THE ZERO-DIVISOR GRAPH OF A MODULE

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Abstract. Let $R$ be a commutative ring with identity and $M$ an $R$-module. In this paper, we associate a graph to $M$, say $\Gamma(RM)$, such that when $M = R$, $\Gamma(RM)$ coincide with the zero-divisor graph of $R$. Many well-known results by D. F. Anderson and P. S. Livingston, have been generalized for $\Gamma(RM)$. We will show that $\Gamma(RM)$ is connected with $\text{diam}(\Gamma(RM)) \leq 3$, and if $\Gamma(RM)$ contains a cycle, then $\text{gr}(\Gamma(RM)) \leq 4$. We will also show that $\Gamma(RM) = \emptyset$ if and only if $M$ is a prime module. Among other results, it is shown that for a reduced module $M$ satisfying DCC on cyclic submodules, $\text{gr}(\Gamma(RM)) = \infty$ if and only if $\Gamma(RM)$ is a star graph. Finally, we study the zero-divisor graph of free $R$-modules.

1. Introduction

Throughout the paper, $R$ is a commutative ring with identity and $RM$ is a unitary $R$-module. Let $Z(R)$ be the set of zero-divisors of $R$. Associating graphs to algebraic structures has become an exciting research topic in the last twenty years. There are many papers on assigning a graph to a ring; see for instance, [2, 3, 6, 9, 10, 18]. Most of the work has focused on the zero-divisor graph. The concept of a zero-divisor graph of a ring $R$ was first introduced by Beck in [7], where he was mainly interested in coloring. This investigation of colorings of a commutative ring was then continued by Anderson and Naseer.

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in [5]. In [3], Anderson and Livingston associate a graph, $\Gamma(R)$, with vertices $Z^*(R) := Z(R) \setminus \{0\}$, the set of non-zero zero-divisors of $R$, and for distinct $x, y \in Z^*(R)$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$. The zero-divisor graphs of commutative rings have been extensively studied by many authors, and become a major field of research; see for instance, [11, 22] and two survey papers [1, 13]. In [22], Redmond extended the zero-divisor graph of a commutative ring to an ideal-based zero-divisor graph of a commutative ring. This notion of zero-divisor graph was also studied in [19, 21, 25]. The graph of zero-divisors for commutative rings has been generalized to modules over commutative rings; see for instance, [8, 16, 23].

In this paper, we introduce a new (and natural) definition of the zero-divisor graph for modules. As any suitable generalization, many of well known results about zero-divisor graph of rings have been generalized to modules. The concept of a zero-divisor elements of a ring, has been generalized to a module (see for example [24], or any other book in commutative algebra):

$$Zdv(RM) = \{r \in R | rx = 0 \text{ for some non-zero } x \in M\}.$$  

Let $N$ and $K$ be two submodules of an $R$-module $M$. Then, $(N : K) := \{r \in R | rK \subseteq N\}$ is an ideal of $R$. The ideal $(0 : M)$ is called the annihilator of $M$ and is denoted by $\text{Ann}(M)$; for $x \in M$, we may write $\text{Ann}(x)$ for the ideal $\text{Ann}(Rx)$. We give a new generalization of the concept of zero-divisor elements in rings to modules:

**Definition 1.1.** Let $M$ be an $R$-module. The set of the zero-divisors of $M$ is:

$$Z(RM) := \{x \in M | x \in \text{Ann}(y)M \text{ or } y \in \text{Ann}(x)M \text{ for some } 0 \neq y \in M\}.$$  

We note that when $M = R$, this concept coincides with the set of zero-divisor elements of $R$.

**Definition 1.2.** Let $M$ be an $R$-module. We define an undirected graph $\Gamma(RM)$ with vertices $Z^*(RM) := Z(RM) \setminus \{0\}$, where $x-y$ is an edge between distinct vertices $x$ and $y$ if and only if $x \in \text{Ann}(y)M$ or $y \in \text{Ann}(x)M$.

We note that, the graph $\Gamma(RM)$ is exactly a generalization of the zero-divisor graph of $R$ (i.e., $\Gamma(RR) = \Gamma(R)$). As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo $n$, respectively. The zero-divisor graphs of some $\mathbb{Z}$-modules are presented in figure 1.

Let $G$ be a graph with the vertex set $V(G)$. For two distinct vertices $x$ and $y$ of $V(G)$ the notation $x-y$ means that $x$ and $y$ are adjacent.
For \( x \in V(G) \), we denote by \( N_G(x) \) the set of all vertices of \( G \) adjacent to \( x \). Also, the size of \( N_G(x) \) is denoted by \( \deg_G(x) \) and it is called the degree of \( x \). A walk of length \( n \) in a graph \( G \) between two vertices \( x, y \) is an ordered list of vertices \( x = x_0, x_1, ..., x_n = y \) such that \( x_{i-1} \) is adjacent to \( x_i \), for \( i = 1, ..., n \). We denote this walk by 

\[ x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n. \]

If the vertices in a walk are all distinct, it defines a path in \( G \). A cycle is a path \( x_0 \rightarrow \cdots \rightarrow x_n \) with an extra edge \( x_0 \rightarrow x_n \). The girth of \( G \), denoted by \( \text{gr}(G) \), is the length of a shortest cycle in \( G \) (\( \text{gr}(G) = \infty \), if \( G \) has no cycle). A graph \( G \) is called connected if for any vertices \( x \) and \( y \) of \( G \) there exists a path between \( x \) and \( y \). For \( x, y \in V(G) \), the distance between \( x \) and \( y \), denoted by \( d(x, y) \), is the length of a shortest path between \( x \) and \( y \). The greatest distance between any two vertices in \( G \), is the diameter of \( G \), denoted by \( \text{diam}(G) \).

A graph \( G \) is called bipartite if \( V(G) \) admits a partition into two classes such that vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent, is called a complete bipartite graph. Let \( K^{m,n} \) denote the complete bipartite graph on two nonempty disjoint sets \( V_1 \) and \( V_2 \) with \( |V_1| = m \) and \( |V_2| = n \) (we allow \( m \) and \( n \) to be infinite cardinals). A \( K^{1,n} \) graph will often be called a star graph.
The motivation of this paper is the study of interplay between the graph-theoretic properties of \( \Gamma(RM) \) and the module-theoretic properties of \( RM \). The organization of the paper is as follows: In Section 2 of this paper, we give some basic properties of \( \Gamma(RM) \). In Section 3, we determine when the graph \( \Gamma(RM) \) is bipartite. In Section 4, we study \( \Gamma(RF) \), where \( F \) is a free \( R \)-module. Finally, in Section 5, we study \( \Gamma(RM) \), where \( M \) is a multiplication \( R \)-module.

We follow standard notations and terminologies from graph theory [12] and module theory [15].

2. Basic Properties of \( \Gamma(RM) \)

We begin with the following evident proposition.

**Proposition 2.1.** Let \( M \) be an \( R \)-module and \( I \) be an ideal of \( R \). Then

1. \( \Gamma(RM) = \Gamma(R/\text{Ann}(M))M \),
2. \( \Gamma(R(R/I)) = \Gamma(R/I) \).

Let \( I \) be an arbitrary index set, and let \( \{M_i|i \in I\} \) be a family of \( R \)-modules. The direct product \( \prod_{i \in I} M_i = \{(x_i)_{i \in I} | x_i \in M_i\} \) is an \( R \)-module. We also note that a direct product of rings is a ring endowed with componentwise operations. We are going to explain the relationship between the zero-divisor graph of \( \prod_{i \in I} R/m_i \) as \( R \)-module and as ring.

**Theorem 2.2.** Let \( \{m_i|i \in I\} \) be a family of maximal ideals of \( R \) and \( RM = \prod_{i \in I} R/m_i \). Then

1. \( \Gamma(RM) \) is a subgraph of \( \Gamma(\prod_{i \in I} R/m_i) \),
2. \( \Gamma(RM) = \Gamma(\prod_{i \in I} R/m_i) \) if and only if the \( m_i \)'s are distinct ideals.

**Proof.** (1): Let \( (x_i)_I \) and \( (y_i)_I \) be two adjacent vertices of \( \Gamma(RM) \). Without loss of generality, we may assume that \( (x_i)_I \in \text{Ann}(y_i)_I \). Then, \( x_i \in \cap_{j \in I} \text{Ann}(y_j)_I \subseteq \text{Ann}(y_i)_I \), for all \( i \in I \). It then follows that \( x_i y_i = 0 \) and hence \( (x_i)_I(y_i)_I = 0 \). So, \( (x_i)_I \) and \( (y_i)_I \) are two adjacent vertices of \( \Gamma(\prod_{i \in I} R/m_i) \).

(2)\( \Rightarrow \): Suppose \( m_r = m_s \), for some distinct elements \( r, s \in I \). Let \( x_r := 1 \) and \( x_i := 0 \) for all \( i \neq r \) and let \( y_s := 1 \) and \( y_i := 0 \) for all \( i \neq s \). Then, \( (x_i)_I(y_i)_I = 0 \), and hence \( (x_i)_I \) and \( (y_i)_I \) are not adjacent in \( \Gamma(RM) \), since \( (x_i)_I \notin \text{Ann}(y_i)_I M \) and \( (y_i)_I \notin \text{Ann}(x_i)_I M \).

\( \Leftarrow \): Let \( (x_i)_I \) and \( (y_i)_I \) be two adjacent vertices of \( \Gamma(\prod_{i \in I} R/m_i) \). We have

\[
\text{Ann}(x_i)_I M = \bigcap_{i \in I} \text{Ann}(x_i)_I M = \prod_{i \in I} N_i,
\]
where $N_i = 0$ if $x_i \neq 0$ and $N_i = M_i$ if $x_i = 0$. Since $(x_i)_I(y_i)_I = 0$, the assumption $y_i \neq 0$ implies that $x_i = 0$ and hence $(y_i)_I \in \prod_{i \in I} N_i$. It then follows that $(x_i)_I$ and $(y_i)_I$ are two adjacent vertices of $\Gamma(\mathcal{R}M)$. Hence, the assertion follows from Part (1).

It is well-known that a ring $R$ is a domain if and only if the zero-divisor graph $\Gamma(R)$ is empty. The following proposition is a natural generalization of this fact. We recall that an $\mathcal{R}$-module $M \neq 0$ is called a prime module if its zero submodule is prime, i.e., $rx = 0$ for $x \in M$, $r \in R$ implies that $x = 0$ or $rM = (0)$ (see [14] and [20]).

**Proposition 2.3.** Let $M$ be an $R$-module. Then the following are equivalent:

1. $\Gamma(\mathcal{R}M) = \emptyset$ i.e., $Z(M) = \{0\}$,
2. $\text{Zdv}(M) = \text{Ann}(M)$,
3. $M$ is a prime $R$-module.

**Proof.** (1)$\Rightarrow$(3) Suppose that $\Gamma(\mathcal{R}M) = \emptyset$. If $M$ is not a prime module, then there exist $r \in R \setminus \text{Ann}(M)$ and non-zero element $x \in M$ such that $rx = 0$. Since $r \notin \text{Ann}(M)$, there exists a non-zero element $y \in M$ such that $ry \neq 0$. It follows that $ry - x$ is an edge of $\Gamma(\mathcal{R}M)$ and hence $\Gamma(\mathcal{R}M) \neq \emptyset$, which is a contradiction.

(3)$\Rightarrow$(1) Suppose that $M$ is a prime $R$-module. If $\Gamma(\mathcal{R}M) \neq \emptyset$, then there exist $x, y \in Z^*(\mathcal{R}M)$ such that $x \in \text{Ann}(y)M$. Therefore, there exist $r_1, \ldots, r_n \in \text{Ann}(y)$ and $z_1, \ldots, z_n \in M$ such that $x = r_1z_1 + \cdots + r_nz_n$. Since, $r_iy = 0$ for all $1 \leq i \leq n$ and $M$ is prime, we have $r_iM = 0$ for all $1 \leq i \leq n$. This implies that $x = 0$, which is a contradiction.

(2)$\Leftrightarrow$(3) Follows easily from the definition of prime modules. □

**Corollary 2.4.** Let $R$ be a ring. Then $R$ is a field if and only if $\Gamma(\mathcal{R}M) = \emptyset$ for every $R$-module $M$.

**Proof.** If $R$ is field, then proposition 2.3 implies that $\Gamma(\mathcal{R}M) = \emptyset$. Now, suppose that $\Gamma(\mathcal{R}M) = \emptyset$, for every $R$-module $M$. Let $\mathfrak{m}$ be a non-zero maximal ideal of $R$ and $0 \neq x \in \mathfrak{m}$. Set $M := R/\mathfrak{m} \times R$. Then, $(0, x) \in \text{Ann}(1 + \mathfrak{m}, 0)M$. Therefore, $(0, x)$ is adjacent to $(1 + \mathfrak{m}, 0)$. Thus, $\Gamma(\mathcal{R}M) \neq \emptyset$, which is a contradiction. Therefore, $\mathfrak{m} = 0$ and hence $R$ is a field. □

A semisimple module $M$ is said to be homogeneous if $M$ is a direct sum of pairwise isomorphic simple submodules.

**Corollary 2.5.** Let $M$ be a homogeneous semisimple $R$-module. Then, $\Gamma(\mathcal{R}M) = \emptyset$. 


Proof. Since Ann(M) is a maximal ideal of R, M is vector space over R/Ann(M). Hence, the assertion follows easily from Proposition 2.3. □

We are now in a good position to bring a generalization of [3, Theorem 2.2].

Theorem 2.6. Let M be an R-module. Then, \( \Gamma(RM) \) is finite if and only if either M is finite or a prime module. In particular, if \( 1 \leq |\Gamma(RM)| < \infty \), then M is finite and is not a prime module.

Proof. (⇒): Suppose that \( \Gamma(RM) \) is finite and nonempty. Then, there are non-zero elements \( x, y \in M \) such that \( x \in \text{Ann}(y)M \). Therefore, there exist \( r_1, \ldots, r_n \in \text{Ann}(y) \) and \( z_1, \ldots, z_n \in M \) such that \( x = r_1z_1 + \cdots + r_nz_n \). Since \( x \neq 0 \), we have \( r_iz_i \neq 0 \) for some \( 1 \leq i \leq n \). Let \( L = r_iM \). Then \( L \subseteq Z(RM) \) is finite. If M is infinite, then there exists \( x_0 \in L \) such that \( A := \{ m \in M | r_i m = x_0 \} \) is infinite. If \( m_0 \) is a fixed element of A, then \( N := \{ m_0 - m | m \in A, m \neq m_0 \} \) is an infinite subset of A. For any element \( m_0 - m \in N \), we have \( r_i(m_0 - m) = 0 \). Thus \( x_0 - (m_0 - m) \) is an edge in \( \Gamma(RM) \) and hence \( \Gamma(RM) \) is infinite, a contradiction. Thus M must be finite.

(⇐): If M is finite, there is nothing to prove, also if M is prime, then the assertion follows from Proposition 2.3. □

Corollary 2.7. Let M be an R-module such that \( \Gamma(RM) \neq \emptyset \). If every vertex of \( \Gamma(RM) \) has finite degree, then M is a finite module.

Proof. The assertion follows from the proof of the theorem 2.6. □

The following lemma has a key role in the proof of our main results in the sequel.

Lemma 2.8. Let M be an R-module, \( x, y \in M \) and \( r \in R \). If \( x \rightarrow y \) is an edge in \( \Gamma(RM) \), then either \( ry \in \{0, x\} \) or \( x \rightarrow ry \) is an edge in \( \Gamma(RM) \).

Proof. Let x and y be two adjacent vertices of \( \Gamma(RM) \) and let \( ry \notin \{0, x\} \). If \( x \in \text{Ann}(y)M \), then \( x \in \text{Ann}(ry)M \), and hence, x and ry are adjacent. If \( y \in \text{Ann}(x)M \), then \( ry \in \text{Ann}(x)M \), and hence, x and ry are adjacent. This completes the proof. □

The next result is a generalization of [3, Theorem 2.3].

Theorem 2.9. Let M be an R-module. Then \( \Gamma(RM) \) is connected with \( \text{diam}(\Gamma(RM)) \leq 3 \).

Proof. Let x and y be distinct vertices of \( \Gamma(RM) \). If either \( x \in \text{Ann}(y)M \) or \( y \in \text{Ann}(x)M \), then \( d(x, y) = 1 \). So, suppose that \( d(x, y) \neq 1 \). There
exists a vertex $x'$ of $\Gamma(RM)$ such that $x \in \text{Ann}(x'M)$ or $x' \in \text{Ann}(xM)$.

We consider the following two cases:

Case 1: There exists a vertex $y'$ of $\Gamma(RM)$ such that $y \in \text{Ann}(y'M)$.

Then, there exist $r_1, \ldots, r_n \in \text{Ann}(y')$ and $z_1, \ldots, z_n \in M$ such that $y = r_1z_1 + \cdots + r_nz_n$. If $r_ix' = 0$ for all $i$, then $x-x'-y$ is a path of length 2. If $r_ix' \neq 0$ for some $1 \leq i \leq n$, then by Lemma 2.8, $x-r_ix'-y'y$ is a walk, and hence $d(x, y) \leq 3$.

Case 2: There exists a vertex $y'$ of $\Gamma(RM)$ such that $y' \in \text{Ann}(y'M)$.

Then, there exist $r_1, \ldots, r_n \in \text{Ann}(y)$ and $z_1, \ldots, z_n \in M$ such that $y' = r_1z_1 + \cdots + r_nz_n$. If $r_ix = 0$ for all $i$, then $x-y-y$ is a path of length 2. If $r_ix \neq 0$ for some $1 \leq i \leq n$, then $x-x'-r_ix-y$ is a walk, and hence $d(x, y) \leq 3$. □

Theorem 2.10. Let $M$ be an $R$-module. If $\Gamma(RM)$ contains a cycle, then

$$\text{gr}(\Gamma(RM)) \leq 4.$$ \[Proof.\] Let $x_0-x_1-x_2-\cdots-x_n-x_0$ be a cycle in $\Gamma(RM)$. If $n \leq 4$, we are done. So, suppose that $n \geq 5$. We consider the following two cases:

Case 1: $x_{n-1} \in \text{Ann}(x_n)M$. Then, there exist $r_1, \ldots, r_m \in \text{Ann}(x_n)$ and $z_1, \ldots, z_m \in M$ such that $x_{n-1} = r_1z_1 + \cdots + r_mz_m$. If $r_ix_i = 0$ for all $1 \leq i \leq n$, then $x_1-x_{n-1}$ is an edge, and hence $x_1-x_{n-1}-x_n-x_0-x_1$ is a cycle of length 4. Suppose that $r_ix_i$ is not an edge for some $1 \leq i \leq m$. If $r_ix_i = x_0$, then $x_0-x_2$ is an edge and hence $x_0-x_1-x_2-x_0$ is a cycle of length 3. If $r_ix_i = x_n$, then $x_2-x_n$ is an edge and hence $x_2-x_1-x_0-x_n-x_2$ is a cycle of length 4. Suppose that $r_ix_i \neq \{x_0, x_n\}$. Then $x_0-r_ix_1-x_n-x_0$ is a cycle of length 3.

Case 2: $x_n \in \text{Ann}(x_{n-1})M$. Then, there exist $r_1, \ldots, r_m \in \text{Ann}(x_n)$ and $z_1, \ldots, z_m \in M$ such that $x_n = r_1z_1 + \cdots + r_mz_m$. If $r_ix_i = 0$ for all $1 \leq i \leq m$, then $x_1-x_n$ is an edge and hence $x_n-x_0-x_1-x_n$ is a cycle of length 3. Suppose that $r_ix_i$ is not an edge for some $1 \leq i \leq m$. If $r_ix_i = x_0$, then $x_0-x_2$ is an edge and hence $x_0-x_1-x_2-x_0$ is a cycle of length 3. If $r_ix_i = x_{n-1}$, then $x_0-x_{n-1}$ is an edge and hence $x_0-x_n-x_{n-1}-x_0$ is a cycle of length 3. So, suppose that $r_ix_i \neq \{x_0, x_{n-1}\}$. Then, $x_0-r_ix_1-x_{n-1}-x_n-x_0$ is a cycle of length 4. □

In the following theorem, we answer to the question that “when does $\Gamma(RM)$ contain a cycle?”.

Theorem 2.11. Let $M$ be an $R$-module. If $\Gamma(RM)$ has a path of length four, then $\Gamma(RM)$ has a cycle.

Proof. Let $x_1-x_2-x_3-x_4-x_5$ be a path of length four. We consider the following two cases:
Case 1: \(x_1 \in \text{Ann}(x_2)M\). Then, there exist \(r_1, \ldots, r_n \in \text{Ann}(x_2)\) and \(y_1, \ldots, y_n \in M\) such that \(x_1 = r_1 y_1 + \cdots + r_n y_n\). If \(r_i x_4 = 0\) for all \(1 \leq i \leq n\), then \(x_1\) and \(x_4\) are adjacent and hence \(x_1-x_2-x_3-x_4-x_1\) is a cycle. Now, let \(z := r_i x_4 \neq 0\) for some \(1 \leq i \leq n\). Then, we have the following subcases:

Subcase 1.1: \(z = x_1\). Then, \(x_1-x_2-x_3-x_4-x_5-x_1\) is a cycle.

Subcase 1.2: \(z = x_2\). Then, \(x_2-x_3-x_4-x_5-x_2\) is a cycle.

Subcase 1.3: \(z = x_3\). Then, \(x_3-x_4-x_5-x_3\) is a cycle.

Subcase 1.4: \(z = x_4\). Then, \(x_2-x_3-x_4-x_2\) is a cycle.

Subcase 1.5: \(z = x_5\). Then, \(x_2-x_3-x_4-x_2\) is a cycle.

Subcase 1.6: \(z \notin \{x_1, x_2, x_3, x_4, x_5\}\). Then, \(x_2-x_3-x_4-x_5-z-x_2\) is a cycle.

Case 2: \(x_2 \in \text{Ann}(x_1)M\). So there exist \(r_1, \ldots, r_n \in \text{Ann}(x_1)\) and \(y_1, \ldots, y_n \in M\) such that \(x_2 = r_1 y_1 + \cdots + r_n y_n\). If \(r_i x_4 = 0\) for all \(1 \leq i \leq n\), then \(x_2\) and \(x_4\) are adjacent and hence \(x_2-x_3-x_4-x_2\) is a cycle. Now, let \(z := r_i x_4 \neq 0\) for some \(1 \leq i \leq n\). Then, we have the following subcases:

Subcase 2.1: \(z = x_1\). Then, \(x_1-x_2-x_3-x_1\) is a cycle.

Subcase 2.2: \(z = x_2\). Then, \(x_2-x_3-x_4-x_2\) is a cycle.

Subcase 2.3: \(z = x_3\). Then, \(x_3-x_4-x_3\) is a cycle.

Subcase 2.4: \(z = x_4\). Then, \(x_1-x_2-x_3-x_1\) is a cycle.

Subcase 2.5: \(z = x_5\). Then, \(x_1-x_2-x_3-x_4-x_5-x_1\) is a cycle.

Subcase 2.6: \(z \notin \{x_1, x_2, x_3, x_4, x_5\}\). Then, \(x_3-x_4-x_5-z-x_3\) is a cycle.

So, the proof is complete. \(\square\)

3. Bipartite Graphs

In [17], the authors showed that a zero-divisor semigroup graph is bipartite if and only if it contains no triangles. The following theorem is an analogous of this result.

**Theorem 3.1.** Let \(M\) be an \(R\)-module. Then \(\Gamma(\text{Ann}(M))\) is bipartite if and only if it contains no triangles.

**Proof.** \(\Rightarrow:\) Follows immediately from the fact that any bipartite graph contains no cycles of odd length.

\(\Leftarrow:\) We will show that for every cycle of odd length \(2n + 1 \geq 5\), there exists a cycle with length \(2m + 1\) such that \(m < n\). Suppose that \(n \geq 2\) and \(x_1-x_2-\cdots-x_{2n+1}-x_1\) is a cycle with odd length \(2n + 1\). Since \(x_1\) is adjacent to \(x_2\), we have the following two cases:

**Case 1:** \(x_1 \in \text{Ann}(x_2)M\). So, there exist \(r_1, \ldots, r_t \in \text{Ann}(x_2)\) and \(y_1, \ldots, y_t \in M\) such that \(x_1 = r_1 y_1 + \cdots + r_t y_t\). If \(r_i x_4 = 0\) for all \(1 \leq i \leq t\), then \(x_1\) is adjacent to \(x_4\) and hence \(x_1-x_4-x_5-\cdots-x_{2n+1}-x_1\) is
a cycle with odd length $2n - 1$. Now, suppose that $r_jx_4 \neq 0$ for some $1 \leq j \leq t$. Let $z := r_jx_4$. We consider the following three subcases:

Subcase 1.1: $z = x_2$. Then $x_1 - z - x_5 - \cdots - x_{2n+1} - x_1$ is a cycle with odd length $2n - 1$.

Subcase 1.2: $z = x_3$. Then $x_3 - x_4 - x_5 - x_3$ is a triangle.

Subcase 1.3: $z \notin \{x_2, x_3\}$. Then $x_3 - z - x_2 - x_3$ is a triangle.

**Case 2:** $x_2 \in \text{Ann}(x_1)M$. So, there exist $r_1, \ldots, r_t \in \text{Ann}(x_1)$ and $y_1, \ldots, y_t \in M$ such that $x_2 = r_1y_1 + \cdots + r_ty_t$. If $r_i x_4 = 0$ for all $1 \leq i \leq t$, then $x_2$ is adjacent to $x_4$ and hence we have a triangle. Now suppose that $r_jx_4 \neq 0$ for some $1 \leq j \leq t$. Let $z := r_jx_4$. Then, $x_1 - z - x_5 - \cdots - x_{2n+1} - x_1$ is a cycle with odd length $2n - 1$.

So, by induction on $n$, $\Gamma(RM)$ contains a triangle. □

We recall that an $R$-module $M$ is called reduced if whenever $r^2x = 0$ (where $r \in R$ and $x \in M$), then $rx = 0$. A submodule $N$ of an $R$-module $M$ is called essential (or large) in $M$ if, for every non-zero submodule $K$ of $M$, we have $N \cap K \neq 0$.

**Theorem 3.2.** Let $M$ be a reduced $R$-module satisfying DCC on cyclic submodules and let $\Gamma(RM)$ be a bipartite graph with parts $V_1$ and $V_2$. Let $\overline{V}_1 = V_1 \cup \{0\}$ and $\overline{V}_2 = V_2 \cup \{0\}$. Then

1. $\overline{V}_1$ and $\overline{V}_2$ are submodules of $M$.
2. $\overline{V}_1 \oplus \overline{V}_2$ is an essential submodule of $M$.

**Proof.** (1): We will show that $\overline{V}_1$ is a submodule of $M$. Let $x, y \in \overline{V}_1$. First we show that $x - y \in \overline{V}_1$. If $x = y$, we are done. Now, let $x \neq y$. If $x$ or $y$ is equal to zero, then $x - y \in \overline{V}_1$. So, we may assume that neither $x$ nor $y$ is zero. There exist $x', y' \in V_2$ such that $x, y$ are adjacent to $x', y'$, respectively. We consider the following two cases:

**Case 1:** $x' \in \text{Ann}(x)M$ and $y' \in \text{Ann}(y)M$. Without loss of generality, we may assume that $x' = rx_1$ and $y' = sy_1$, where $r \in \text{Ann}(x)$, $s \in \text{Ann}(y)$ and $x_1, y_1 \in M$. Let $z := sx_1$. We claim that $z \neq 0$. If $z = 0$, then $x'$ and $y'$ are adjacent and hence $x' = y'$, since $x', y' \in V_2$. It then follows that $s^2y_1 = sx_1 = 0$ and hence $y' = sy_1 = 0$, which is a contradiction. So, $z \neq 0$. Since $z \in \text{Ann}(x)M \cap \text{Ann}(y)M$, we must have $z \in V_2$. If $z = x - y$, then $r^2s^2x_1 = rsx - rsy = 0$. Since $M$ is reduced, we have $z = 0$, a contradiction. So $z \neq x - y$. On the other hand, $z \in \text{Ann}(x - y)M$, and hence $x - y \in \overline{V}_1$.

**Case 2:** $x \in \text{Ann}(x')M$ and $y' \in \text{Ann}(y)M$. Then there exist are $r_1, \ldots, r_n \in \text{Ann}(x')$ and $x_1, \ldots, x_n \in M$ such that $x = r_1x_1 + \cdots + r_nx_n$ and again without loss of generality, we may assume that $y' = sy_1$, for some $s \in \text{Ann}(y)$ and $y_1 \in M$. Let $z_0 := sx$. If $z_0 = 0$, then $0 \neq y' \in \text{Ann}(x - y)M$ and hence $x - y \in \overline{V}_1$. Now, let $z_0 \neq 0$. 
Consider the following ascending chain of cyclic submodules:

\[ Rz_0 \supseteq Rr_1z_0 \supseteq Rr_1^2z_0 \supseteq \cdots. \]

Suppose that \( Rz_0 = Rr_1z_0 \). Then, there exists \( a \in R \) such that \( z_0 = ar_1z_0 \). Since \( M \) is reduced, \( z_0 \neq y \) and hence \( z_0 - y \) is an edge in \( V_1 \), which is a contradiction. Let \( n_1 \geq 1 \) be the smallest integer number such that \( Rr_1^{n_1}z_0 = Rr_1^{n_1+1}z_0 \). There exists \( a_1 \in R \) such that \( r_1^{n_1}z_0 = a_1r_1^{n_1+1}z_0 \). Set \( z_1 = (r_1^{n_1-1} - a_1r_1^{n_1})z_0 \). Then, \( z_1 \neq 0 \) and we have the following ascending chain of cyclic submodules:

\[ Rz_1 \supseteq Rr_1z_1 \supseteq Rr_1^2z_1 \supseteq \cdots. \]

Let \( n_2 \geq 1 \) be the smallest integer number such that \( Rr_2^{n_2}z_1 = Rr_2^{n_2+1}z_1 \). There exists \( a_2 \in R \) such that \( r_2^{n_2}z_1 = a_2r_2^{n_2+1}z_1 \). Set \( z_2 = (r_2^{n_2-1} - a_2r_2^{n_2})z_1 \). By continuing this process, we have \( z_n = (r_n^{n_2-1} - a_nr_n^{n_2})z_{n-1} \). We have \( z_n \neq 0 \) and

\[
\begin{aligned}
z_n &\in (\text{Ann}(r_1x_1) \cap \cdots \cap \text{Ann}(r_nx_n) \cap \text{Ann}(y))M \\
&\subseteq (\text{Ann}(r_1x_1 + \cdots + r_nx_n) \cap (\text{Ann}(y)))M \\
&\subseteq (\text{Ann}(x) \cap \text{Ann}(y))M \\
&\subseteq \text{Ann}(x - y)M.
\end{aligned}
\]

It follows that \( z_n \in V_2 \) and hence \( x - y \in V_1 \).

**Case 3**: \( x^* \in \text{Ann}(x)M \) and \( y \in \text{Ann}(y')M \). The proof of this case is similar to that of Case 2.

**Case 4**: \( x \in \text{Ann}(x')M \) and \( y \in \text{Ann}(y')M \). Then there exist \( r_1, \ldots , r_n \in \text{Ann}(x') \) and \( x_1, \ldots , x_n \in M \) such that \( x = r_1x_1 + \cdots + r_nx_n \). Let \( z_0 := y' \). Consider the following ascending chain of cyclic submodules:

\[ Rz_0 \supseteq Rr_1z_0 \supseteq Rr_1^2z_0 \supseteq \cdots. \]

Suppose that \( Rz_0 = Rr_1z_0 \). Then, there exists \( a \in R \) such that \( z_0 = ar_1z_0 \). Since \( M \) is reduced, \( z_0 \neq y' \) and hence \( z_0 - y' \) is an edge in \( V_2 \), which is a contradiction. Let \( n_1 \geq 1 \) be the smallest integer number such that \( Rr_1^{n_1}z_0 = Rr_1^{n_1+1}z_0 \). There exists \( a_1 \in R \) such that \( r_1^{n_1}z_0 = a_1r_1^{n_1+1}z_0 \). Set \( z_1 = (r_1^{n_1-1} - a_1r_1^{n_1})z_0 \). We have \( z_1 \neq 0 \) and the following ascending chain of cyclic submodules:

\[ Rz_1 \supseteq Rr_1z_1 \supseteq Rr_1^2z_1 \supseteq \cdots. \]

Let \( n_2 \geq 1 \) be the smallest integer number such that \( Rr_2^{n_2}z_1 = Rr_2^{n_2+1}z_1 \). There exists \( a_2 \in R \) such that \( r_2^{n_2}z_1 = a_2r_2^{n_2+1}z_1 \). Set \( z_2 = (r_2^{n_2-1} - a_2r_2^{n_2})z_1 \). By continuing this process we have \( z_n = (r_n^{n_2-1} - a_nr_n^{n_2})z_{n-1} \). We have \( x \in \text{Ann}(z_n)M \), \( r_i \in \text{Ann}(z_n) \), for all \( 1 \leq i \leq n \). We also have \( y \in \text{Ann}(y')M \subseteq \text{Ann}(z_n)M \). Therefore, \( z_n \in V_2 \). If \( z_n \neq x - y \), then
$x - y \in V_1$, since $x - y \in \text{Ann}(z_n)M$. If $z_n = x - y$, then $x \in \text{Ann}(x - y)$ and $y \in \text{Ann}(x - y)$. It follows that $x - y \in V_1$. Now, let $r \in R$ and $x \in V_1$ such that $rx \neq 0$. We show that $rx \in V_1$. There exists $y \in V_2$ such $x$ is adjacent to $y$. We have the following two cases:

**Case 1:** $y \in \text{Ann}(x)M$. Without loss of generality, we may assume that $y = r_1z_1$, for some $r_1 \in \text{Ann}(x)$ and $z_1 \in M$. If $rx = r_1z_1$, then $r_1^2z_1 = rr_1x = 0$ and hence $rx = 0$, which is a contradiction. So, $rx \neq r_1z_1$. Since $rx$ is adjacent to $r_1z_1$ and $r_1z_1 \in V_2$, we have $rx \in V_1$.

**Case 2:** $x \in \text{Ann}(x)M$. Then there exist $r_1, \ldots, r_n \in \text{Ann}(x)$ and $z_1, \ldots, z_n \in M$ such that $x = r_1z_1 + \cdots + r_nz_n$. We may assume $r_i z_i \neq 0$ for all $1 \leq i \leq n$. Let $1 \leq i \leq n$. We claim that $r_i z_i \in V_1$. If $r_i z_i = y$, then $r_i^2z_i = 0$, and hence $r_i z_i = 0$, a contradiction. Since $r_i z_i$ is adjacent to $y$, we must have $r_i z_i \in V_1$. So, $x = r_1z_1 + \cdots + r_nz_n \in \overline{V}_1$. It then follows that $\overline{V}_1$ is a submodule of $M$ and a similar argument shows that $\overline{V}_2$ is a submodule of $M$.

(2): Let $x \in M \setminus (\overline{V}_1 \oplus \overline{V}_2)$. Since $\Gamma(RM)$ is bipartite, there exist $x_0, y_0 \in \overline{V}_1 \cup \overline{V}_2$ such that $x_0 \in \text{Ann}(y_0)M$. So, there exist $r_1, \ldots, r_n \in \text{Ann}(y_0)$ and $x_1, \ldots, x_n \in M$ such that $x_0 = r_1x_1 + \cdots + r_nx_n$. There exists $1 \leq i \leq n$ such that $r_i x_i \neq 0$. Since $M$ is reduced, the assumption $r_i x = 0$ implies that $x \in V_1 \cup V_2$, which is a contradiction. So, $r_i x \neq 0$.

Consider the following ascending chain of cyclic submodules:

$$Rx \supseteq Rr_i x \supseteq Rr_i^2 x \supseteq \cdots.$$ 

Suppose that $Rx = Rr_i x$. Then, $x \in V_1 \cup V_2$, which is a contradiction. Let $n \geq 1$ be the smallest integer number such that $Rr_i^n x = Rr_i^{n+1} x$. There exists $a \in R$ such that $r_i^n x = ar_i^{n+1} x$. Set $z = (r_i^{n+1} - ar_i^n)x$. We have $0 \neq z \in (V_1 \cup V_2)$ and so $\overline{V}_1 \oplus \overline{V}_2$ is an essential submodule of $M$. \hfill \Box

**Theorem 3.3.** Let $M$ be a reduced $R$-module satisfying DCC on cyclic submodules. If $\Gamma(RM)$ is a bipartite graph, then it is a complete bipartite graph.

**Proof.** Let $\Gamma(RM)$ be a bipartite graph with parts $V_1$ and $V_2$. Let $x \in V_1$ and $y \in V_2$. We will show that $x$ and $y$ are adjacent. We consider the following three cases:

**Case 1:** $\text{Ann}(x) \not\subset \text{Ann}(y)$. Let $r \in \text{Ann}(x)$ such that $r \notin \text{Ann}(y)$. If $Ry = Rry$, then $y = ray$ for some $a \in R$ and hence $x$ is adjacent to $y$.

Now, suppose that $Ry \neq Rry$. Consider the following ascending chain of cyclic submodules:

$$Ry \supseteq Rry \supseteq Rr_i^2 y \supseteq \cdots.$$ 


Let \( n \geq 1 \) be the smallest integer number such that \( Rr^ny = Rr^{n+1}y \).
There exists \( b \in R \) such that \( r^ny = br^{n+1}y \). Set \( z = (r^{n-1} - br^n)y \). By the definition of \( n \), we have \( 0 \neq z \in V_2 \). Now, we consider the following two subcases:

**Subcase 1.1:** \( z = ry \). Then, \( r^2y = 0 \) and hence \( z = 0 \), which is a contradiction.

**Subcase 1.2:** \( z \neq ry \). Then, \( z \) and \( ry \) are adjacent vertices of \( V_2 \), which is again a contradiction.

**Case 2:** \( \text{Ann}(y) \nsubseteq \text{Ann}(x) \). The proof of this case is similar to that of Case 1.

**Case 3:** \( \text{Ann}(x) = \text{Ann}(y) \). There exists \( \alpha \in V_2 \) such that \( \alpha \) is adjacent to \( x \). Since \( \alpha, y \in V_2 \), the assumption \( \alpha \in \text{Ann}(x)M = \text{Ann}(y)M \), implies that \( \alpha = y \). Hence, \( x \) and \( y \) are adjacent. Now, suppose that \( x \in \text{Ann}(\alpha)M \). Then, there exist \( r_1, \ldots, r_n \in \text{Ann}(\alpha) \) and \( x_1, \ldots, x_n \in M \) such that \( x = r_1x_1 + \cdots + r_nx_n \). If \( r_iy = 0 \) for all \( 1 \leq i \leq n \), then \( x \) and \( y \) are adjacent, and we are done. Now, suppose that there exists \( 1 \leq i \leq n \) such that \( r_iy \neq 0 \). Since \( M \) is reduced, \( r_iy \) and \( \alpha \) are adjacent vertices in \( V_2 \), which is a contradiction. This completes the proof. □

If \( M = R = \mathbb{Z}_3 \times \mathbb{Z}_4 \), then \( \Gamma(RM) \) is bipartite which is not complete bipartite. So, the reduced condition in Theorem 3.3 is essential. We have not found any example of a module \( M \) to show that the DCC condition in Theorem 3.3 is essential, which motivates to ask the following question.

**Question 3.4.** Let \( M \) be a reduced \( R \)-module such that \( \Gamma(RM) \) is a bipartite graph. Is \( \Gamma(RM) \) a complete bipartite graph?

In [4, Theorem 2.2], it has been proved that for a reduced commutative ring \( R \), \( \text{gr}(R) = 4 \) if and only if \( \Gamma(R) = K^{m,n} \) with \( m, n \geq 2 \). In the following corollary, we prove an analogous result for \( \Gamma(RM) \).

**Corollary 3.5.** Let \( M \) be a reduced \( R \)-module satisfying DCC on cyclic submodules. Then, \( \text{gr}(\Gamma(RM)) = 4 \) if and only if \( \Gamma(RM) = K^{m,n} \) with \( m, n \geq 2 \).

**Proof.** Let \( \text{gr}(\Gamma(RM)) = 4 \). By Theorem 3.1, \( \Gamma(RM) \) has no cycle of odd length, and hence it is a bipartite graph. Now, by Theorem 3.3, we observe that \( \Gamma(RM) \) is a complete bipartite graph. Since \( \Gamma(RM) \) has a cycle of length four, we have \( \Gamma(RM) = K^{m,n} \) with \( m, n \geq 2 \). The converse is trivial. □

In [4, Theorem 2.4], it has been proved that for a reduced commutative ring \( R \), \( \Gamma(R) \) is nonempty with \( \text{gr}(R) = \infty \) if and only if
\[ \Gamma(R) = K^{1,n} \text{ for some } n \geq 1. \] In the following corollary, we prove an analogous result for \( \Gamma(RM) \).

**Corollary 3.6.** Let \( M \) be a reduced \( R \)-module satisfying DCC on cyclic submodules. Then, \( \text{gr}(\Gamma(RM)) = \infty \) if and only if \( \Gamma(RM) \) is a star graph.

**Proof.** Let \( \text{gr}(\Gamma(RM)) = 1 \). Then, \( \Gamma(RM) \) has no cycle and hence it is a bipartite graph. By Theorem 3.3, \( \Gamma(RM) \) is a complete bipartite graph. Let \( \Gamma(RM) = K^{m,n} \), where \( m, n \geq 1 \). Since \( \Gamma(RM) \) has no cycle, then either \( m = 1 \) or \( n = 1 \), which implies that \( \Gamma(RM) \) is a star graph. The converse is trivial. \( \square \)

4. **Zero-divisor graphs of free modules**

We recall that an \( R \)-module \( F \) is called free if it is isomorphic to a direct sum of copies of \( R \). We write \( R(I) \) for the direct sum \( \bigoplus_{i \in I} R_i \), where each \( R_i \) is a copy of \( R \), and \( I \) is an arbitrary indexing set. If \( I \) is a finite set with \( n \) elements, then the direct sum and the direct product coincide; in this case, we write \( R^n \) for \( R(I) = R \times \cdots \times R \) (\( n \) times).

We begin this section with the following useful and evident proposition.

**Proposition 4.1.** Let \( RF = R(I) \) be a free \( R \)-module and \( (x_i)_{I_1}, (y_i)_{I_1} \in Z^*(RF) \). Then

1. \( Z(RF) = \{(x_i)_{I_1} \in F \mid \exists \, 0 \neq y \in R \text{ such that } yx_i = 0 \text{ for all } i \in I_1\} \),
2. \( (x_i)_{I_1} - (y_i)_{I_1} \) is an edge in \( \Gamma(RF) \) if and only if \( x_i y_j = 0 \) for all \( i, j \in I \).

**Theorem 4.2.** Let \( F = R(I) \) be a free \( R \)-module. Then, \( \Gamma(RF) \) is complete if and only if \( F = R = Z_2 \times Z_2 \) or \( (Z(R))^2 = 0 \).

**Proof.** If \( F = R = Z_2 \times Z_2 \) or \( (Z(R))^2 = 0 \), then it is easy to see that \( \Gamma(RF) \) is complete.

Conversely, suppose that \( \Gamma(RF) \) is complete. Let \( i_0 \in I \) and \( x, y \) be two distinct elements of \( Z^*(R) \). Let \( x_i = y_i = 0 \) for all \( i \in I \setminus \{i_0\} \), \( x_{i_0} = x \) and \( y_{i_0} = y \). Then, \( (x_i)_{I_1}, (y_i)_{I_1} \in Z^*(RF) \) and hence \( xy = 0 \). Thus, \( \Gamma(R) \) is complete. Then, [3, Theorem 2.8] implies that \( R = Z_2 \times Z_2 \) or \( (Z(R))^2 = 0 \). We show that \( |I| = 1 \), if \( R = Z_2 \times Z_2 \). Suppose on the contrary that \( |I| \geq 2 \). Let \( i_1, i_2 \) be two distinct elements of \( I \). Put

\[
 x_i := \begin{cases} 
 (1, 0) & \text{if } i = i_1, \\
 (1, 0) & \text{if } i = i_2, \\
 (0, 0) & \text{otherwise,}
\end{cases}
\]
and

\[ y_i := \begin{cases} (1, 0) & \text{if } i = i_1, \\ (0, 0) & \text{otherwise.} \end{cases} \]

Then, \( x := (x_i)_{l}, y := (y_i)_{l} \in Z^*(R F) \) and \( x \) and \( y \) are not adjacent in \( \Gamma(R F) \), a contradiction. This completes the proof. \( \square \)

Let \( F = R^{(l)} \). In the following three theorems, we study the relationship between the properties of \( \Gamma(R F) \) and \( \Gamma(R) \).

**Theorem 4.3.** Let \( F = R^n \) be a finitely generated free \( R \)-module. Let \( a \in Z^*(R) \), \( t = \deg_{\Gamma(R)} a \), \( A = \{(x_1, \ldots, x_n) \in Z^*(R F) | x_i = 0 \text{ or } x_i = a \} \) and \( x \in A \). Then,

\[
\deg_{\Gamma(R F)}(x) = \begin{cases} (t + 1)^n - 1 & \text{if } a^2 \neq 0, \\ (t + 2)^n - 2 & \text{otherwise.} \end{cases}
\]

**Proof.** Let \( t = \deg_{\Gamma(R)}(a) \) and \( N_{\Gamma(R)}(a) = \{a_1, \ldots, a_t\} \). If \( a^2 \neq 0 \), then

\[ N_{\Gamma(R F)}(x) = \{(x_1, \ldots, x_n) | x_i \in \{0, a_1, \ldots, a_t\}\} \setminus \{0\}. \]

Therefore, \( \deg_{\Gamma(R F)}(x) = |N_{\Gamma(R F)}(x)| = (t + 1)^n - 1 \). Now, suppose that \( a^2 = 0 \). Then,

\[ N_{\Gamma(R F)}(x) = \{(x_1, \ldots, x_n) | x_i \in \{0, a, a_1, \ldots, a_t\}\} \setminus \{0, x\}. \]

Hence, \( \deg_{\Gamma(R F)}(x) = |N_{\Gamma(R F)}(x)| = (t + 2)^n - 2 \). \( \square \)

**Theorem 4.4.** Let \( F = R^{(l)} \) such that \( |I| \geq 2 \). Then

\[ gr(\Gamma(R F)) = \begin{cases} gr(\Gamma(R)) & \text{if } R \text{ is reduced,} \\ 3 & \text{otherwise.} \end{cases} \]

**Proof.** First suppose that \( R \) is not reduced. Then, there exists \( 0 \neq a \in R \) such that \( a^2 = 0 \). Let \( i_1, i_2 \) be two distinct elements of \( I \). Put

\[
x_i := \begin{cases} a & \text{if } i = i_1, \\ 0 & \text{otherwise,} \end{cases} \quad y_i := \begin{cases} a & \text{if } i = i_2, \\ 0 & \text{otherwise.} \end{cases}
\]

and \( z_i := a \) for all \( i \in I \). Then \((x_i)_{l} - (y_i)_{l} - (z_i)_{l} - (x_i)_{l}\) is a cycle of length three and hence \( gr(\Gamma(R F)) = 3 \). Now, suppose that \( R \) is reduced. Let \( a_1 - a_2 - \cdots - a_t - a_1 \) be a cycle in \( \Gamma(R) \). Let \( j \in \{1, 2, \ldots, t\} \) and \( i_0 \in I \). Put

\[
x^j_i := \begin{cases} a_j & \text{if } i = i_0, \\ 0 & \text{otherwise.} \end{cases}
\]

Then, \((x^1_i)_{l} - (x^2_i)_{l} - \cdots - (x^t_i)_{l} - (x^1_i)_{l}\) is a cycle in \( \Gamma(R F) \) and hence, \( gr(\Gamma(R)) \leq gr(\Gamma(R F)) \). Now, let

\[(x^1_i)_{l} - (x^2_i)_{l} - \cdots - (x^t_i)_{l} - (x^1_i)_{l}, \]
be a cycle in $\Gamma(F)$. For all $j \in \{1, 2, \ldots, t\}$, there exists $i_j \in I$ such that $x_{i_j}^j \neq 0$. Then, $x_{i_1}^1 - x_{i_2}^2 - \cdots - x_{i_t}^t - x_{i_1}^1$ is a cycle in $\Gamma(R)$ and hence, $\text{gr}(\Gamma(R)) \leq \text{gr}(\Gamma(F))$. This completes the proof. \hfill \Box

A clique in a graph $G$ is a subset of pairwise adjacent vertices. The supremum of the size of cliques in $G$, denoted by $\omega(G)$, is called the clique number of $G$.

**Theorem 4.5.** Let $F = R^n$ be a finitely generated free $R$-module. Then $\omega(\Gamma(R)) = \omega(\Gamma(F))$.

**Proof.** Let $\{(x_1^1)_I, (x_1^2)_I, \ldots, (x_1^t)_I\}$ be a clique in $\Gamma(R)$. For each $1 \leq j \leq t$, there exists $i_j \in I$ such that $x_{i_j}^j \neq 0$. Then, $\{x_{i_1}^1, x_{i_2}^2, \ldots, x_{i_t}^t\}$ is a clique in $\Gamma(F)$ and hence $\omega(\Gamma(R)) \leq \omega(\Gamma(F))$. Now, let $\{x_1, x_2, \ldots, x_t\}$ be a clique in $\Gamma(R)$. Let $1 \leq j \leq t$ and $i_0 \in I$. Put

$$x_{i_j}^j := \begin{cases} x_j & \text{if } i = i_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\{(x_1^1)_I, (x_1^2)_I, \ldots, (x_1^t)_I\}$ is a clique in $\Gamma(R)$ and hence $\omega(\Gamma(R)) \leq \omega(\Gamma(F))$. This completes the proof. \hfill \Box

The next theorem shows that the structure of a finitely generated free $R$-module $F$ can be determined by $\Gamma(F)$. We denote the maximum degree of vertices of a graph $G$ by $\Delta(G)$.

**Theorem 4.6.** Let $M$ and $N$ be two finitely generated free $R$-module. If $\Gamma(R M) \cong \Gamma(R N)$, then $M \cong N$ as $R$-modules.

**Proof.** Let $M = R^m$ and $N = R^n$, for some natural numbers $m, n$. Suppose that $m > n$. Let $x = (x_1, x_2, \ldots, x_n)$ be a vertex of $\Gamma(R N)$ such that $\deg_{\Gamma(R N)}(x) = \Delta(\Gamma(R N))$. Since $x \in \text{Z}^*(\Gamma(N))$, there exists $0 \neq a \in R$ such that $ax_1 = ax_2 = \cdots = ax_n = 0$. Let $y = (x_1, x_2, \ldots, x_n, 0, \ldots, 0) \in M$. Then, the set

$$\{(y_1, \ldots, y_n, z_1, \ldots, z_{m-n}) \in R^M \mid (y_1, \ldots, y_n) \in N_{\Gamma(R N)}(x), z_i \in \{0, a\}\},$$

is a subset of $N_{\Gamma(R N)}(y)$. It then follows that $\Delta(\Gamma(R M)) \geq \deg_{\Gamma(R M)}(y) > \deg_{\Gamma(R N)}(x) = \Delta(\Gamma(R N))$, a contradiction. So, $m \leq n$. A similar argument shows that $n \leq m$. This completes the proof. \hfill \Box

5. Further Notes

In this short section, we study $\Gamma(R M)$, where $M$ is a multiplication $R$-module. We recall that an $R$-module $M$ is called a multiplication module if for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$. Let $N = IM$ and $K = JM$, for some ideals $I$ and $J$ of $R$. The product of $N$ and $K$, is denoted by $N \ast K$, and
defined by $IJM$. It is easy to see that the product of $N$ and $K$, is independent of presentations of $N$ and $K$. In [16], Lee and Varmazyar have given a generalization of the concept of zero-divisor graph of rings to multiplication modules. For a multiplication $R$-module $M$, they defined an undirected graph $\Gamma_*(R)$, with vertices $\{0 \neq x \in M | Rx \neq Ry = 0 \text{ for some non-zero } y \in M\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $Rx \neq Ry = 0$.

The following theorem shows that, in multiplication modules, this generalization and the one given in this paper are the same.

**Theorem 5.1.** Let $M$ be a multiplication $R$-module. Then, $\Gamma(RM) = \Gamma_*(RM)$.

**Proof.** Let $x$ and $y$ be two non-zero element of $M$ and suppose that $Rx = IM$ and $Ry = JM$, for some ideals $I$ and $J$ of $R$. Let $x - y$ be an edge in $\Gamma_*(R)$). Since $Rx \neq Ry = 0$, we have $IJM = 0$ and hence $I \subseteq \text{Ann}(JM)$. It then follows that $IM \subseteq \text{Ann}(JM)$. Therefore, $Rx \subseteq \text{Ann}(Ry)M$ and hence, $x - y$ is an edge in $\Gamma(RM)$.

Now, suppose that $x - y$ is an edge in $\Gamma(R)$. It then follows that $Rx \subseteq \text{Ann}(Ry)M$. So $IM \subseteq \text{Ann}(JM)$. In view of [26, Theorem 9], we have the following two cases:

**Case 1:** $I \subseteq \text{Ann}(JM) + \text{Ann}(M)$. In this case, $I \subseteq \text{Ann}(JM)$, since $\text{Ann}(M) \subseteq \text{Ann}(JM)$. It then follows that $IJM = 0$ and hence, $x - y$ is an edge in $\Gamma_*(R)$.

**Case 2:** $M = ((\text{Ann}(JM) + \text{Ann}(M)) : I)M$. In this case, we have $M = (\text{Ann}(JM) : I)M$ and hence, $IJM = [(\text{Ann}(JM) : I)I]JM \subseteq \text{Ann}(JM)JM = 0$. Therefore, $x - y$ is an edge in $\Gamma_*(R)$. This completes the proof. \hfill \Box

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**References**


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