A NOTE ON THE COMMUTING GRAPHS OF A CONJUGACY CLASS IN SYMMETRIC GROUPS

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Abstract. The aim of this paper is to obtain the automorphism group of the commuting graph of a conjugacy class in the symmetric groups. The clique number, coloring number, independence number and diameter of these graphs are also computed.

1. Introduction

Let \( L = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). The (open) neighborhood \( N(a) \) of a vertex \( a \in V \) is the set of all vertices that are adjacent to \( a \), and the closed neighborhood of \( a \) is defined as \( N[a] = N(a) \cup \{a\} \). The distance between vertices \( x \) and \( y \) in \( L \), denoted by \( d(x, y) \), is defined as the length of a shortest path connecting them. Note that \( d(x, x) = 0 \), and \( d(x, y) = \infty \) if there is no path connecting \( x \) and \( y \). The diameter \( \text{diam}(L) \) is the maximum of \( d(x, y) \) taken over all pairs of vertices of \( L \). A subset \( X \subseteq V \) is called an independence set if there exists no edge with both endpoints in \( X \). The independent number \( \alpha(L) \) is the maximum cardinality among all independent sets in \( L \) and the chromatic number \( \omega(L) \) is the maximum number of vertices that are mutually adjacent, that is the order of a maximum complete subgraph of \( L \). The chromatic number \( \chi(L) \) is the minimum number of colors which are used for the coloring of the vertices of \( L \), where any two adjacent vertices have distinct colors. Let us denote the clique number of \( L \) by \( \chi(L) \). The Kneser graph \( K_{n,m} \) is the graph whose vertices are the \( m \)-subsets of a fixed \( n \)-set, and two vertices are adjacent if the


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corresponding $m$-subsets are disjoint. We refer to [6] for other graph
theory notations of this paper.

Let $G$ be a group and $S \subseteq G$. The \textit{commuting graph} $C(G, S)$ is the
graph with vertex set $S$ such that two vertices are adjacent if and only
if they commute. Akbari and his co-workers [1] studied the commuting
graph of a ring and Araujo et al. [2] investigated this graph for
semigroups. In [3], Bates et al. determined the diameter of $C(S_n, X)$,
where $X$ is the set of $m$-cycles and $n \geq 2m + 1 \geq 7$. In some special
cases, they obtained upper bounds for diameter of commuting graph.
In [4, 5], the authors, among others, obtained a number of results on
the diameter of commuting graph of a finite group.

In this paper, we apply the main properties of the Kneser graphs to
obtain the automorphism group of the commuting graph of a conjugacy
class in symmetric groups and then determine the clique number, the
independence number and the diameter of these graphs. For the sake
of completeness, we mention here the main properties of Kneser graphs
which is crucial throughout this paper.

\textbf{Theorem 1.1.} Suppose $L = K_{n;m}$, $n > 2m$ and $n = 2m + k$. Then we have

(1) $\text{Aut}(L) = S_n$, where $S_n$ is the symmetric group of degree $n$,
(2) $\omega(L) = \left\lceil \frac{n}{m} \right\rceil$,
(3) $\chi(L) = n - 2k + 2$,
(4) $\alpha(L) = \binom{n-1}{m-1}$,
(5) $\text{diam}(L) = \left\lceil \frac{n-1}{k} \right\rceil + 1$.

\section{Preliminary Results}

Let $L$ be a graph. A subgraph $H$ of $L$ is called an \textit{EN-subgraph} if any
two vertices of $H$ have equal closed neighborhood. An \textit{EN-subgraph}
of $L$ is called \textit{MEN-subgraph} if it is maximal among the set of all \textit{EN-}
subgraphs of $L$. It can be seen that any \textit{EN-subgraph} is complete and
the set of vertices of $L$ can be partitioned into the vertex sets of all \textit{MEN-}
subgraphs of $L$. Suppose $x \in V(L)$. The set of \textit{MEN-subgraph}
$B$ with $x \in V(B)$ is denoted by $\mathcal{E}$. Define the weighted graph $\overline{L}$ as
follows: $V(\overline{L}) = \{ \mathcal{E} | x \in V(L) \}$, $\text{weight}(\mathcal{E}) = |\mathcal{E}|$ and two vertices $\mathcal{E}$ and
$\mathcal{F}$ are adjacent if and only if $x$ and $y$ are adjacent in $L$.

We start our result, by the following elementary lemma:

\textbf{Lemma 2.1.} Let $L$ be a graph and $\varphi \in \text{Aut}(L)$. Then $\varphi(\mathcal{E}) \in \text{Aut}(\overline{L})$, where $\varphi(\mathcal{E}) = \varphi(x)$.
Proof. Let $L$ be a graph and $\varphi \in Aut(L)$. Since $\varphi(N[x]) = N[\varphi(x)]$ for all $x$, hence, $B \in \overline{L}$ if and only if $\varphi(B) \in \overline{L}$. On the other hand, if $x \in B$ then $\varphi(B) = \overline{\varphi(x)}$, proving the lemma. □

**Theorem 2.2.** For a graph $L$ with $|V(L)| < \infty$, we have $Aut(L) \cong Aut(\overline{L}) \rtimes \prod_{B \in V(\overline{L})} S_{|B|}$.

*Proof.* Define $\psi : Aut(L) \rightarrow Aut(\overline{L})$, by $\psi(\varphi) = \overline{\varphi}$ such that $\overline{\varphi}(B) = \varphi(B)$. Then $\psi$ is homomorphism and $|Aut(L)| \leq |kerl(\psi)||Aut(\overline{L})|$. But $kerl(\psi) = \{\varphi | \varphi(B) = B, \text{ for all } B \in \overline{L}\}$. On the other hand, all functions where induced a bijection function on each $B$ are in $kerl(\psi)$ and so $kerl(\psi) \cong \prod_{B \in V(\overline{L})} S_{|B|}$, where $S_n$ is the symmetric group of degree $n$.

We now find a subgroup $H$ of $Aut(G)$ such that $H \cap kerl(\varphi) = 1$. Since $L$ is finite, there exists a total order $\leq$ on $V(L)$. Set $H = \{\varphi | \varphi(B) \in V(\overline{L}), \text{ for all } B \in V(\overline{L}) \text{ and } \varphi \text{ preserve the partial order }\}$. It is clear that $H$ is a subgroup of $Aut(L)$. Let $\varphi \in H$, $B = \{x_1, \ldots, x_t\} \in V(\overline{L})$ and $\varphi(B) = B$ where $x_1 < x_2 < \cdots < x_t$. So, $\{\varphi(x_1), \ldots, \varphi(x_t)\} = B$ and consequently $\varphi(x_i) = x_i$, for all $i$. Thus, if $\varphi \in H$ then $\varphi(B) = B$ if and only if $\varphi(x) = x$, for all $x \in B$. Thus, $H \cap kerl(\psi) = 1$ and then $|Aut(L)| \geq |kerl(\psi)||H|$. Now $f : H \rightarrow Aut(\overline{L})$ given by $f(\varphi) = \psi(\varphi)$ is an homomorphism. Assume that $h \in Aut(\overline{L})$. For $B = \{x_1, \ldots, x_t\} \in \overline{L}$, $x_1 < \ldots < x_t$, we define $\varphi(x_i) = y_i$, where $h(B) = \{y_1, \ldots, y_t\}$, $y_1 < \ldots < y_t$. If $x, y$ are adjacent and $x, y \in B$ for some $B$, then $\varphi(x), \varphi(y) \in \varphi(B)$ and are adjacent. If $x, y$ are adjacent and $x \in B_1, y \in B_2$ for some $B_1 \neq B_2$, then $\varphi(x) \in \varphi(B_1), \varphi(y) \in \varphi(B_2)$ and, since $B_1, B_2$ are adjacent, $\varphi(B_1), \varphi(B_2)$ are adjacent and consequently $\varphi(x), \varphi(y)$ are adjacent.

Thus $\varphi \in Aut(L)$ and $f(\varphi) = h$. Consequently, $f$ is an automorphism, which completes the proof. □

For a graph $L$, we define the relation $\sim$ on $V(L)$ as follows: $a \sim b$ if and only if $N(a) = N(b)$. It is easy to see that $\sim$ is an equivalence relation on $V(L)$. Denote the equivalence class of $x$ by $[x]$. Moreover, define the weighted graph $\overline{L}$ with vertex set $\{[x] \mid x \in V(L)\}$, $weight([x]) = |[x]|$ and $[x]$ is adjacent to $[y]$ if and only if $x$ is adjacent to $y$. Then we can see that no pair of elements of $[x]$ are adjacent in $L$. Similar to the Theorem 2.2, we can obtain the following:

**Theorem 2.3.** For a finite graph $L$, we have $Aut(L) \cong Aut(\overline{L}) \rtimes \prod_{[x] \in V(\overline{L})} S_{|[x]|}$.

We have the following important theorem:
Theorem 2.4. For a finite graph $L$ with $|V(L)| \geq 2$, the following hold:

1. $\alpha(\overline{L}) = \alpha(L)$ and if $L, \overline{L}$ are regular then $\alpha(L) = r\alpha(\overline{L})$, where $r$ is the weight of each vertex of $\overline{L}$.
2. $\chi(L) = \chi(\overline{L})$ and if $L, \overline{L}$ are regular then $\chi(L) = r\chi(\overline{L})$, where $r$ is the weight of each vertex of $\overline{L}$.
3. $\omega(L) = \omega(\overline{L})$ and if $L, \overline{L}$ are regular then $\omega(L) = r\omega(\overline{L})$, where $r$ is the weight of each vertex of $\overline{L}$.
4. If $L$ is not complete, then $\text{diam}(L) = \text{diam}(\overline{L})$.
5. Let $L$ be connected but not complete. If $\overline{L}$ is complete, then $\text{diam}(L) = \text{diam}(\overline{L}) + 1$ otherwise $\text{diam}(L) = \text{diam}(\overline{L})$.

Proof. (1) It is clear that $\alpha(L) \geq \alpha(\overline{L})$. Let $\{x_1, \ldots, x_\delta\} \subseteq V(L)$ be an independent set. Then $x_i, x_j$ are not adjacent and, then $\overline{x_i}, \overline{x_j}$ are not adjacent. Thus $\alpha(L) \leq \alpha(\overline{L})$, as required. Also, if $\{[x_1], \ldots, [x_\delta]\} \subseteq \overline{L}$ is an independent set then any element of $[x_i]$ is not adjacent to any element of $[x_j]$, therefore $[x_1] \cup \ldots \cup [x_\delta]$ is an independent set of $L$. On the other hand, if $A$ is a maximal independent set of $L$ and $x \in A$, then $[x] \subseteq A$ and $\{[x] | x \in A\}$ is an independent set of $\overline{L}$. Thus, $\alpha(\overline{L}) \geq s/r$, which completes (1).

(2) The proof of $\chi(L) = \chi(\overline{L})$ is trivial. Assume that $\{u_1, \ldots, u_\epsilon\}$ is a coloring set of $\overline{L}$. We use $r$ distinct colors $u_1', \ldots, u_\epsilon'$ for vertices in $\overline{\pi}$, where $x$ has color $u_x$. Thus, $\chi(L) \leq r\chi(\overline{L})$. Let $S$ be a coloring set for $L$ and $u_x$ is the color of $x$. We consider the color of $b \in \pi$ for $\pi$ and obtain a coloring set for $\overline{L}$. Assume that $A$ is an arbitrary subset of $V(L)$, where $|A \cap \pi| = 1$. Then $\{u_x | x \in A\}$ is a coloring set of $\overline{L}$. Consequently, $|S| \geq r\chi(\overline{L})$, as required.

(3) Is similar to (1).

(4) Is elementary.

(5) We see that $\overline{L}$ is complete if and only if $L$ is a complete $k$-partite graph, where $k = |\overline{L}|$. So, we assume that $\overline{L}$ is not complete. Let $diam(\overline{L}) = s \geq 2$, $d([a], [b]) = s$ and, $[a] = [a_0] \cdots [a_s] = [b]$ is a path. Then $a = a_0 - a_1 - \cdots - a_s = b$ is a path and thus $d(a, b) \leq s$. If $d(a, b) = t$ and $a = x_0 - x_1 - \cdots - x_t = b$ is a path, then we can obtain a path with length less than or equal to $t$. From which $d(a, b) = s$ and $diam(\overline{L}) \leq diam(L)$. Since $\overline{L}$ is not complete, $diam(\overline{L}) \geq 2$. Let $d(x, y) = diam(L) = t$ for $t \geq 2$ and $x = x_0 - x_1 - \cdots - x_t = y$ is a path. Because $d(x, y) \geq 2$, $[x] \neq [y]$ and thus $d(x, y) = d([x], [y])$, which completes the proof. \[\square\]
Note. For a graph $L$, if $\overline{L}$ or $\overline{eL}$ is regular with equal weights, then we can consider this graphs un-weighted.

3. Main results

Let $\sigma = (12\ldots m)$ be a cycle of length $m$ in $S_n$ and $S = \sigma^{S_n}$. We obtain the automorphism group, chromatic number, clique number and diameter of $C(S_n, S)$. We start by the following elementary lemma.

Lemma 3.1. $C_{S_n}(\sigma) = \langle \sigma \rangle \times \text{Sym} \{\{m + 1, \ldots, n\}\}$.

Proof. It is enough to see that $n(n-1)\ldots(n-m+1)/m = |\sigma^{S_n}| = |S_n:C_{S_n}(\sigma)|$ and $\langle \sigma \rangle \times \text{Sym} \{\{m + 1, \ldots, n\}\} \subseteq C_{S_n}(\sigma)$.

Corollary 3.2. Consider the graph $C(S_n, S)$. If $(m, n) \neq (2, 4)$, then the vertices of any MEN-subgraph are the generators of $\langle \alpha \rangle$, for some $\alpha \in S$.

Proof. Suppose $\alpha \in S$. Without loss of generality, we can assume that $\alpha = (12\ldots m)$. By Lemma 3.1, $N[\alpha] = S \cap (\langle \alpha \rangle \cup \text{Sym} \{\{m + 1, \ldots, n\}\}) = (S \cap (\langle \alpha \rangle \cup \text{Sym} \{\{m + 1, \ldots, n\}\}))$. If $\beta \in N(\alpha) - \langle \alpha \rangle$, then $\beta \in \text{Sym} \{\{m + 1, \ldots, n\}\}$. We see that $\overline{\alpha} = \overline{\beta}$ if and only if $(m, n) = (2, 4)$, and hence the result follows.

We are now ready to present our main result.

Theorem 3.3. Suppose $n > 2m$ and $n = 2m + k$. Then,

1. $\alpha(C(S_n, S)) = \frac{(n-1)!}{m-1)!}$,
2. $\chi(C(S_n, S)) = \phi(m)(n-2m + 2)$,
3. $\omega(C(S_n, S)) = \phi(m)\frac{n}{m}$,
4. $\text{diam}(C(S_n, S)) = \lceil \frac{m-1}{k} \rceil + 1$,
5. $\text{Aut}(C(S_n, S)) = (S_n \times S_d^a) \ltimes S_b^c$, where $a = \phi(m)$, $b = \frac{(n-1)!}{m-1)!}$, $c = \frac{(m-1)!}{\phi(m)}$, $d = \frac{b}{a}$ and $\phi$ is the Euler function.

Proof. Since $C(S_n, S)$ is a graph with equal weights $\phi(m)$, we can assume that it is un-weighted. Let $L$ be such a graph and $x = (a_1\ldots a_m) \in S$. Since $(m, n) \neq (2, 2)$, hence $x = \text{gen}(x)$, where gen($x$) is the set of all generators of $\langle x \rangle$. Since $N((12\ldots m)) = \{\overline{\beta} \mid \beta \in \text{Sym}(m + 1, \ldots, n)\}$, $N((a_1\ldots a_m)) = N((b_1\ldots b_m))$ if and only if $\{a_1, \ldots, a_m\} = \{b_1, \ldots, b_m\}$. Therefore, $\overline{\mathcal{X}} = \{(b_1\ldots b_m)|\{b_1, \ldots, b_m\} = \{a_1, \ldots, a_m\}\}$. We now consider the graph $\overline{L}$. Then two vertices $a = \overline{(a_1\ldots a_m)}$ and $b = \overline{(b_1\ldots b_m)}$ are adjacent if and only if $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$ are disjoint. But $|S \cap S_m| = (m - 1)!$ and $|(12\ldots m)| = \varphi(m)$, thus
all weights of vertices of $\tilde{L}$ are equal to $\frac{(m-1)!}{\phi(m)}$ and so we can consider this graph to be un-weighted, say $L'$. Hence, $V(L')$ is all subsets of $\{1, \ldots, n\}$ with $m$ elements such that two vertices are adjacent if and only if they are disjoint. This concludes that $L'$ is the Kneser graph $K_{n,m}$. By above assumption and Theorems 1.1, 2.2, 2.3 and 2.4, the proof will be proved.

It is merit to mention here that the part (d) of Theorem 3.3 is a main result of [3].

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