، ایدآل $I$ از $R$ موجود باشد به طوری‌که $N = (0 :_M I)$. در این مقاله نشان خواهیم داد که، مدول‌های هم-ضربی تعمیمی از مدول‌های قویا دئو هستند. مدول‌های تک‌رشته‌ای که دارای طول متناهی هستند و حلقه‌های ارزه آرتینی مثال‌های مهمی از مدول‌های هم-ضربی هستند. نشان می‌دهیم، اگر $R$ یک حلقه آرتینی شبه‌تزریقی باشد، آنگاه $R$ یک حلقه قویا دئو است اگر و تنها اگر $R$ یک حلقه هم-ضربی باشد. همچنین نشان می‌دهیم، حلقه‌های $J$-نیم‌ساده قویا دئو دقیقا حلقه‌های نیم‌ساده هستند. علاوه بر آن، اگر $R$ یک حلقه تام باشد، آن‌گاه تمام $R$-مدول‌های تک‌رشته‌ای، $R$-مدول‌هایی هم-ضربی و با طول متناهی هستند. در پایان نشان خواهیم داد، $\mathbb{Z}$-مدول‌های هم-ضربی، کاهش یافته هستند. علاوه بر آن گروه‌های آبلی هم-ضربی مشخص می‌شوند.

کلمات کلیدی: مدول‌های قویا دئو، مدول‌های هم-ضربی، گروه‌های آبلی.