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MOST RESULTS ON A-IDEALS IN MV-MODULES

S. SAIDI GORAGHANI*, AND R. A. BORZOOEI

Abstract. In the present paper, by considering the notion of MV -modules which is the structure that naturally correspond to lu-modules over lu-rings, we prove some results on prime A-ideals and state some conditions to obtain a prime A-ideal in MV-modules. Also, we state some conditions that an A-ideal is not prime and investigate conditions that $K \subseteq \bigcup_{i=1}^{n} K_i$ implies $K \subseteq K_j$, where $K, K_1, \cdots, K_n$ are A-ideals of A-module M and $1 \leq j \leq n$.

1. Introduction

MV-algebras were defined by C. C. Chang [2, 3] as algebras corresponding to the Łukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-Von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang, that non-trivial MV-algebras are sub-direct products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. A *product MV-algebra*

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∗Corresponding author.

(or PMV-algebra, for short) is an MV-algebra which has an associative binary operation ".". It satisfies an extra property which will be explained in Preliminaries section. During last years, PMV-algebras were considered and their equivalence with a certain class of l-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible MV-algebras and the MV-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra by A. Di Nola [6]. Recently, Forouzesh, Eslami and Borumand Saeid [7] defined prime A-ideals in MV-modules. Since MV-modules are in their infancy, stating and opening of any subject in this field can be useful. Hence, in this paper, we study prime A-ideals and state some conditions to obtain a prime A-ideal (or no prime A-ideal) in MV-modules. Also, in special case, we prove that if $K \subseteq \bigcup_{i=1}^{n} K_i$, then $K \subseteq K_j$, where $K, K_1, \ldots, K_n$ are A-ideals of A-module M and $1 \leq j \leq n$. In fact, our results in this paper gives new insights to anyone who is interested in studying and development of MV-modules.

2. Preliminaries

In this section, we review related lemmas and theorems that we will use in the next sections.

Definition 2.1. [4] An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that
$(MV1)$ $(M, \oplus, 0)$ is an abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $a \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in M$. It is clear that $(M, \odot, 1)$ is an abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice*. An MV-algebra M is a *Boolean* algebra if and only if the operation " $\oplus$ " is idempotent, i.e., $a \oplus a = a$, for every $a \in M$. In every MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in M$ such that

$a \oplus c = b$, for every $a, b \in M$. For any two elements a, b of MV-algebra M, $a \leq b$ if and only if a, b satisfy in the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following conditions: $(I1)$ $0 \in I$, $(I2)$ $x \leq y$ and $y \in I$ imply that $x \in I$, $(I3)$ $x \oplus y \in I$, for every $x, y \in I$. A proper ideal I of M is a prime ideal if and only if $x \ominus y \in I$ or $y \ominus x \in I$, for every $x, y \in M$. A proper ideal I of M is a maximal ideal of M if and only if no proper ideal of M strictly contains I. In MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv) $d(x, y) = d(x', y')$, (v) $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of MV-algebra M. Then, we denote $x \sim y$ $(x \equiv_I y)$ if and only if $d(x, y) \in I$, for every $x, y \in M$. So, $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in M\}$. Then, $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$. Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an MV-*homomorphism* if $(H1)$ $f(0) = 0$, $(H2)$ $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$ $f(x') = (f(x))'$, for every $x, y \in M$. If f is one to one (resp. onto), then f is called an MV-monomorphism (resp. epimorphism) and if f is onto and one to one, then f is called an MV-isomorphism (see [6]).

Proposition 2.2. [4] *Let M be an MV-algebra and $z \in M$. Then the principal ideal generated by z is denoted by $\langle z \rangle$ and* $\langle z \rangle = \{x \in M : nz = \underbrace{z \oplus \cdots \oplus z}_{n\ times} \geq x,\ for\ some\ n \geq 0\}$.

Lemma 2.3. [4] *In every MV-algebra M, the natural order "$\leq$" has the following properties:*

(i) *$x \leq y$ if and only if $y' \leq x'$,*
(ii) *if $x \leq y$, then $x \oplus z \leq y \oplus z$, for every $z \in M$.*

Definition 2.4. [5] In MV-algebra M, a partial addition is defined as following:
$x + y$ is defined iff $x \leq y'$ and in this case, $x + y = x \oplus y$, for any $x, y \in M$.

Lemma 2.5. [6] *In MV-algebra M,*

(i) *$x + 0 = x$,*
(ii) *if $x + y = z$, then $y = x' \odot z$,*
(iii) *if $z + x = z + y$, then $x = y$,*
(iv) *if $z + x \leq z + y$, then $x \leq y$, where "$+$" is the partial addition on M.*

Definition 2.6. [5] A *product MV-algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x + y$ is defined, then $x.z + y.z$ and $z.x + z.y$ are defined and $(x + y).z = x.z + y.z$, $z.(x + y) = z.x + z.y$, for every $x, y, z \in A$, where " + " is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then A is called a unital PMV-algebra. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation.

Lemma 2.7. [5] *If A is a unital PMV-algebra, then;*

(i) *the unity for product is $e = 1$,*
(ii) *$x.y \leq x \wedge y$, for every $x, y \in A$.*

Lemma 2.8. [5] *Let A be a PMV-algebra. Then, $1.a = a$ and $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$, for any $a, b, c \in A$.*

Definition 2.9. [6] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, m) = am$, which satisfies the following axioms:

$(AM1)$ if $x + y$ is defined in M, then $ax + ay$ is defined in M and $a(x + y) = ax + ay$,
$(AM2)$ if $a + b$ is defined in A, then $ax + bx$ is defined in M and $(a + b)x = ax + bx$,
$(AM3)$ $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.

Then M is called a (left) *MV-module* over A or briefly an A-module. We say that M is a *unitary MV*-module if A has a unity 1_A for the product and

$(AM4)$ $1_A x = x$, for every $x \in M$.

Lemma 2.10. [6] *Let A be a PMV-algebra and M be an A-module. Then;*

(a) $0x = 0$,
(b) $a0 = 0$,
(c) $ax' \leq (ax)'$,
(d) $a'x \leq (ax)'$,
(e) $(ax)' = a'x + (1x)'$,
(f) *$x \leq y$ implies that $ax \leq ay$,*
(g) *$a \leq b$ implies that $ax \leq bx$,*
(h) $a(x \oplus y) \leq ax \oplus ay$,
(i) $d(ax, ay) \leq ad(x, y)$,
(j) *if $x \equiv_I y$, then $ax \equiv_I ay$, where I is an ideal of A,*

(k) *if M is a unitary MV-module, then $(ax)' = a'x + x'$, for every $a, b \in A$ and $x, y \in M$.*

Definition 2.11. [6] Let A be a PMV-algebra and M_1, M_2 be two A-modules. A map $f : M_1 \to M_2$ is called an A-*module* homomorphism or (A-*homomorphism*, for short) if f is an MV-homomorphism and $(H4)$: $f(ax) = af(x)$, for every $x \in M_1$ and $a \in A$.

Definition 2.12. [6] Let A be a PMV-algebra and M be an A-module. Then, an ideal $N \subseteq M$ is called an A-*ideal* of M if $(I4)$ $ax \in N$, for every $a \in A$ and $x \in N$.

Definition 2.13. [7] Let M be an A-module and N be a proper A-ideal of M. Then, N is called a *prime* A-ideal of M, if $am \in N$ implies that $m \in N$ or $a \in (N : M)$, for any $a \in A$ and $m \in M$, where $(N : M) = \{a \in A : aM \subseteq N\}$. Moreover, the set of all prime A-ideals of M is denoted by $Spec(M)$.

Note. From now onwards, A denotes a PMV-algebra.

3. Some results on prime A-ideals in MV-modules

In this section, we state and prove some conditions to obtain a prime A-ideal in $MV-$modules.

Example 3.1. Let $A = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "." on A are defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$. Then, it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra and $(A, \oplus, ', 0)$ is an MV-algebra. Now, let the operation $\bullet : A \times A \longrightarrow A$ be defined by $a \bullet b = a.b$, for every $a, b \in A$. It is easy to show that A is an MV-module on A and $I = \{0, 1\}$, $J = \{0, 2\}$ are prime A-ideals of A. $\{0\}$ is not a prime A-ideal of A. Note that $1 \bullet 2 = 0$, but $2 \notin \{0\}$ and $1 \notin (\{0\} : A) = \{0\}$.

Proposition 3.2. *Let M be an A-module and N, L be A-ideals of M. Then;*

(i) *$(N : M) = \{a \in A : aM \subseteq N\}$ is an ideal of A,*
(ii) *$(N : m)$ is an ideal of A, for every $m \in M$,*
(iii) *N is a prime A-ideal of M if and only if $(N : m) = (N : M)$, where $m \notin N$.*

Proof. (i) It is clear that $0 \in (N : M)$. Let $\alpha, \beta \in (N : M)$. Then, $\alpha m, \beta m \in N$, for every $m \in N$. Since $\beta m \leq (\alpha m)' \oplus \beta m$, by Lemma 2.3(i), we get $(\alpha m) \odot (\beta m)' = ((\alpha m)' \oplus \beta m)' \leq (\beta m)'$ and so $(\alpha m) \odot (\beta m)' + \beta m$ is defined, where " $+$ " is the partial addition on M. Similarly, $\alpha \odot \beta' + \beta$ is defined, too. Also, since $\alpha \odot \beta' \leq \beta'$, by Lemma 2.10 (d) and (g), we have $(\alpha \odot \beta')m \leq \beta' m \leq (\beta m)'$ and so $(\alpha \odot \beta')m + \beta m$ is defined. Now, $\alpha \leq \alpha \vee \beta$ implies that $\alpha m \leq (\alpha \vee \beta)m$ and similarly, $\beta m \leq (\alpha \vee \beta)m$. Then, $\alpha m \vee \beta m \leq (\alpha \vee \beta)m$ and so

$$\begin{aligned}(\alpha m) \odot (\beta m)' + \beta m &= \alpha m \vee \beta m \leq (\alpha \vee \beta)m = (\alpha \odot \beta' \oplus \beta)m \\ &= (\alpha \odot \beta' + \beta)m = (\alpha \odot \beta')m + \beta m.\end{aligned}$$

By Lemma 2.5 (iv), we have $\alpha m \odot (\beta m)' \leq (\alpha \odot \beta')m$. If we set $\alpha \oplus \beta$ instead of α, then by Lemma 2.10 (g), we get $(\alpha \oplus \beta)m \odot (\beta m)' \leq ((\alpha \oplus \beta) \odot \beta')m = (\alpha \wedge \beta')m \leq \alpha m$. Since

$$(\alpha \oplus \beta)m = (\alpha \oplus \beta)m \vee \beta m = (\alpha \oplus \beta)m \odot (\beta m)' \oplus \beta m \leq \alpha m \oplus \beta m \in N,$$

hence $\alpha \oplus \beta \in (N : M)$. Now, let $\alpha \leq \beta$ and $\beta \in (N : M)$. Then, by Lemma 2.10(g), we have $\alpha m \leq \beta m \in N$ and so $\alpha m \in N$, for every $m \in M$. It means that $\alpha \in (N : M)$.
(ii) By (i), the proof is clear.
(iii) By (i) and (ii), the proof is straight forward. □

Lemma 3.3. *Let M be a unitary A-module and $m \in M$. Then;*

$$\begin{aligned} I_m &= \{\sum_{i=1}^{k} t_i m : \sum_{i=1}^{k} t_i m \leq nm, \ for\ some\ n, k \in \mathbb{N} \cup \{0\}, \\ & \qquad where\ t_i \in A\ and\ t_1 m + \cdots + t_k m\ is\ defined\ \}\end{aligned}$$

is an A-ideal of M.

Proof. (I_1) It is clear that $0 \in I_m$.
(I_2) Let $x \leq \sum_{i=1}^{k} t_i m \in I_m$, for some $x \in M$. Then, $x = 1x \leq \sum_{i=1}^{k} t_i m \leq nm \in I_m$, where $n \geq 0$ and so $x \in I_m$.
(I_3) Let $\sum_{i=1}^{k} t_i m, \sum_{i=1}^{w} s_i m \in I_m$. Then, there exist $n_1, n_2 \geq 0$ such that $\sum_{i=1}^{k} t_i m \leq n_1 m$ and $\sum_{i=1}^{w} s_i m \leq n_2 m$ and so

$$\begin{aligned}\sum_{i=1}^{k+w} c_i m &= \sum_{i=1}^{k} t_i m \oplus \sum_{i=1}^{w} s_i m \leq n_1 m \oplus n_2 m = \underbrace{m \oplus \cdots \oplus m}_{n_1\ times} \\ &\oplus \underbrace{m \oplus \cdots \oplus m}_{n_2\ times} = (n_1 + n_2)m,\end{aligned}$$

where

$$c_i = \begin{cases} t_i & 1 \leq i \leq k \\ s_{i-k} & k+1 \leq i \leq k+w \end{cases},$$

It means that $\sum_{i=1}^{k} t_i m \oplus \sum_{i=1}^{w} s_i m \in I_m$.
(I_4) Let $a \in A$ and $\sum_{i=1}^{k} t_i m \in I_m$. Then, there exists $n \geq 0$ such that $\sum_{i=1}^{k} t_i m \leq nm$. Since $\sum_{i=1}^{k} t_i m \leq nm = \underbrace{m \oplus \cdots \oplus m}_{n\ times}$, by Lemma 2.10(f) and (h), hence

$$a(\sum_{i=1}^{k} t_i m) \leq a(m \oplus \cdots \oplus m) \leq \underbrace{am \oplus \cdots \oplus am}_{n\ times}.$$

By Lemma 2.10(k), since $(am)' \oplus m = a'm \oplus m' \oplus m = 1$, and $am \leq m$, so $a(\sum_{i=1}^{k} t_i m) \leq \underbrace{m \oplus \cdots \oplus m}_{n\ times} = nm$. It results that $\sum_{i=1}^{k} (a.t_i)m = \sum_{i=1}^{k} a(t_i m) \in I_m$. □

Notation. For A-module M, non-empty subset I of A and A-ideal N of M, we let $IN = \{xm : x \in I, m \in N\}$.

Definition 3.4. A PMV-algebra A is called *commutative*, if $x.y = y.x$, for every $x, y \in A$.

Example 3.5. In Example 3.1, A is a commutative PMV-algebra.

Theorem 3.6. *Let A be commutative $MV-$algebra, M be a unitary A-module, N be a proper A-ideal of M and $x \oplus x = x$, for every $x \in A$. Then, N is a prime A-ideal of M if and only if for every ideal I of A and A-ideal D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.*

Proof. $(\Rightarrow)$ Let N be a prime A-ideal of M, I be an ideal of A and D be an A-ideal of M such that $ID \subseteq N$. We will show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then, there exist $x \in A$ and $d \in D$ such that $xM \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $xd \in N$. Since N is a prime A-ideal of M and $d \notin N$, $xM \subseteq N$, which is a contradiction.
$(\Leftarrow)$ For every ideal I of A and A-ideal D of M, let $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$. Then suppose that there exist $x \in A$ and $m \in M$ such that $xm \in N$ and $m \notin N$. By Proposition 2.2 and Lemma 3.3, let $I = \langle x \rangle$ and $D = I_m$. Then for $y \in I$, by Proposition 2.2, there exists $n \geq 0$ such that $y \leq nx$ and so $y \ominus nx = 0$. Hence,

$$\begin{aligned} ym &= (y \ominus 0)m = (y \ominus (y \ominus nx))m = (y \odot (y \odot (nx)')')m \\ &= (y \odot (y' \oplus nx))m = (y \wedge nx)m. \end{aligned}$$

By Lemma 2.10 (g), since $y \wedge nx \leq nx$ and $x \oplus x = x$, we get

$$ym = (y \wedge nx)m \leq (nx)m = (\underbrace{x \oplus x \oplus \cdots \oplus x}_{n\ times})m = xm \in N.$$

Hence, $ym \in N$ and then we get $ID = \{y(\sum_{i=1}^{k} t_i m) : y, t_i \in A\} = \{\sum_{i=1}^{k} t_i(ym) : y, t \in A\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, hence $I \subseteq (N : M)$ and so $xM \subseteq N$. Therefore, N is a prime A-ideal of M. □

Definition 3.7. Let M be an A-module. Then M is called a *Boolean A-module* if $ax \oplus ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$.

Example 3.8. If A is a Boolean algebra, then every A-module M is a Boolean A-module.

Proposition 3.9. [1, 10] *Let M be a Boolean A-module.*

(i) *If I is an A-ideal of M, then $\frac{M}{I}$ is an A-module.*
(ii) *If N and K are two A-ideals of M such that $N \subseteq K$, then $\frac{K}{N} = \{\frac{k}{N} : k \in K\}$ is an A-ideal of $\frac{M}{N}$.*

Proposition 3.10. *Let M be a Boolean A-module and N be an A-ideal of M. Then P is a prime A-ideal of M if and only if $\frac{P}{N}$ is a prime A-ideal of $\frac{M}{N}$, where $N \subseteq P$.*

Proof. ($\Rightarrow$) Let P be a prime A-ideal of M. By Proposition 3.9, $\frac{M}{N}$ is an A-module and $\frac{P}{N}$ is an A-ideal of $\frac{M}{N}$. Let $x\frac{m}{N} \in \frac{P}{N}$, where $x \in A$ and $m \in M$. Then there exists $q \in P$ such that $\frac{xm}{N} = \frac{q}{N}$ and so $d(xm, q) \in N \subseteq P$. Since $xm = d(xm, 0) \leq d(xm, q) \oplus d(q, 0) \in P$, $xm \in P$ and so $x \in (P : M)$ or $m \in P$. It results that $x\frac{M}{N} \subseteq \frac{P}{N}$ or $\frac{m}{N} \in \frac{P}{N}$. Therefore, $\frac{P}{N}$ is a prime A-ideal of $\frac{M}{N}$.
($\Leftarrow$) The proof is straight forward. □

Lemma 3.11. *Consider A as A-module. Let I be an ideal of A and P be a prime A-ideal of A containing I. Then $\frac{P}{I}$ is a prime A-ideal of $\frac{A}{I}$.*

Proof. Note that if the operation $\bullet : A \times \frac{A}{I} \to \frac{A}{I}$ is defined by $x \bullet \frac{y}{I} = \frac{x.y}{I}$, for any $x, y \in A$, then $\frac{A}{I}$ is an A-module. By Proposition 3.9, $\frac{P}{I}$ is an A-ideal of $\frac{A}{I}$, and it is easy to show that $\frac{P}{I}$ is a prime A-ideal of $\frac{A}{I}$. □

Lemma 3.12. *Let M_1 and M_2 be two A-modules, $\Phi : M_1 \to M_2$ be an MV-homomorphism and N be a prime A-ideal of M_2 such that $\phi(M_1) \nsubseteq N$. Then, $\phi^{-1}(N)$ is a prime A-ideal of M_1.*

Proof. The proof is straight forward. □

Notation. If M_1 and M_2 are two MV-algebras, then $hom(M_1, M_2)$ denotes the set of all MV-homomorphisms from M_1 to M_2.

Theorem 3.13. *Let M be an A-module, $rad(A)$ be the intersection of all prime A-ideals of A as A-module and $hom(M, \frac{A}{rad(A)}) \neq 0$. Then M contains a prime A-ideal.*

Proof. Since $hom(M, \frac{A}{rad(A)}) \neq 0$, then there exists an $MV-$homomorphism $\phi : M \to \frac{A}{rad(A)}$ such that $\phi(m) = \frac{a}{rad(A)} \neq \frac{0}{rad(A)}$, for some $m \in M$ and $a \in A$. Hence, $a \notin rad(A)$ and then there exists a prime A-ideal P of M such that $a \notin P$. Since $\frac{a}{rad(A)} \notin \frac{P}{rad(A)}$, $\phi(M) \nsubseteq \frac{P}{rad(A)}$. Therefore, by Lemmas 3.11 and 3.12, $\phi^{-1}(\frac{P}{rad(A)})$ is a prime A-ideal of M. □

4. Most results on A-ideals in MV-modules

In this section, we obtain some conditions that an A-ideal is not prime. Also, we investigate if $K, K_1, \ldots, K_n$ are A-ideals of A-module M such that $K \subseteq \bigcup_{i=1}^{n} K_i$, then $K \subseteq K_j$, for some $1 \leq j \leq n$.

Definition 4.1. Let M be an A-module and $K, K_1, \ldots, K_n$ be A-ideals of M. Then, $\bigcup_{i=1}^{n} K_i$ is called an *efficient covering* of K, if $K \subseteq \bigcup_{i=1}^{n} K_i$ and $K \nsubseteq \bigcup_{j\neq i=1}^{n} K_i$, for every $1 \leq j \leq n$. Moreover, $K = \bigcup_{i=1}^{n} K_i$ is called an *efficient union*, if $K \neq \bigcup_{j\neq i=1}^{n} K_i$, for every $1 \leq j \leq n$.

Example 4.2. Let $A = M = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "$'$" be defined on M as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

$'$	0	1	2	3
	3	2	1	0

.

Also, for every $a, b \in A$,

$$a.b = \begin{cases} 0 & a \neq y \\ x & a = b \end{cases} .$$

Then, it is easy to show that $(M, \oplus, ', 0)$ is an MV-algebra and $(A, \oplus, ', ., 0)$ is a PMV-algebra. Now, let the operation $\bullet : A \times M \longrightarrow M$ be defined by $a \bullet b = a.b$, for every $a \in A$ and $b \in M$. It is easy to see that M is an A-module and $K_1 = \{0, 1\}$, $K_2 = \{0, 2\}$, $K = \{0, 1, 2\}$ are A-ideals of M. Also, $K_1 \cup K_2$ is an efficient covering of K and it is an efficient union.

Lemma 4.3. *Let M be an A-module, $K, K_1, \ldots, K_n$ be A-ideals of M and $K = \bigcup_{i=1}^{n} K_i$ be an efficient union of A-ideals of M, where $n > 1$. Then, $\bigcap_{j\neq i=1}^{n} K_i = \bigcap_{i=1}^{n} K_i$, for every $1 \leq j \leq n$.*

Proof. Without loss of generality, let $j = 1$ and $a \in \bigcap_{i=2}^{n} K_i$. Since K has an efficient covering, then there exists $b \in K$ such that $b \notin \bigcup_{i=2}^{n} K_i$. Now, if $a \oplus b \in \bigcup_{i=2}^{n} K_i$, then there exists $2 \leq t \leq n$ such that $a \oplus b \in K_t$.

Since $b \leq a \oplus b \in K_t$, hence $b \in K_t$, which is a contradiction. Hence, $a \oplus b \in K - \bigcup_{i=2}^{n} K_i$ and so $a \oplus b \in K_1$. Since $a \leq a \oplus b \in K_1$, we get $a \in K_1$ and then $a \in \bigcap_{i=1}^{n} K_i$. It results that $\bigcap_{i=2}^{n} K_i \subseteq \bigcap_{i=1}^{n} K_i$, and therefore $\bigcap_{i=2}^{n} K_i = \bigcap_{i=1}^{n} K_i$. □

Theorem 4.4. *(Prime avoidance of A-ideals) Let M be a unitary A-module and $K, K_1, \ldots, K_n$ be A-ideals of M. (i) If $K \subseteq \bigcup_{i=1}^{n} K_i$ is an efficient covering of K and $(K_t : M) \nsubseteq (K_j : M)$, for any $j \neq t$, where $1 \leq j, t \leq n$, then K_j is not a prime A-ideal of M, for every $1 \leq j \leq n$.*
(ii) If $K \subseteq \bigcup_{i=1}^{n} K_i$, at most two of K_i 's are not prime and $(K_i : M) \nsubseteq (K_j : M)$, where $n \geq 3$, $j \neq i$ and $1 \leq i, j \leq n$, then there exists $1 \leq j \leq n$ such that $K \subseteq K_j$.

Proof. (i) We first show that $K = \bigcup_{i=1}^{n}(K \cap K_i)$ is an efficient union of K. Since $K \subseteq \bigcup_{i=1}^{n} K_i$ is an efficient covering of K, then there exists $a \in K$ such that $a \notin \bigcup_{j \neq i=1}^{n} K_i$, for any $j \neq i$, where $1 \leq j \leq n$. Hence, $a \notin K_i$ and so $a \notin K \cap K_i$, for any $i \neq j$. It then follows that $a \notin \bigcup_{j \neq i=1}^{n}(K \cap K_i)$ and so $K \neq \bigcup_{j \neq i=1}^{n}(K \cap K_i)$. Hence, $K = \bigcup_{i=1}^{n}(K \cap K_i)$ is an efficient union of K. Let j be a constant number, where $1 \leq j \leq n$. If $i \neq j$, then $(K_i : M) \nsubseteq (K_j : M)$ and so there exists $a_i \in (K_i : M) - (K_j : M)$, where $1 \leq i \leq n$. We set $a = a_1.a_2.\ldots.a_{j-1}.a_{j+1}.\ldots.a_n$. Since A is unital, by Lemma 2.7 (ii), we have $a \leq a_i$, where $1 \leq i \leq n$. Since $a \leq a_i \in (K_i : M)$, $a \in (K_i : M)$, for any $i \neq j$. Now, we show that K_j is not a prime A-ideal of M. Since $K = \bigcup_{i=1}^{n}(K \cap K_i)$ is an efficient union of K, there exists $x \in K - K_j$ and so by Lemma 4.3, we get $ax \in \bigcap_{j \neq i=1}^{n}(K \cap K_i) = \bigcap_{i=1}^{n}(K \cap K_i) \subseteq K_j$. If K_j is a prime A-ideal, then $x \in K_j$ or $a \in (K_j : M)$, which in any of two cases is a contradiction. Therefore, K_j is not a prime A-ideal of M, for every $1 \leq j \leq n$.
(ii) We have $K \subseteq \bigcup_{i=1}^{n} K_i$. Let $K \subseteq \bigcup_{t=1}^{m} K_{i_t}$ be an efficient covering of K, where $1 \leq m \leq n$ and $m \neq 2$. If $m > 2$, then at least one of the K_{i_t} 's is prime A-ideal of M and so by (i), that is a contradiction. Hence, $m = 1$ and therefore $K \subseteq K_j$, for some $1 \leq j \leq n$. □

Example 4.5. By Example 4.2, we have $(K_1 : M) = \{0, 1\}$ and $(K_2 : M) = \{0, 2\}$. It is clear that $(K_1 : M) \nsubseteq (K_2 : M)$ and $(K_2 : M) \nsubseteq (K_1 : M)$. Note that K_1 and K_2 are not prime A-ideals of M. For example, $2.3 = 0 \in K_1$, but $3 \notin K_1$ and $2 \notin (K_1 : M)$.

Note. Now, we want to state a different shape of the theorem of "prime avoidance of $A-$ideals". Let $K, K_1, \ldots, K_n$ be A-ideals of M and $m_1+K_1, \cdots, m_n+K_n$ be cosets in M, for $m_i \in M$, where $1 \leq i \leq n$. We say $\bigcup_{i=1}^{n}(m_i+K_i)$ is an efficient covering of K, if $K \subseteq \bigcup_{i=1}^{n}(m_i+K_i)$

and $K \nsubseteq \bigcup_{j\neq i=1}^{n}(m_i + K_i)$, for every $1 \leq j \leq n$. Moreover, $K = \bigcup_{i=1}^{n}(m_i + K_i)$ is an efficient union, if $K \neq \bigcup_{j\neq i=1}^{n}(m_i + K_i)$, for every $1 \leq j \leq n$.

Lemma 4.6. *Let M be an A-module, N be an A-ideal of M and $m \oplus N = \{m \oplus n : n \in N\}$. Then, $m \oplus N = N$, where $m \in M$ and $m \leq n$, for every $0 \neq n \in N$.*

Proof. Since $m \leq n \in N$, by (I_2), we get $m \in N$ and so $m \oplus N \subseteq N$. Since $n' \leq n' \oplus m$, by Lemma 2.3 (i), we have $(n' \oplus m)' \leq n \in N$ and hence $(n' \oplus m)' \in N$. Now, by $(MV4)$, we have

$$n = n \oplus 0 = n \oplus 1' = n \oplus (m' \oplus n)' = m \oplus (n' \oplus m)' \in m \oplus N,$$

for every $n \in N$ and then $N \subseteq m \oplus N$. Therefore, $m \oplus N = N$. □

Lemma 4.7. *Let M be an A-module, $K, K_1, \ldots, K_n$ be A-ideals of M and $K \subseteq \bigcup_{i=1}^{n}(K_i + m_i)$ be an efficient covering of K, where $n \geq 2$ and $m_i \leq k_i$, for every $0 \neq k_i \in K_i$, $1 \leq i \leq n$ and "$+$" is the partial addition on M. Then $K \cap (\bigcap_{j\neq i=1}^{n} K_i) \subseteq K_j$, but $K \nsubseteq K_j$, for any $1 \leq j \leq n$.*

Proof. Without loss of generality, we accept $j = 1$. Let $a \in K \cap \bigcap_{i=2}^{n} K_i$ and $b \in K - \bigcup_{i=2}^{n}(K_i + m_i)$. Then, $b \in K_1 + m_1$. If there exits $j \geq 2$ such that $a + b \in K_j + m_j$, then $a \in K_j$ implies that $b \in K_j + m_j$, which is a contradiction. Hence, $a + b \in K - \bigcup_{i=2}^{n}(K_j + m_j)$ and so $a + b \in K_1 + m_1$. It then results that $a + b = k_1 + m_1$, for some $k_1 \in K_1$. On the other hand, $b = k + m_1$, for some $k \in K_1$. Then, $a + k + m_1 = k_1 + m_1$ and so by Lemma 2.5 (iii), we get $a + k = k_1$. By Lemma 2.5 (ii), we have $a = k' \odot k_1 = (k_1' \oplus k)'$. Since $k_1' \leq k_1' \oplus k$, $(k_1' \oplus k)' \leq k_1 \in K_1$ so $a = (k_1' \oplus k)' \in K_1$. Hence, $K \cap (\bigcap_{i\neq 1} K_i) \subseteq K_1$. Now, let there exists $1 \leq j \leq n$ such that $K \subseteq K_j$. If $m_j \in K_j$, then by Lemma 4.6, we have $K \subseteq K_j = K_j + m_j$, which is a contradiction. Which the fact that $\bigcup_{i=1}^{n}(K_i + m_i)$ is an efficient covering of K. If $m_j \notin K_j$, then we will show that $K \cap (K_j + m_j) = \emptyset$. Let $x \in K \cap (K_j + m_j)$. Then there exists $k_j \in K_j$ such that $x = k_j + m_j \in K \subseteq K_j$. Since $m_j \leq k_j + m_j$, then $m_j \in K_j$, which is a contradiction. Hence, $K \cap (K_j + m_j) = \emptyset$ and so $K \subseteq \bigcup_{i\neq j}^{n}(K_i + m_i)$, which is a contradiction. Which the fact that $\bigcup_{i=1}^{n}(K_i + m_i)$ is an efficient covering of K. Therefore, $K \nsubseteq K_j$, for any $1 \leq j \leq n$. □

Theorem 4.8. *Let M be an A-module, $K, K_1, \ldots, K_n$ be A-ideals of M and $K + m \subseteq \bigcup_{i=1}^{n} K_i$ be an efficient covering of $K + m$ and $(K_j : M) \nsubseteq (K_t : M)$, for every $j \neq t$, where $1 \leq j, t \leq n$ and $m \in M$. Then K_j is not a prime A-ideal of M, for every $1 \leq j \leq n$.*

Proof. By Lemma 4.7, we have$K \cap (\bigcap_{j\neq i=1}^{n} K_i) \subseteq K_j$ and $K \nsubseteq K_j$, for every $1 \leq j \leq n$. Let $I = (\bigcap_{j\neq i=1}^{n} K_i : M)$. Then, $IK \subseteq K \cap (\bigcap_{j\neq i=1}^{n} K_i) \subseteq K_j$. Now, let K_j be a prime A-ideal of M. Then, $K \subseteq K_j$ or $IM \subseteq K_j$. Since $K \nsubseteq K_j$, $I \subseteq (K_j : M)$. On the other hand, $I = (\bigcap_{j\neq i=1}^{n} K_i : M) = \bigcap_{j\neq i=1}(K_i : M) \subseteq (K_j : M)$, for every $i \neq j$. Hence, there exists $i \neq j$ such that $(K_i : M) \subseteq (K_j : M)$, which is a contradiction. Therefore, K_i is not a prime A-ideal of M, for every $1 \leq i \leq n$. □

5. Conclusions

Our results in this paper about the $A-$ideals of $MV-$modules gives new insights for anyone who is interested in studying and development of ideals in $MV-$modules. One can study of ideals in $MV-$modules and obtain some new methods to study and characterize the $A-$ideals of $MV-$modules. Furthermore, one can define another types of $A-$ideals in $MV-$modules and study many other subjects in this field.

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Simin Saidi Goraghani
Department of Mathematics, University of Farhangian, Tehran, Iran.
Email: siminsaidi@yahoo.com

Rajab Ali Borzooei
Department of Mathematics, University of Shahid Beheshti, Tehran, Iran.
Email: borzooei@sbu.ac.ir

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MOST RESULTS ON A-IDEALS IN MV-MODULES

S. SAIDI GORAGHANI, R. A. BORZOOEI

نتایجی بیشتر روی A-ایده‌آل‌ها در MV-مدول‌ها

سیمین سعیدی گراغانی[۱] و رجب علی برزویی[۲]
دانشگاه فرهنگیان، تهران، ایران[۱]، دانشگاه شهید بهشتی، تهران، ایران[۲]

در مقاله ارائه شده، با در نظر گرفتن MV-مدول‌ها که به طور طبیعی ساختاری متناظر با lu-مدول‌ها روی lu-حلقه‌ها است، نتایجی را روی A-ایده‌آل‌های اول ثابت کرده و شرایطی را برای یافتن A-ایده‌آل‌های اول در MV-مدول‌ها بیان می‌کنیم. همچنین شرایط را برای داشتن یک A-ایده‌آل غیراول بیان و برای A-ایده‌آل‌های $K, K_{۱}, \ldots, K_n$ از A-مدول M شرایطی را مورد بررسی قرار می‌دهیم که از $K \subseteq \bigcup_{i=۱}^{n} K_i$ نتیجه شود $K \subseteq K_j$، جایی که $۱ \leq j \leq n$.

کلمات کلیدی: MV-جبر، MV-مدول، A-ایده‌آل اول.