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MORE ON EDGE HYPER WIENER INDEX OF GRAPHS

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ABSTRACT. Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The (first) edge-hyper Wiener index of the graph G is defined as:

$$\begin{aligned} WW_e(G) &= \sum_{\{f,g\}\subseteq E(G)} (d_e(f,g|G) + d_e^2(f,g|G)) \\ &= \frac{1}{2}\sum_{f\in E(G)} (d_e(f|G) + d_e^2(f|G)), \end{aligned}$$

where $d_e(f,g|G)$ denotes the distance between the edges $f = xy$ and $g = uv$ in $E(G)$ and $d_e(f|G) = \sum_{g\in E(G)} d_e(f,g|G)$. In this paper, we use a method, which applies group theory to graph theory, to improving mathematically computation of the (first) edge-hyper Wiener index in certain classes of graphs. We give also upper and lower bounds for the (first) edge-hyper Wiener index of a graph in terms of its size and Gutman index. Our aim in last section is to investigate products of two or more graphs, and compute the second edge-hyper Wiener index of the some classes of graphs.

1. Introduction and preliminaries

Unless stated otherwise, the graphs considered in this paper are undirected, connected, simple and finite, i.e., connected graphs on a finite

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number of vertices without multiple edges or loops. A graph is (usually) denoted by $G = (V(G), E(G))$, where $V(G)$ is its vertex set and $E(G)$ its edge set. The *order* of G is the number $n = |V(G)|$ of its vertices and its *size* is the number $m = |E(G)|$ of its edges. Two vertices of G, connected by an edge, are said to be adjacent. The number of vertices of G, adjacent to a given vertex v, is the *degree* of this vertex, and will be denoted by $deg_G(v)$. The *distance* between two vertices u and v in a connected graph G is the length of any shortest path between these vertices, and it is denoted by $d(u, v|G)$ or $d(u, v)$. For undefined terminology and notation for graphs, we refer to [2].

Graph invariants are properties of graphs that are invariant under graph isomorphisms. The first, and most well-known parameter, the Wiener index, was introduced in the late 1940's by Harold Wiener in an attempt to analyze the chemical properties of paraffins (alkanes) and connection with the modeling of various physico-chemical, biological and pharmacological properties of organic molecules in chemistry [13]. This is a distance-based index, whose mathematical properties and chemical applications have been widely researched (see [11]). In our notation, it can be described as follows:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d(u, v|G) = \frac{1}{2} \sum_{u\in V(G)} d(u|G),$$

where $d(u|G) = \sum_{v\in V(G)} d(u, v|G)$. In mathematical research, the Wiener index has been first studied in [6], and for along time mathematicians were not aware of the importance of the Wiener index in mathematical chemistry. Milan Randić introduced a modification of the Wiener index for trees (acyclic graphs), and it is known as the hyper-Wiener index. Then Klein et al., generalized Randić's definition for all connected (cyclic) graphs, as a generalization of the Wiener index, denoted by $WW(G)$ and defined as

$$WW(G) = \frac{1}{2} \sum_{u\in V(G)} [d(u|G) + d^2(u|G)].$$

The Wiener index and the hyper-Wiener index are based on the distances between pairs of vertices in a graph and therefore, similar concepts have been introduced for distances between pairs of edges under the names of the edge-Wiener index [9] and the edge-hyper-Wiener index [10], respectively. Let $f = xy$ and $g = uv$ be two edges of G. The distance between f and g is denoted by $d_e(f, g|G)$, and defined as the distance between the vertices f and g in the line graph of G. This distance is equal to $D(f, g) + 1$, where $D(f, g) =$

$min\{d(x,u),(x,v),(y,u),(y,v)\}$. For example, distance 1 means that the edges share a vertex and distance 2 means that at least two of the four end vertices of two edges are adjacent.

The (first) edge Wiener index of the graph G, is denoted by $W_{0e}(G)$, and defined as the sum of distances between all pairs of edges of the graph G. That is,

$$W_{0e}(G) = \sum_{\{f,g\}\subseteq E(G)} d_e(f,g|G) = \frac{1}{2}\sum_{f\in E(G)} d_e(f|G), \tag{1.1}$$

where $d_e(f|G) = \sum_{g\in E(G)} d_e(f,g|G)$.

It is purposeful to generalize Eq. (1.1) in the following manner [10]:

$$W_{0e}^{\lambda}(G) = \sum_{\{f,g\}\subseteq E(G)} d_e^{\lambda}(f,g|G) = \frac{1}{2}\sum_{f\in E(G)} d_e^{\lambda}(f|G),$$

where λ is some parameter and we name it the (first) edge-Wiener type index. Also, by [10], the (first) edge hyper Wiener index of the graph G is defined as:

$$\begin{aligned} WW_{0e}(G) &= \sum_{\{f,g\}\subseteq E(G)} (d_e(f,g|G) + d_e^2(f,g|G)) \\ &= \frac{1}{2}\sum_{f\in E(G)} (d_e(f|G) + d_e^2(f|G)). \end{aligned}$$

In this paper, we are concerned with some variants of the Wiener index. The paper is organized as follows. In Section 2, our aim is to use a method which applies group theory to graph theory. We improve only mathematically computation of the (first) edge-hyper Wiener index in certain graphs by this method. We encourage the interested readers to consult also the paper by Darafsheh [5], and references therein, for more information on this topic.

First, we need some concepts from the theory of groups and graph theory, that we will give in the following. Let $A = Aut(G)$ be the automorphism group of a graph G. A acts *transitive* on $V(G)$ (or $E(G)$), if for any pair u, v of vertices (or f, g of edges) in G, there exists an automorphism σ such that $\sigma(u) = v$ (or $\sigma(f) = g$). In this case, G is called *vertex-transitive* (or *edge-transitive*) (see [2]).

The *triangular graph* $T(n)$ is the line graph of the complete graph K_n. The vertices of $T(n)$ may be identified with the 2-subsets of $\Omega = \{1, 2, \ldots, n\}$, in fact $V = \{\{a, b\}|\ a, b \in \Omega, a \neq b\}$. Two distinct vertices $\{a, b\}$ and $\{c, d\}$ are adjacent if the 2-subsets have a non-empty intersection. We have $|V| = \binom{n}{2}$, the degree of each vertex is $2n - 4$

and hence $|E| = (n-2)\binom{n}{2}$. We can see easily that $T(n)$ is an edge-transitive graph [4]. We will express explicit formula for the (first) edge-hyper Wiener index of this graph, based on the properties of the automorphism group of the graph.

In Section 3, we investigate some relations between Gutman index and the (first) edge-hyper Wiener index. In Section 4, we study the second edge-hyper Wiener index, and its behavior under the join of graphs. Some results of this paper are analogous to the results obtained in [12].

2. On the (first) edge-hyper Wiener index

In this section, we calculate the (first) edge-hyper Wiener index of some classes of graphs such as triangular graphs. Also, we obtain an explicit formula for the (first) edge-hyper Wiener index of the Cartesian product of two graphs using the group automorphisms of graphs. Before proceeding further, let us first set some notations and terminologies. Let E' and E'' be two subsets of $E = E(G)$. Then, define $d_e(E', E'')$ and $d_e^2(E', E'')$ as follows:

$$d_e(E', E'') = \sum_{f \in E'} \sum_{g \in E''} d_e(f, g),$$

$$d_e^2(E', E'') = \sum_{f \in E'} \sum_{g \in E''} d_e^2(f, g).$$

According to the above notations, we can rewrite:

$$WW_{0e}(G) = \frac{1}{2}(d_e(E, E) + d_e^2(E, E)).$$

Define distance number $\delta_e(\sigma)$ and second distance number $\delta_e^{(2)}(\sigma)$ of $\sigma \in A = Aut(G)$, as follows:

$$\delta_e(\sigma) = \frac{1}{|E|} \sum_{f \in E} d_e(f, \sigma(f)),$$

and

$$\delta_e^{(2)}(\sigma) = \frac{1}{|E|} \sum_{f \in E} d_e^2(f, \sigma(f)).$$

If Γ is a subgroup of A, then we define $\delta_e(\Gamma)$ and $\delta_e^{(2)}(\Gamma)$ as follows:

$$\delta_e(\Gamma) = \frac{1}{|\Gamma|} \sum_{\sigma \in \Gamma} \delta_e(\sigma) = \frac{1}{|\Gamma||E|} d_e(f, \sigma(f)),$$

and

$$\delta_e^{(2)}(\Gamma) = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}\delta_e^{(2)}(\sigma) = \frac{1}{|\Gamma||E|}d_e^2(f,\sigma(f)).$$

When $\Gamma = A$, the distance number and second distance number of a graph G, are denoted by $\delta_e(A)$ and $\delta_e^2(A)$, respectively. Since Γ acts on the set E, we denote the orbits of this action by $E_i = O(f_i) = \{f_i^\sigma | \sigma \in A, 1 \leq i \leq r\}$. Therefore, we have $E = E_1 \cup \ldots \cup E_r$. We bring the following lemma, which was proved in [7], and will be used frequently in the sequel.

Lemma 2.1. (*Orbit-stabilizer*) *Let G be a permutation group acting on Ω and let ω be a point in Ω. Then, $|G| = |G_\omega||\omega^G|$, where $\omega^G = \{\omega^G | g \in G\}$ is an orbit of G and $G_\omega = \{g \in G | \omega^G = \omega\}$ is the stabilizer of ω in G.*

So, if $\Gamma_i = \{\sigma \in \Gamma | f_i^\sigma = f_i\}$ denotes a stabilizer of an edge f_i from E_i, then by the Lemma 2.1, we get $|\Gamma| = |E_i||\Gamma_i|$.

In the following theorem, we show that the (first) edge-hyper Wiener index of an edge-transitive graph G, can be expressed in terms of the distance number and the second distance number of G.

Theorem 2.2. *Let G be a connected graph with the edge set $E = E(G)$. If we assume that $A = Aut(G)$, and for $1 \leq i \leq r$, E_i, is an orbit of the action of A on the set E, then*

$$|E|(\delta_e(A) + \delta_e^2(A)) = \sum_{i=1}^{r}\frac{2WW_{0e}(E_i)}{|E_i|}.$$

Proof. Take $\Gamma = A$ and for two edges f and g of G, let $\sum = \{\sigma \in \Gamma | f^\sigma = g\}$ and $n(f,g) = |\sum|$. We denote by Γ_f the stabilizer group for f. If f and g belong to the same orbit E_i, we can construct a bijection between Γ_f and Γ_g, and hence $n(f,g) = |\Gamma_f| = |\Gamma_g|(= |\Gamma_i|)$.

Now, we have

$$\begin{aligned}\sigma_e(A)+\sigma_e^{(2)}(A) &= \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}(\delta_e(\sigma)+\delta_e^{(2)}(\sigma))\\ &= \frac{1}{|\Gamma||E|}\sum_{f\in E}\sum_{\sigma\in\Gamma}(d_e(f,\sigma(f))+d_e^2(f,\sigma(f)))\\ &= \frac{1}{|\Gamma||E|}\sum_{f\in E}\sum_{g\in E}n(f,g)(d_e(f,g)+d_e^2(f,g))\\ &= \frac{1}{|\Gamma||E|}\sum_{i=1}^{r}\sum_{f\in E_i}\sum_{g\in E_i}(d_e(f,g)+d_e^2(f,g))|\Gamma_i|\\ &= \frac{1}{|E|}\sum_{i=1}^{r}\frac{|\Gamma_i|}{|\Gamma|}(d(E_i,E_i)+d^2(E_i,E_i)).\end{aligned}$$

Therefore,

$$|E|(\delta_e(A)+\delta_e^2(A))=\sum_{i=1}^{r}\frac{1}{|E_i|}(d(E_i,E_i)+d^2(E_i,E_i)=\sum_{i=1}^{r}\frac{2WW_{0e}(E_i)}{|E_i|},$$

and hence we get the desired result. □

Corollary 2.3. *If G is an edge-transitive graph, then*

$$WW_{0e}(G)=\frac{|E|^2(\delta_e(A)+\delta_e^2(A))}{2}.$$

For a subset U of $E=E(G)$, define

$$\omega_{0e}(U)=\frac{1}{|U|^2}\sum_{f\in U}\sum_{g\in U}d_e(f,g)=\frac{1}{|U|^2}d_e(U,U),$$

and

$$\omega_{0e}^{(2)}(U)=\frac{1}{|U|^2}\sum_{f\in U}\sum_{g\in U}d_e^2(f,g))=\frac{1}{|U|^2}d_e^2(U,U).$$

Clearly, we have

$$\omega_{0e}(U)+\omega_{0e}^2(U)=\frac{2WW_{0e}(U)}{|U|^2}.$$

Theorem 2.4. *If for $1\le i\le r$, E_i, is an orbit, then for each edge $f\in E_i$, we have*

$$\omega_{0e}(E_i)+\omega_{0e}^{(2)}(E_i)=\frac{1}{|E_i|}(d_e(f|E_i)+d_e^2(f|E_i)).$$

Proof. We have

$$\omega_{0e}(E_i) = \frac{1}{|E_i|^2} d_e(E_i, E_i),$$

and

$$\omega_{0e}^{(2)}(E_i) = \frac{1}{|E_i|^2} d_e^2(E_i, E_i).$$

However, if f and g are two elements from E_i, then for some $\sigma \in Aut(G)$, we have $f^\sigma = g$ and also $d_e(f|E_i) = d_e(g|E_i)$ and $d_e^2(f|E_i) = d_e^2(g|E_i)$. therefore,

$$\omega_{0e}(E_i) + \omega_{0e}^{(2)}(E_i) = \frac{1}{|E_i|^2}|E_i|(d_e(f|E_i) + d_e^2(f|E_i)).$$

□

Corollary 2.5. *If G is an edge-transitive graph, then for each $f \in E = E(G)$, we have*

$$\delta_e(A) + \delta_e^{(2)}(A) = \omega_{0e}(E) + \omega_{0e}^{(2)}(E) = \frac{1}{|E|}(d_e(f|G) + d_e^2(f|G)).$$

Corollary 2.6. *If G is an edge-transitive graph, then for each $f \in E = E(G)$, we have*

$$WW_{0e}(G) = \frac{1}{2}|E|(d_e(f|G) + d_e^2(f|G)).$$

Example 2.7. The automorphism group of K_n is the symmetric group S_n. So, the complete graph K_n is an edge-transitive graph.Thus, we have

$$\begin{aligned} WW_{0e}(K_n) &= \frac{1}{2}|E(K_n)|((d_e(f|G) + d_e^2(f|G)) \\ &= \frac{1}{2}\binom{n}{2}((n^2 - 3n + 2) + (n^2 - 3n + 2)^2) \\ &= \frac{1}{2}\binom{n}{2}(n^4 - 6n^3 + 14n^2 - 15n + 6). \end{aligned}$$

Example 2.8. The Petersen graph P is edge-transitive. We have $WW_{0e}(P) = \frac{1}{2}|E|(d_e(f|P) + d_e^2(f|P))$, where f is an arbitrary edge of P. One can see that $d_e(f|P) = 26$, and then we conclude that $WW_{0e}(P) = 5265$.

The next theorem shows that, an edge-transitive graph which is not vertex-transitive is necessarily bipartite.

Theorem 2.9. [2] *If a connected graph G is edge-transitive but not vertex-transitive, then it is bipartite.*

Example 2.10. Since the complete bipartite graph $K_{m,n}$ with $m \neq n$, is edge-transitive, we have

$$WW_{0e}(K_n) = \frac{1}{2}|E(K_{m,n})|((d_e(f|K_{m,n}) + d_e^2(f|K_{m,n})).$$

Then, we get $d_e(f|K_{m,n}) = \sum_{g\in E} d_e(f,g) = 2mn - n - m$, and hence

$$WW_{0e}(K_{m,n}) = 2(nm)^2(nm-n-m+1)+\frac{1}{2}nm(n(n-1)+m(m-1)).$$

Now, in the following, we give the formula of the (first) edge-hyper Wiener index of the triangular graph $T(n)$, according to concepts in transitive graphs.

Theorem 2.11. *The (first) edge-hyper Wiener index of the triangular graph $G = T(n)$ is*

$$WW_{0e}(G) = \frac{1}{2}(n-2)\binom{n}{2}(6n^3 - 45n^2 + 131n - 146).$$

Proof. The distance between any two distinct vertices of V is either 1 or 2. The vertices z whose distance from $u = \{a, b\} \in V$ is 1 should meet u in one element, hence the number of them is $2n - 4$. If v is another vertex of V with $u \cup v = \emptyset$, then $v = \{c, d\}$, where c and d are distinct elements of Ω disjoint from a and b. Now, if we take $w = \{a, c\}$, then $u \to w \to v$ is a path of length 2 from u to v.

Now, according to definition of the distance between $f, g \in E$, we have $d_e(f, g|G) = 1, 2$ or 3. Fixing $f = uv \in E$, where $u = \{a, b\}$ and $v = \{c, d\}$. The edges whose distance from f is 1 should share with f in one vertex. So, the number of these edges is $4n - 10$. Suppose that $g = xy$ is a different edge from f. We let $x = \{r, s\}$ and $y = \{r, t\}$, where r, s, t are distinct elements of Ω disjoint from a, b, c. Therefore, by the above arguments, $d(x, u) = d(x, v) = d(y, u) = d(y, v) = 2$, and so we have $d_e(f, g|G) = min\{d(x, u), d(x, v), d(y, u), d(y, v)\} + 1 = 3$. The number of these edges is equal to the number of selections of $g = xy$, which is equal to $\frac{(n-3)(n-4)(n-5)}{2}$. In this manner the number of edges at distance 2 of f is $\frac{1}{2}(9n^2 - 53n + 78)$, and then we get:

$$\begin{aligned} WW_{0e}(G) &= \frac{1}{2}|E|(d_e(f|G) + d_e^2(f|G)) \\ &= \frac{1}{2}(n-2)\binom{n}{2}[\frac{1}{2}(3n^3 - 18n^2 + 43n - 44) \\ &+ \frac{1}{2}(9n^3 - 72n^2 + 219n - 248)] \\ &= \frac{1}{2}(n-2)\binom{n}{2}(6n^3 - 45n^2 + 131n - 146), \end{aligned}$$

Hence we conclude the desired result. □

Many interesting classes of graphs arise from simpler graphs via binary operations known as graph products. The *Cartesian product* of G and H is a graph, denoted by $G \times H$, with vertex set is $V(G \times H) = \{(u, v)|\ u \in V(G), v \in V(H)\}$. Two vertices (u, x) and (v, y) are adjacent precisely if $u = v$ and $xy \in E(H)$, or $uv \in E(G)$ and $x = y$. The graphs G and H are called factors of the product $G \times H$. In the case that $G_1 = G_2 = \cdots = G_n = G$, we denote $G_1 \times G_2 \times \cdots \times G_n$ by G^n. In this part, we study the Cartesian product of vertex-transitive graphs and then give another formula for this product. This formula is better in vertex-transitive graphs. In [8], we have the following results in relation to this product.

Lemma 2.12. [8] *Let G and H be two graphs. Then we have*

1. $|V(G \times H)| = |V(G)||V(H)|$,
2. $|E(G \times H)| = |V(G)||E(H)| + |E(G)||V(H)|$,
3. *If (u, x) and (v, y) are vertices of $G \times H$,*

then

$$d((u, x), (v, y)|G \times H) = d(u, v|G) + d(x, y|H).$$

Lemma 2.13. [8] *A cartesian product has transitive automorphism group if and only if every factor has transitive automorphism group.*

Lemma 2.14. [8] *Let G and H be two vertex-transitive graphs. Then, $G \times H$ is a vertex-transitive graph.*

Lemma 2.15. [8] *For any transitive graph G, the graph G^n is a vertex-transitive graph.*

Theorem 2.16. *Let G and H be two vertex-transitive graphs. Then, for each two vertices $w \in G$ and $x \in H$, we have:*

$$\begin{aligned} WW(G \times H) &= \frac{1}{2}|V(G \times H)|^2 (\frac{d(w|G) + d^2(w|G)}{|V(G)|} \\ &+ \frac{d(x|G) + d^2(x|G)}{|V(H)|} + 2\frac{d(w|G)}{|V(G)|}.\frac{d(x|H)}{|V(H)|}). \end{aligned}$$

Proof. For each vertex (w, z) in $G \times H$, first we have

$$d((w, x)|G \times H) = |V(G \times H)|(\frac{d(w|G)}{|V(G)|} + \frac{d(x|H)}{|V(H)|}),$$

and

$$\begin{aligned} d^2((w,x)|G\times H) &= |V(G\times H)|(\frac{d^2(w|G)}{|V(G)|}+\frac{d^2(x|H)}{|V(H)|} \\ &+ 2\frac{d(w|G)}{|V(G)|}.\frac{d(x|H)}{|V(H)|}). \end{aligned}$$

Now, according to Lemma 2.14, $G\times H$ is a vertex-transitive graph, and we get

$$\begin{aligned} WW(G\times H) &= \frac{1}{2}|V(G\times H)|.(d((w,x)|G\times H)+d^2((w,x)|G\times H)) \\ &= \frac{1}{2}|V(G\times H)|^2.(\frac{d(w|G)+d^2(w|G)}{|V(G)|} \\ &+ \frac{d(x|H)+d^2(x|H)}{|V(H)|}+2\frac{d(w|G)}{|V(G)|}.\frac{d(x|H)}{|V(H)|}). \end{aligned}$$

Now, the theorem is proved. □

Corollary 2.17. *Let $G_1,G_2,\dots,G_n$ be vertex-transitive graphs with $V_i=V(G_i)$, for $1\le i\le n$, and $V=V(G)$ such that $G=G_1\times\cdots\times G_n$. Then,*

$$WW(G)=\frac{1}{2}|V|^2\sum_{i=1}^{n}[\frac{d(u_i|G_i)+d^2(u_i|G_i)}{|V_i|}+2\prod_{i=1}^{n}\frac{d(u_i|G_i)}{|v_i|^2}].$$

3. Relations between Gutman index and the (first) edge-hyper Wiener index

In this section, we give some bounds on $WW_{0e}(G)$ in terms of order and size of a graph G, and characterize the extremal graphs. We also bring some relations between Gutman index and the (first) edge-hyper Wiener index.

Lemma 3.1. *Let G be a connected graph of order n. Then*

$$WW_{0e}(G)\ge (n-1)(n-2),$$

with equality holding if and only if G is a star.

Proof. First note that G has at least $n-1$ edges, and the distance between any two edges is at least 1. Hence

$$WW_{0e}(G)=\sum_{\{f,g\}\subseteq E(G)}(d_e(f,g)+d_e^2(f,g))\ge 2\binom{|E(G)|}{2}\ge (n-1)(n-2).$$

If the equality holds, then G must have $n-1$ edges, and therefore G is a tree. Moreover, the line graph of G is complete, since the distance between any two edges is 1. Hence, G is a star, as desired. □

Definition 3.2. Let G be a connected graph and c be a real valued weight function on the vertices of G. Then, the hyper Wiener index of G with respect to c is

$$WW(G,c) = \sum_{\{x,y\}\subseteq V} c(x)c(y)(d(x,y) + d^2(x,y)).$$

The (first) edge-hyper Wiener index of a graph is connected to its Gutman index, which is defined as:

$$Gut(G) = \frac{1}{2}\sum_{u,v\in V(G)} deg(u)deg(v)d(u,v).$$

We have the Gutman type index, as well

$$Gut^{\lambda}(G) = \frac{1}{2}\sum_{u,v\in V(G)} deg(u)deg(v)d^{\lambda}(u,v).$$

Evidently, if $\lambda = 1$, then $Gut^{\lambda}(G)$ coincides with the ordinary Gutman index $Gut(G)$ (see also [14]).

Theorem 3.3. *Let G be a connected graph of order n. Then*

$$|8(WW_{0e}(G) + W_{0e}^{(2)}(G)) - 2(Gut(G) + 2Gut^{(2)}(G))| \leq 3n^4.$$

Proof. Consider the graph H obtained from G be subdividing each edge once. Consider the following functions a and b on $V(H)$, defined as follows:

$$a(v) = \begin{cases} deg(v) & v\in V(G),\\ 0 & v\in V(H)\setminus V(G), \end{cases} \qquad b(v) = \begin{cases} 0 & v\in V(G),\\ 2 & v\in V(H)\setminus V(G). \end{cases}$$

Since for any two vertices u,v of G, we have $d_H(u,v) = 2d_G(u,v)$, it follows that

$$\begin{aligned} WW(H,a) &= \sum_{\{x,y\}\subseteq V(H)} a(x)a(y)(d_H(x,y) + d_H^2(x,y)) \qquad (3.1)\\ &= \sum_{\{x,y\}\subseteq V(G)} 2deg(x)deg(y)(d_G(x,y) + 2d_G^2(x,y))\\ &= 2Gut(G) + 4Gut^{(2)}(G). \end{aligned}$$

Denote the vertex of degree 2 in $V(H)-V(G)$ that subdivides the edge $f\in E(G)$ by v_f. Then,$b(x)\neq 0$ only if $x = v_f$, for some edge f of G.

For any two edges f, g of G, we have $d_H(v_f, v_g) = 2d_G(f, g)$, and hence

$$\begin{aligned} WW(H,b) &= \sum_{\{x,y\}\subseteq V(H)-V(G)} b(x)b(y)(d_H(x,y)+d_H^2(x,y)) \quad (3.2)\\ &= \sum_{\{f,g\}\subseteq E(G)} 4(2d_{eG}(f,g)+4d_{eG}^2(f,g))\\ &= \sum_{\{f,g\}\subseteq E(G)} [4(2d_{eG}(f,g)+2d_{eG}^2(f,g))+8d_{eG}^2(f,g)]\\ &= 8WW_{0e}(G)+8W_{0e}^{(2)}(G). \end{aligned}$$

We now compare $WW(H, a)$ and $WW(H, b)$. Clearly, the weight function a is obtained from the weight function b by moving one weight unit of a vertex v_{uw} to vertex u and the other weight unit to vertex w for all $uv \in E(G)$. Hence, no weight has been moved over a distance of more than one, so no distance between two weights has been changed by more than 2. Since, we have $2|E(G)|$ weight units in total, the sum of the distances between the weight units has changed by at most $2\binom{2|E(G)|}{2}$. Hence,

$$|WW(H,a) - WW(H,b)| \leq 6\binom{2|E(G)|}{2} \leq 3n^4,$$

which in view of Eq. (3.1) and Eq. (3.2), completes the proof. □

Theorem 3.4. [3] *Let G be a connected graph of order n. Then*

$$Gut(G) \leq \frac{2^4}{5^5}n^5 + O(n^{\frac{9}{2}}),$$

and the coefficient of n^5 is best possible.

Corollary 3.5. *Let G be a connected graph of order n. Then*

$$Gut^{(2)}(G) \leq \frac{2^4}{5^6}n^6 + O(n^{\frac{11}{2}}),$$

and the coefficient of n^6 is best possible.

Corollary 3.6. *Let G be a connected graph of order n. Then*

$$WW_{0e}(G) \leq \frac{2^2}{5^6}n^6 + O(n^{\frac{11}{2}}),$$

and the coefficient of n^6 is best possible.

Definition 3.7. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let N^* denote the set of non-negative integers. The following notations are used

$$\begin{aligned} d: &\quad V \times V \to N^* \\ D: &\quad E \times E \to N^* \\ d': &\quad V \times E \to N^* \\ D': &\quad E \times V \to N^*, \end{aligned}$$

where distance between $u, v \in V$, denoted by $d(u,v)$, is defined as the length of a shortest path between u and v and for edges $e = ab$ and $f = xy$,

$$d'(u,e) = D'(e,u) = min\{d(u,a), d(u,b)\},$$

and

$$D(e,f) = min\{D'(e,x), D'(e,y)\}.$$

Similarly, we define,

$$d'^2(u,e) = D'^2(e,u) = min\{d^2(u,a), d^2(u,b)\},$$

and

$$D^2(e,f) = min\{D'^2(e,x), D'^2(e,y)\}.$$

This allows us to define the vertex-edge Wiener index of a graph as:

$$WW_{ev}(G) = \frac{1}{2} \sum_{f \in E(G)} \sum_{v \in V(G)} (d'(v,f) + d'^2(v,f)).$$

In the next theorem, we give an upper bound for the first edge-hyper Wiener index of a graph, in terms of its Gutman index and size.

Theorem 3.8. *Let G be a connected graph of size m. Then we have*

$$WW_{0e}(G) \leq \frac{1}{4}(3Gut(G) + Gut^2(G)) + m(m-2).$$

Proof. Let $f = uv$ and $g = xy$ be any two edges of G. By the definition of $D(f,g)$ and $D^2(f,g)$, it is obvious that

$$\begin{aligned} &\frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) - 1 \\ \leq &D(f,g) \\ \leq &\frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)), \end{aligned}$$

and

$$\begin{aligned} &D^2(f,g) + 2D(f,g) + 1 \\ \leq &\frac{1}{16}(d(u,x) + d(u,y) + d(v,x) + d(v,y))^2 \\ + &\frac{1}{2}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) + 1. \end{aligned}$$

Then,

$$
\begin{aligned}
WW_{0e}(G) &= \sum_{\{f,g\}\subseteq E(G)} (d_e(f,g) + d_e^2(f,g)) \\
&\leq \sum_{\{f,g\}\subseteq E(G)} (\frac{1}{4}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) + 1) \\
&+ \sum_{\{f,g\}\subseteq E(G)} \frac{1}{16}(d^2(u,x) + d^2(u,y) + d^2(v,x) + d^2(v,y)) \\
&+ \sum_{\{f,g\}\subseteq E(G)} \frac{3}{16}(d^2(u,x) + d^2(u,y) + d^2(v,x) + d^2(v,y)) \\
&+ \sum_{\{f,g\}\subseteq E(G)} \frac{1}{2}(d(u,x) + d(u,y) + d(v,x) + d(v,y)) \\
&+ \sum_{\{f,g\}\subseteq E(G)} 1 \\
&= \frac{1}{4}\sum_{uv\in E(G)} (d(u)d(v) - 1)(d(u,v) + d^2(u,v)) \\
&+ \frac{1}{4}\sum_{uv\notin E(G)} d(u)d(v)(d(u,v) + d^2(u,v)) + \sum_{\{f,g\}\subseteq E(G)} 2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\sum_{uv\in E(G)} (d(u)d(v) - 1)d(u,v) \\
&+ \frac{1}{2}\sum_{uv\notin E(G)} d(u)d(v)d(u,v) \\
&= \frac{1}{4}\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)(d(u,v) + d^2(u,v)) \\
&- \frac{1}{4}\sum_{uv\subseteq E(G)} (d(u,v) + d^2(u,v)) \\
&+ \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d(u)d(v)d(u,v) - \frac{1}{2}\sum_{uv\subseteq E(G)} d(u,v) \\
&+ \sum_{\{f,g\}\subseteq E(G)} 2 \\
&= \frac{1}{4}(3Gut(G) + Gut^2(G)) + m(m-2).
\end{aligned}
$$

□

Lemma 3.9. *Let T be a tree with n vertices. Then*

$$WW_{ev}(T) = WW(T) - (n-1)W(T) + (n-2)\binom{n}{2}.$$

Proof. Consider $v \in V(G)$. Define $f : E(T) \to V(T) - v$ such that $f(e)$ is the end vertex of e with greater distance to v. Then, f is bijective and so $d'(v,T) = d(v,T) - (n-1)$. This implies that

$$\begin{aligned} WW_{ev}(T) &= \frac{1}{2}\sum_{v\in V(T)} (d'(v,T) + d'^2(v,T)) \\ &= \frac{1}{2}\sum_{v\in V(T)} (d(v,T) + d^2(v,T) - 2(n-1)d(v,T) \\ &+ (n-1)^2 - (n-1)) \\ &= WW(T) - (n-1)W(T) + (n-2)\binom{n}{2}. \end{aligned}$$

□

4. **The second edge-hyper Wiener index**

Edge versions of the Wiener index, based on the distance between all pairs of edges in a simple connected graph G, were introduced in 2009 [9]. Two possible distances between the edges $g = uv$ and $f = zt$ of a graph G can be considered. Each of them gives rise to a corresponding (first) edge-Wiener index. The first distance, is the one based on the distance between the corresponding vertices in the line graph of G, and obviously, its related (first) edge-Wiener index is equal to the ordinary Wiener index of the line graph of G. The second distance $d_{e|G}(g,f)$, between the edges $g = uv$ and $f = zt$ of the graph G, is defined in [9] as

$$d_{e|G}(g,f) = \begin{cases} 0 & \text{if } g = f \\ max\{d_G(u,z), d_G(u,t), d_G(v,z), d_G(v,t)\} & \text{if } g \neq f \end{cases}.$$

Related to this distance, the second edge-hyper Wiener index $WW_{1e}(G)$ of G, is defined in [9] as

$$WW_{1e}(G) = \sum_{\{g,f\}\subseteq E(G)} (d_{e|G}(g,f) + d^2_{e|G}(g,f)).$$

Let $N_G(u)$ denotes the neighborhood of a vertex u in G, i.e., the set of all vertices of G adjacent to u. The degree of u in G, is the cardinality of $N_G(u)$ and is denoted by $deg_G(u)$. Let $T(G)$ and $R(G)$ denote the number of triangles in G and the number of subgraphs of G isomorphic to the 4-vertex complete graph K_4, respectively. It is easy to see that

$$T(G) = \frac{1}{3} \sum_{uv \in E(G)} |N_G(u) \cap N_G(v)|,$$

$$R(G) = \frac{1}{12} \sum_{uv \in E(G)} \sum_{z \in N_G(u) \cap N_G(v)} |N_G(u) \cap N_G(v) \cap N_G(z)|.$$

Corresponding to each triangle in G, there are 3 pairs of adjacent edges which are at distance 1 in G. So, the number of such pairs of edges in G is equal to $3T(G)$. Also, corresponding to each subgraph of G isomorphic to K_4, there are 3 pairs of nonadjacent edges which are at distance 1 in G. So, the number of such pairs of edges in G is equal to $3R(G)$. Hence, the total number of pairs of edges which are at distance 1 in G is equal to $3(T(G) + R(G))$.

Let x be a vertex of G and $g = uv$ be an edge of G. The distance $D_G(x, g)$, between the vertex x and the edge g of the graph G, is defined in [1] as

$$D_G(x, g) = max\{d_G(x, u), d_G(x, v)\}.$$

In the sequel, we will let n_i and e_i to denote the numbers of vertices and edges of simple connected graphs G_i, respectively, where $i \in \{1, 2\}$. Our aim is to compute the second edge-hyper Wiener index of the join of G_1 and G_2.

Definition 4.1. The join of two vertex-disjoint graphs G_1 and G_2, denoted $G_1 \nabla G_2$, is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 \nabla G_2) = E(G_1) \cup E(G_2) \cup S,$$

where $S = \{u_1u_2 |\ u_1 \in V(G_1), u_2 \in V(G_2)\}$. All distinct vertices of $G_1 \nabla G_2$ are either at distance 1 or 2.

Theorem 4.2. *Let G_1 and G_2 be two simple connected graphs. Then*

$$\begin{aligned} WW_{1e}(G_1 \nabla G_2) &= 6[\binom{n_1n_2}{2} + \binom{e_1}{2} + \binom{e_2}{2}] - 12(n_1e_2 + n_2e_1) \\ &+ 6n_1n_2(e_1 + e_2) - 6e_1e_2 - 12(n_2 + 1)T(G_1) \\ &- 12(n_1 + 1)T(G_2) - 12(R(G_1) + R(G_2)). \end{aligned}$$

Proof. Let E be the set of all pairs of edges of $G_1 \nabla G_2$. We partition E into six disjoint sets, as follows:

$$\begin{aligned}
E_1 &= \{\{g,f\}|\ g,f \in E(G_1)\};\\
E_2 &= \{\{g,f\}|\ g,f \in E(G_2)\};\\
E_3 &= \{\{g,f\}|\ g \in E(G_1), f \in E(G_2)\};\\
E_4 &= \{\{g,f\}|\ g \in E(G_1), f \in S\};\\
E_5 &= \{\{g,f\}|\ g \in E(G_2), f \in S\};\\
E_6 &= \{\{g,f\}|\ g,f \in S\}.
\end{aligned}$$

The second edge-hyper Wiener index of $G_1 \nabla G_2$ is obtained by summing the contributions of all pairs of edges over those six sets. We proceed to evaluate their contributions in order of increasing complexity.

The case of E_3 is the simplest. Let $\{g,f\} \in E_3$, where $g = u_1v_1 \in E(G_1)$ and $f = u_2v_2 \in E(G_2)$. Then,

$$\begin{aligned}
d_{e|G_1\nabla G_2}(g,f) &= max\{d_{G_1\nabla G_2}(u_1,u_2), d_{G_1\nabla G_2}(u_1,v_2), d_{G_1\nabla G_2}(v_1,u_2),\\
&\quad d_{G_1\nabla G_2}(v_1,v_2)\} = max\{1,1,1,1\} = 1.
\end{aligned}$$

There are e_1e_2 such pairs of edges in E_3 and each of them contributes 2 to the second edge- hyper Wiener index. Hence, the total contribution of pairs from E_3 is equal to $2e_1e_2$.

The set E_6 contains pairs of edges from S. Let $\{g,f\} \in E_6$ and $g = u_1u_2, f = v_1v_2$, where $u_1, v_2 \in V(G_1), u_2, v_2 \in V(G_2)$. Then

$$\begin{aligned}
d_{e|G_1\nabla G_2}(g,f) &= max\{d_{G_1\nabla G_2}(u_1,v_1), d_{G_1\nabla G_2}(u_1,v_2), d_{G_1\nabla G_2}(u_2,v_1),\\
&\quad d_{G_1\nabla G_2}(u_2,v_2)\}\\
&= max\{d_{G_1\nabla G_2}(u_1,v_1), 1, 1, d_{G_1\nabla G_2}(u_2,v_2)\}.
\end{aligned}$$

The total number of pairs of edges in E_6 is equal to $\binom{n_1n_2}{2}$. Among them, there are $n_1e_2+n_2e_1+2e_1e_2$ pairs that contribute 2 to the second edge-hyper Wiener index, and all other pairs contribute 6. Hence, the total contribution of pairs from E_6 is also equal to

$$6\binom{n_1n_2}{2} - 4n_1e_2 - 4n_2e_1 - 8e_1e_2.$$

Now, we compute the contribution of pairs from E_4. Let $\{g,f\} \in E_4$ and $g = u_1v_1 \in E(G_1)$ and $f = z_1u_2 \in S$, where $u_1, v_1, z_1 \in V(G_1), u \in V(G_2)$. Then,

$$\begin{aligned}
d_{e|G_1\nabla G_2}(g,f) &= max\{d_{G_1\nabla G_2}(u_1,z_1), d_{G_1\nabla G_2}(u_1,u_2), d_{G_1\nabla G_2}(v_1,z_1),\\
&\quad d_{G_1\nabla G_2}(v_1,u_2)\}\\
&= max\{d_{G_1\nabla G_2}(u_1,z_1), 1, d_{G_1\nabla G_2}(v_1,z_1), 1\}.
\end{aligned}$$

The total number of pairs from E_4 is equal to $e_1n_1n_2$. Among them there are $2e_1n_2 + 3n_2T(G_1)$ pairs that contribute 2 to the second edge-hyper Wiener index, and all other pairs contribute 6. Hence, the total contribution of pairs from E_4 is equal to

$$6e_1n_1n_2 - 8e_1n_2 - 12n_2T(G_1).$$

By symmetry, the total contribution of pairs from E_5 is equal to

$$6e_2n_1n_2 - 8e_2n_1 - 12n_1T(G_2).$$

Let $\{g, f\} \in E_1$, where $g = u_1v_1, f = z_1t_1$. Then

$$\begin{aligned} d_{e|G_1\nabla G_2}(g, f) &= max\{d_{G_1\nabla G_2}(u_1, z_1), d_{G_1\nabla G_2}(u_1, t_1), d_{G_1\nabla G_2}(v_1, z_1), \\ &\quad d_{G_1\nabla G_2}(v_1, t_1)\}. \end{aligned}$$

The total number of pairs in E_1 is equal to $\binom{e_1}{2}$. As mentioned before, $3(T(G) + R(G))$ pairs contribute 2 to the second edge-hyper Wiener index, and all other pairs contribute 6. Hence the total contribution of pairs from E_1 is equal to

$$6\binom{e_1}{2} - 12(T(G_1) + R(G_1)).$$

Again, the total contribution of E_2 is obtained by the symmetry as

$$6\binom{e_2}{2} - 12(T(G_2) + R(G_2)).$$

Now, by adding the contributions of $E_1, \ldots, E_6$ and simplifying the resulting expression, the result follows. □

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References

[1] M. Azari, A. Iranmanesh and A. Tehranian, Two topological indices of three chemical structures, *MATCH Commun. Math. Comput. Chem.* **69**(1) (2013), 69–86.

[2] N. L. Biggs, *Algebraic Graph Theory*, Cambridge Tracts in Mathematics, Cambridge University Press, London, 1974.

[3] P. Dankelmann, I. Gutman, S. Mukwembi and H. C. Swart, The edge-Wiener index of a graph, *Discrete Math.* **309**(10) (2009), 3452–3457.

[4] M. R. Darafsheh, The Wiener, Szeged and PI-index of the triangle graph, *In Lee See Keong and Hailiza Kamarul Haili (eds) 5th Asian Mathematica Conference Proceedings*, **1** (2009), 81–89.

[5] M. R. Darafsheh, Computation of topological indices of some graphs, *Acta Appl. Math.* **110**(3) (2010), 1225–1235.

[6] R. C. Entringer, D. E. Jackson and D. A. Snyder, Distance in graphs, *Czech. Math. J.* **26**(2) (1976), 283–296.

[7] C. Godsil and G. Royle, *Algebraic graph theory*, Graduate Texts in Mathematics, 207. Springer-Verlag, New York, 2001.

[8] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Second Edition. Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2011.

[9] A. Iranmanesh, I. Gutman, O. Khormali and A. Mahmiani, The edge versions of Wiener index, *MATCH Commun. Math. Comput. Chem.* **61**(3) (2009), 663–672.

[10] A. Iranmanesh, A. S. Kafrani and O. Khormali, A new version of hyper-Wiener index, *MATCH Commun. Math. Comput. Chem.* **65**(1) (2011), 113–122.

[11] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi and S. G. Wagner, Some new results on distance-based graph invariants, *European J. Combin.* **30**(5) (2009), 1149–1163.

[12] A. Soltani and A. Iranmanesh, On the edge Wiener index, *Filomat*, **28**(3) (2014), 541–549.

[13] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.* **69**(1) (1947), 17–20.

[14] B. Wu, Wiener index of line graphs, *MATCH Commun. Math. Comput. Chem.* **64**(3) (2010), 699–706.

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Journal of Algebraic Systems

MORE ON EDGE HYPER WIENER INDEX OF GRAPHS

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درباره شاخص هایپر-وینر یالی گراف‌ها

عبدالله آل‌هوز و مریم باغی‌پور
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فرض کنید $G=(V(G),E(G))$ یک گراف ساده همبند با مجموعه راسی $V(G)$ و مجموعه یالی $E(G)$ باشد. (اولین) شاخص هایپر-وینر یالی گراف G به صورت زیر تعریف می‌شود:

$$\begin{aligned} WW_e(G) &= \sum_{\{f,g\}\subseteq E(G)} (d_e(f,g|G)+d_e^{۲}(f,g|G)) \\ &= \frac{۱}{۲}\sum_{f\in E(G)} (d_e(f|G)+d_e^{۲}(f|G)), \end{aligned}$$

جایی‌که $d_e(f,g|G)$ فاصله بین یال‌های $f=xy$ و $g=uv$ در $E(G)$ بوده و $d_e(f|G)=\sum_{g\in E(G)} d_e(f,g|G)$. در این مقاله ما از روشی استفاده می‌کنیم که نظریه گروه‌ها را برای نظریه گراف به کار گرفته و محاسبه‌ی ریاضی (اولین) شاخص هایپر-وینر یالی را در رده ای خاص از گراف‌ها تسهیل می‌بخشد. همچنین، کران‌هایی بالا و پایین برای (اولین) شاخص هایپر-وینر یالی برحسب اندازه و شاخص گوتمن یک گراف ارایه می‌کنیم. در بخش آخر نیز به بررسی حاصل‌ضرب گراف‌ها پرداخته و دومین شاخص هایپر-وینر یالی را برای رده‌هایی خاص از گراف‌ها محاسبه می‌نماییم.

کلمات کلیدی: فاصله، شاخص هایپر-وینر یالی، گراف یالی، پیوند دو گراف، شاخص گوتمن، همبندی، گراف‌های یال انتقالی.