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ARTINIANNESS OF COMPOSED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, and let $\underline{a}$ and $\underline{b}$ be two ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian. Let M and N be two finitely generated R-modules. We prove that $H^j_{\underline{b}}(H^t_{\underline{a}}(M,N))$ is Artinian for $j=0,1$, where $t=\inf\{i\in\mathbb{N}_0 : H^i_{\underline{a}}(M,N)$ is not finitely generated$\}$. Also, we prove that if $\dim\operatorname{Supp}(H^i_{\underline{a}}(M,N))\leq 2$, then $H^1_{\underline{b}}(H^i_{\underline{a}}(M,N))$ is Artinian for all i. Moreover, we show that if $\dim N=d$, then $H^j_{\underline{b}}(H^{d-1}_{\underline{a}}(N))$ is Artinian for all $j\geq 1$.

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring with non-zero identity, and let M and N be two finitely generated R-modules. For an ideal $\underline{a}$ of R, let $H^i_{\underline{a}}(M,N)$, $i\in\mathbb{N}_0$ denote the generalized local cohomology modules of two R-modules M and N with respect to $\underline{a}$ (see [7], [14], and [3] for the definitions and basic properties). With $M=R$, one clearly obtains the ordinary local cohomology, which was introduced by Grothendieck (see [6] and [4]). One of the main problems in the study of local cohomology modules is to determine when they are Artinian. Recently, some results have been proved about the Artinianness of local cohomology modules (see [8], [9], [13], [15] , [5], [1], and [11]).

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Mafi and the author in [11] have proved the Artinianness of graded composed local cohomology modules. Aghapournahr and Melkersson, in [1], have shown that if $\underline{a}$ and $\underline{b}$ are ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian and $\dim N/\underline{a}N \leq 2$, then $H^1_{\underline{b}}(H^i_{\underline{a}}(N))$ is Artinian for all i. In [2], Bahmanpour, Naghipour, and Sedghi have proved that if R is local with maximal ideal $\underline{m}$ and N of dimension d, then $H^1_{\underline{m}}(H^{d-1}_{\underline{a}}(N))$ is Artinian. We prove that with uniform proofs, some general results about Artinianness of generalized local cohomology modules.

Namely, our main aim in this paper is to establish the following theorem:

Theorem 1.1. *Let a, b be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian.*
(i) Then $H^j_{\underline{b}}(H^t_{\underline{a}}(M,N))$ is Artinian for $j=0,1$, provided that $t = \inf\{i \in \mathbb{N}_0 : H^i_{\underline{a}}(M,N)$ is not finitely generated$\}$.
(ii) If $\dim \operatorname{Supp}(H^i_{\underline{a}}(M,N)) \leq 2$, then $H^1_{\underline{b}}(H^i_{\underline{a}}(M,N))$ is Artinian for all i. Moreover, $H^0_{\underline{b}}(H^i_{\underline{a}}(M,N))$ is Artinian for all i if and only if $H^2_{\underline{b}}(H^i_{\underline{a}}(M,N))$ is Artinian for all i.
(iii) $H^j_{\underline{b}}(H^{d-1}_{\underline{a}}(N))$ is Artinian for all $j \geq 1$, where $\dim N = d$.

2. RESULTS

Theorem 2.1. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian. Then $H^j_{\underline{b}}(H^t_{\underline{a}}(M,N))$ is Artinian for $j=0,1$ and all $t \leq \inf\{i \in \mathbb{N}_0 : H^i_{\underline{a}}(M,N)$ is not finitely generated$\}$.*

Proof. Let $F(.) := \Gamma_{\underline{b}}(.)$ and $G(.) := \operatorname{Hom}_R(M, \Gamma_{\underline{a}}(.))$. We claim that $H^i_{\underline{b}}(\operatorname{Hom}_R(M, \Gamma_{\underline{a}}(E))) = 0$ for all injective R-module E and all $i \geq 1$. Since $\Gamma_{\underline{a}}(E)$ is an injective R-module and any injective R-module decomposes into a direct sum of indecomposable injective R-modules, we may and do assume that $\Gamma_{\underline{a}}(E) = E(R/\underline{p})$, for some prime ideal $\underline{p}$ of R. (Note that the functor $H^i_{\underline{b}}(.)$ commutes with direct sums, and as M is finitely generated the functor $\operatorname{Hom}_R(M,.)$ also commutes with direct sum.) Since $\operatorname{Hom}_R(M, E(R/\underline{p})) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{R_{\underline{p}}}(R_{\underline{p}}, E(R/\underline{p}))) \cong \operatorname{Hom}_{R_{\underline{p}}}(M_{\underline{p}}, E(R_{\underline{p}}/\underline{p}R_{\underline{p}}))$, we deduce that $\operatorname{Hom}_R(M, E(R/\underline{p}))$ is an Artinian $R_{\underline{p}}$-module. Thus $H^i_{\underline{b}}(\operatorname{Hom}_R(M, E(R/\underline{p}))) \cong H^i_{\underline{b}R_{\underline{p}}}(\operatorname{Hom}(M, E(R/\underline{p}))) = 0$, as claimed. Thus since $(FG)(.) = \Gamma_{\underline{b}+\underline{a}}(M,.)$, by [12, Theorem 11.38], there is the Grothendieck's spectral sequence

$$E_2^{p,q} := H^p_{\underline{b}}(H^q_{\underline{a}}(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\underline{a}+\underline{b}}(M,N).$$

Since $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $r \geq 2$, by [4, Exercises 2.1.9 and 7.1.4] and our hypotheses, we have that $E_r^{p,q}$ is Artinian for all

$r \geq 2$, $p \geq 0$, and $q < t$. For each $r \geq 2$, and $p, q \geq 0$, let $Z_r^{p,q} = \operatorname{Ker}(E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})$ and $B_r^{p,q} = \operatorname{im}(E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})$. For all $r \geq 2$ and $p = 0, 1$, we have the exact sequences

$$0 \longrightarrow B_r^{p,q} \longrightarrow Z_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow 0,$$

and

$$0 \longrightarrow Z_r^{p,q} \longrightarrow E_r^{p,q} \longrightarrow B_r^{p+r,q-r+1} \longrightarrow 0.(\dagger)$$

Notice that $B_r^{p,t} = 0$ and $B_r^{p+r,t-r+1}$ is Artinian for all $r \geq 2$ and $p = 0, 1$. Hence, we have that $Z_r^{p,t} \cong E_{r+1}^{p,t}$ (‡) for all $r \geq 2$ and $p = 0, 1$. Now, $E_\infty^{p,t}$ is isomorphic to a subquotient of $H_{\underline{a}+\underline{b}}^{p+t}(M, N)$, and so it is Artinian for all $p \geq 0$, by [8, Corollary 2.6]. Since $E_\infty^{p,t} \cong E_r^{p,t}$ for r sufficiently large, we have that $E_r^{p,t}$ is Artinian for all $p \geq 0$ and all large r. Fix r, and suppose $E_{r+1}^{p,t}$ is Artinian for $p = 0, 1$. From the isomorphism (‡), we have that $Z_r^{p,t}$ is Artinian for $p = 0, 1$. From the exact sequence (†) we get that $E_r^{p,t}$ is Artinian. Continuing in this fashion, we see that $E_r^{p,t}$ is Artinian for all $r \geq 2$ and $p = 0, 1$. In particular, $E_2^{p,t} = H_{\underline{b}}^p(H_{\underline{a}}^t(M, N))$ is Artinian for $p = 0, 1$. □

The following corollaries immediately follow by Theorem 2.1.

Corollary 2.2. *(see [11, Corollary 2.2]) Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a} + \underline{b})$ is Artinian. Then $H_{\underline{b}}^j(H_{\underline{a}}^1(M, N))$ is Artinian for $j = 0, 1$.*

Corollary 2.3. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a} + \underline{b})$ is Artinian, and let t be a non-negative integer such that $\operatorname{grade}(M/\underline{a}M, N) = t$. Then $H_{\underline{b}}^j(H_{\underline{a}}^t(M, N))$ is Artinian for $j = 0, 1$.*

Proposition 2.4. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a} + \underline{b})$ is Artinian, and t be a non-negative integer. Let $H_{\underline{b}}^j(H_{\underline{a}}^i(M, N))$ be Artinian for all $i \neq t$ and for all j. Then $H_{\underline{b}}^j(H_{\underline{a}}^t(M, N))$ is Artinian for all j.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := H_{\underline{b}}^p(H_{\underline{a}}^q(M, N)) \underset{p}{\Longrightarrow} H_{\underline{a}+\underline{b}}^{p+q}(M, N).$$

For each $r \geq 2$, we consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} d_r^{p,t} \longrightarrow E_r^{p,t} \xrightarrow{d_r^{p,t}} E_r^{p+r,t-r+1}. \quad (\natural)$$

It follows from our hypotheses that the R-module $E_r^{p+r,t-r+1}$ is Artinian. Note that $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \geq 0$. There is an integer s such that $E_\infty^{p,q} = E_r^{p,q}$ for all p, q and all $r \geq s$. Also, for each $n \geq 0$, there is a finite filtration

$$0 = \phi^{n+1}H^n \subseteq \phi^n H^n \subseteq \ldots \subseteq \phi^1 H^n \subseteq \phi^0 H^n = H_{\underline{a}+\underline{b}}^n(M, N),$$

such that $E_\infty^{p,n-p} \cong \phi^p H^n/\phi^{p+1}H^n$ for all $0 \le p \le n$. Thus $E_\infty^{p,q}$ is Artinian for all $p, q \ge 0$. Since $E_s^{p,t} \cong \operatorname{Ker} d_{s-1}^{p,t}/\operatorname{im} d_{s-1}^{p-s+1,t+s-2}$, it follows that $\operatorname{Ker} d_{s-1}^{p,t}$ is Artinian for all $p \ge 0$. Hence, using the exact sequence (♮) for $r = s-1$, we deduce that $E_{s-1}^{p,t}$ is Artinian for all $p \ge 0$. By continuing this argument repeatedly for integer $s-1, s-2, \dots, 3$ instead of s, we obtain that $E_2^{p,t}$ is Artinian for $p \ge 0$. This completes the proof. □

Definition 2.5. We denote by $\operatorname{cd}(\underline{a}, M, N)$ the *cohomological dimension* of M and N with respect to $\underline{a}$, which is $\sup\{i \in \mathbb{N}_0 : H_{\underline{a}}^i(M, N) \neq 0\}$. One can easily see that $\operatorname{cd}(\underline{a}, M, N) = \operatorname{cd}(\underline{a}, N)$ if $M = R$.

Corollary 2.6. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian, and let $\operatorname{cd}(\underline{a}, M, N) = 1$. Then $H_{\underline{b}}^j(H_{\underline{a}}^i(M, N))$ is Artinian for all i, j.*

Proof. This is clear by Proposition 2.4. □

The following result extends [1, Corollary 2.10].

Proposition 2.7. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian and let $\dim \operatorname{Supp}(H_{\underline{a}}^i(M, N)) \le 1$ for all i. Then $H_{\underline{b}}^j(H_{\underline{a}}^i(M, N))$ is Artinian for all i,j.*

Proof. Consider the Grothendieck's spectral sequence

$$E_2^{p,q} := H_{\underline{b}}^p(H_{\underline{a}}^q(M, N)) \underset{p}{\Longrightarrow} H_{\underline{a}+\underline{b}}^{p+q}(M, N).$$

Hence, for each n, there is a finite filtration

$$0 = \phi^{n+1}H^n \subseteq \phi^n H^n \subseteq \dots \subseteq \phi^1 H^n \subseteq \phi^0 H^n = H_{\underline{a}+\underline{b}}^n(M, N),$$

such that $E_\infty^{p,n-p} \cong \phi^p H^n/\phi^{p+1}H^n$ for all $p = 0, 1, \dots, n$. Thus $E_\infty^{p,q}$ is Artinian for all p, q. Since $\dim \operatorname{Supp}(H_{\underline{a}}^i(M, N)) \le 1$, we get that $H_{\underline{b}}^j(H_{\underline{a}}^i(M, N)) = 0$ for all $j \ge 2$ and all i. Hence, it is enough for us to prove that $H_{\underline{b}}^j(H_{\underline{a}}^i(M, N))$ is Artinian for all i and $j = 0, 1$. Using the exact sequence (†) as in the proof of Theorem 2.1, we obtain $E_2^{0,i} \cong Z_2^{0,i} \cong E_3^{0,i} \cong \dots \cong E_\infty^{0,i}$ and $E_2^{1,i} \cong Z_2^{1,i} \cong E_3^{1,i} \cong \dots \cong E_\infty^{1,i}$ for all i. Therefore, $E_2^{0,i}$ and $E_2^{1,i}$ are Artinian for all i, and so the result follows. □

The following theorem extends [1, Theorem 2.11].

Theorem 2.8. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian, and let $\dim \operatorname{Supp}(H_{\underline{a}}^i(M, N)) \le 2$. Then*
(i) $H_{\underline{b}}^1(H_{\underline{a}}^i(M, N))$ is Artinian for all i.
(ii) $H_{\underline{b}}^0(H_{\underline{a}}^i(M, N))$ is Artinian for all i if and only if $H_{\underline{b}}^2(H_{\underline{a}}^i(M, N))$ is Artinian for all i.

Proof. (i) Since $\dim \operatorname{Supp}(H^i_{\underline{a}}(M,N)) \leq 2$, by using the exact sequence (†) as in the proof of Theorem 2.1, we obtain $E_2^{1,i} \cong Z_2^{1,i} \cong E_3^{1,i} \cong ... \cong E_\infty^{1,i}$ for all i. Hence, $H^1_{\underline{b}}(H^i_{\underline{a}}(M,N))$ is Artinian for all i.
(ii) For each i, consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} d_2^{0,i} \longrightarrow E_2^{0,i} \stackrel{d_2^{0,i}}{\longrightarrow} E_2^{2,i-1} \longrightarrow \operatorname{coker} d_2^{0,i} \longrightarrow 0.$$

by using the exact sequence (†) as in the proof of Theorem 2.1, $\operatorname{Ker} d_2^{0,i} \cong E_3^{0,i} \cong ... \cong E_\infty^{0,i}$ for all i and also $coker d_2^{0,i} = E_2^{2,i-1}/\operatorname{im} d_2^{0,i}$. Since $E_2^{2,i-1}/\operatorname{im} d_2^{0,i} \cong \operatorname{Ker} d_2^{2,i-1}/\operatorname{im} d_2^{0,i} \cong E_3^{2,i-1} \cong ... \cong E_\infty^{2,i-1}$, it follows that $coker d_2^{0,i} \cong E_\infty^{2,i-1}$. Hence, $\operatorname{Ker} d_2^{0,i}$ and $coker d_2^{0,i}$ are Artinian, and so the results follow. □

The following corollary is a generalization of [2, Theorem 2.7], and immediately follows by Theorem 2.8.

Corollary 2.9. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian. If $\dim R/\underline{a} \leq 2$, then $H^1_{\underline{b}}(H^i_{\underline{a}}(N))$ is Artinian for all i.*

The following theorem is a generalization of [2, Theorem 2.8].

Theorem 2.10. *Let $\underline{a}$, $\underline{b}$ be ideals of R such that $R/(\underline{a}+\underline{b})$ is Artinian. Then $H^i_{\underline{b}}(H^{d-1}_{\underline{a}}(N))$ is Artinian for all $i \geq 1$, where $\dim N = d$.*

Proof. By the Grothendieck's spectral sequence, for all p,q, we have

$$E_2^{p,q} := H^p_{\underline{b}}(H^q_{\underline{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\underline{a}+\underline{b}}(N).$$

By [10, Theorem 2.3], $\dim \operatorname{Supp}(H^{d-1}_{\underline{a}}(N)) \leq 1$, and so $H^i_{\underline{b}}(H^{d-1}_{\underline{a}}(N)) = 0$ for all $i \geq 2$. Hence, it is enough to prove that $H^1_{\underline{b}}(H^{d-1}_{\underline{a}}(N))$ is Artinian. There is a finite filtration

$$0 = \phi^{d+1}H^d \subseteq \phi^d H^d \subseteq ... \subseteq \phi^1 H^d \subseteq \phi^0 H^d = H^d_{\underline{a}+\underline{b}}(N),$$

such that $E_\infty^{p,d-p} \cong \phi^p H^d/\phi^{p+1}H^d$ for all $p = 0,1,...,d$. Hence, $E_\infty^{p,q}$ is Artinian for all p,q. For each $r \geq 2$, consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} d_r^{1,d-1} \longrightarrow E_r^{1,d-1} \stackrel{d_r^{1,d-1}}{\longrightarrow} E_r^{1+r,d-r}.$$

Since $E_2^{1+r,d-r} = 0$ and $E_r^{1+r,d-r}$ is subquotient of $E_2^{1+r,d-r}$, it follows that $E_r^{1,d-1} \cong \operatorname{Ker} d_r^{1,d-1}$ for all r. Since $\operatorname{im} d_r^{1-r,d+r-2} = 0$ for all $r \geq 2$, we get that $E_{r+1}^{1,d-1} \cong \operatorname{Ker} d_r^{1,d-1}$ for all $r \geq 2$. Hence, $E_2^{1,d-1} \cong \operatorname{Ker} d_2^{1,d-1} \cong E_3^{1,d-1} \cong ... \cong E_\infty^{1,d-1}$, and so $H^1_{\underline{b}}(H^{d-1}_{\underline{a}}(N))$ is Artinian, as required. □

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ARTINIANNESS OF COMPOSED LOCAL COHOMOLOGY MODULES

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آرتینی بودن مدول‌های کوهمولوژی موضعی ترکیبی

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فرض کنید R یک حلقه جابجایی نوتری و فرض کنید $\underline{a}$، $\underline{b}$ دو ایده‌آل R بطوری‌که $R/(\underline{a}+\underline{b})$ آرتینی باشد. فرض کنید M، N دو $R-$ مدول با تولید متناهی باشند. ثابت می‌کنیم برای $j=۰,۱$، $H_{\underline{b}}^{j}(H_{\underline{a}}^{t}(M,N))$ آرتینی است، که در آن

$$t=\inf\{i\in\mathbb{N}_۰ : H_{\underline{a}}^{i}(M,N) \;\; is \;\; not \;\; finitely \;\; generated\}.$$

همچنین ثابت می‌کنیم اگر $dim\, Supp(H_{\underline{a}}^{i}(M,N)) \leq ۲$، آن‌گاه برای هر i، $H_{\underline{b}}^{۱}(H_{\underline{a}}^{i}(M,N))$ آرتینی است. علاوه براین، نشان می‌دهیم که اگر $dimN=d$، آن‌گاه برای هر $j\leq ۱$، $H_{\underline{b}}^{j}(H_{\underline{a}}^{d-۱}(N))$ آرتینی است.

کلمات کلیدی: مدول‌های کوهمولوژی موضعی، مدول‌های کوهمولوژی تعمیم‌یافته، مدول‌های آرتینی.