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GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

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ABSTRACT. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and Γ be a finite group. It is proved that if R admits a dualizing module, then the group ring $R\Gamma$ has a dualizing bimodule, as well. Moreover, it is shown that a finitely generated $R\Gamma$-module M has generalized Gorenstein dimension zero if and only if it has generalized Gorenstein dimension zero as an R-module.

1. Introduction

Semi-dualizing modules arise naturally in the investigations of various duality theories in commutative algebras. One instance of this, is Grothendieck and Hartshorne's local duality wherein a canonical module, or more generally a dualizing complex, is employed to study local cohomology [13]. Another instance is Auslander and Bridger's methodical study of duality properties with respect to a rank one free module that gives rise to the Gorenstein dimension [3]. Foxby [10], Golod [12] and Vasconcelos [16] independently initiated the study of semi-dualizing modules (under different names) over commutative noetherian local rings. This notion then has been the subject of several expositions; see, for example, [8, 11, 17]. A finitely generated R-module C is said to be *semi-dualizing*, provided that $R \cong \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}^i_R(C, C) = 0$, for any $i > 0$. A rank one free module and a canonical module, are examples of semi-dualizing modules. The definition of semi-dualizing modules has been extended in various settings by

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many authors. In particular, Araya, Takahashi and Yoshino [1] have extended this definition to a pair of non-commutative, but noetherian rings, while White [18] extended to the non-noetherian, but commutative settings. This definition is defined for a pair of associative rings by Holm and White in [14]. Associated to a semi-dualizing R-module C, of most importance is the notion of G_C-dimension of a finitely generated module, which is studied by several authors; see [12, 8, 18]. As this invariant is a generalization of Gorenstein dimension (which is defined by Auslander and Bridger [3]), hence it is called a *generalized Gorenstein dimension.*

In this paper, we study generalized Gorenstein dimension of finitely generated modules over noetherian algebras. To be precise, assume that $(R, \mathfrak{m})$ is a commutative noetherian local ring and Λ is a noetherian R-algebra. Assume that ω is a semi-dualizing Λ-bimodule; see Definition 2.1. Among others, it is proved that if ω is dualizing, in the sense that it has finite injective dimension as a left and as a right Λ-module, then any finitely generated left Λ-module M has finite generalized Gorenstein dimension; see Proposition 2.9. This result, enables us to infer that any finitely generated left Λ-module admits a right $\mathrm{G}_\omega(\Lambda)$-approximation, where $\mathrm{G}_\omega(\Lambda)$ stands for the class of all left Λ-modules with generalized Gorenstein dimension zero; see Corollary 2.11. Section 3 is devoted to study the generalized Gorenstein dimension over a group algebra $R\Gamma$, where Γ is a finite group. At first, it will turn out that if R admits a dualizing module ω, then $R\Gamma \otimes_R \omega$ is a dualizing bimodule for $R\Gamma$; see Proposition 3.2. The main result of this paper asserts that if M is a finitely generated (left) $R\Gamma$-module, then M has generalized Gorenstein dimension zero if and only if it has generalized Gorenstein dimension zero as an R-module; see Theorem 3.5.

Throughout this paper, $(R, \mathfrak{m})$ denotes a commutative noetherian local ring and Λ is a notherian R-algebra. All modules will be considered are finitely generated left modules. Also, right Λ-modules will be identified with left modules over the *opposite algebra* Λ^{op}.

2. Generalized Gorenstein dimension of modules

In this section, by defining the notion of semi-dualizing bimodule for the noetherian algebra Λ, we investigate the generalized Gorenstein dimension of finitely generated Λ-modules. It is proved that if Λ admits a dualizing bimodule, then generalized Gorenstein dimension of each Λ-module is finite.

Definition 2.1. A finitely generated Λ-bimodule ω is called *semi-dualizing* for Λ, if:

(1) The homothety morphisms $\Lambda \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\omega, \omega)$ and $\Lambda^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{\Lambda}(\omega, \omega)$ are bijections,

(2) $\mathrm{Ext}^i_{\Lambda}(\omega, \omega) = 0 = \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\omega, \omega)$, for all $i > 0$.

Definition 2.2. Let ω be a semi-dualizing bimodule for Λ. Recall from [4] that a Λ-module M is said to have *generalized Gorenstein dimension zero* (with respect to ω), denoted by $\mathrm{G}_\omega - \dim_\Lambda M = 0$, if the following conditions are satisfied: (1) M is ω-reflexive, that is, the canonical Λ-homomorphism $M \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M^*, \omega)$ is an isomorphism and (2) $\mathrm{Ext}^i_{\Lambda}(M, \omega) = 0 = \mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M^*, \omega)$, for all $i > 0$, whereas $(-)^* = \mathrm{Hom}_\Lambda(-, \omega)$.

It follows from the definition that any finitely generated projective Λ-module and also any module belonging to $\mathsf{add}\omega$ have generalized Gorenstein dimension zero.

Here, by $\mathsf{add}\omega$ we mean the subcategory of all direct summands of finite direct sums of copies of ω. We use $\mathrm{G}_\omega(\Lambda)$ to denote the full subcategory of $\mathsf{mod}\Lambda$ consisting of all modules with generalized Gorenstein dimension zero. It is easy to see that the class of all modules in $\mathrm{G}_\omega(\Lambda)$ is closed under direct summands and finite direct sums. Moreover, according to [15, Lemma 2.4(1)], $\mathrm{G}_\omega(\Lambda)$ is projectively resolving; that is, it contains projective modules, and in any short exact sequence of Λ-modules; $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ with $M'' \in \mathrm{G}_\omega(\Lambda)$, we have $M \in \mathrm{G}_\omega(\Lambda)$ if and only if $M' \in \mathrm{G}_\omega(\Lambda)$.

Definition 2.3. Let M be a non-zero Λ-module. We write $\mathrm{G}_\omega - \dim_\Lambda M = 0$, if it belongs to $\mathrm{G}_\omega(\Lambda)$. It is clear that ω and Λ always belong to $\mathrm{G}_\omega(\Lambda)$. We say that G_ω-dimension of M is $n \geq 0$, if n is the least integer for which there exists an exact sequence of Λ-modules;

$$0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

where $GG_i \in \mathrm{G}_\omega(\Lambda)$, for any i. If there is no such a resolution of finite length, we set $\mathrm{G}_\omega - \dim_\Lambda M = \infty$. By convention, $\mathrm{G}_\omega - \dim_\Lambda 0 = -\infty$.

The following result will be used later.

Lemma 2.4. *Let M be a Λ-module. Then M is in $\mathrm{G}_\omega(\Lambda)$ if and only if there exists an exact sequence of Λ-modules*

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \omega_{-1} \longrightarrow \omega_{-2} \longrightarrow \cdots ,$$

where $\omega_i \in \mathsf{add}\omega$ and $P_j^{,}s$ are projective Λ-modules, for each i, j, such that $\mathbf{P}_\bullet$ remains exact after applying the functor $\mathrm{Hom}_\Lambda(-, \omega)$, and $M = \mathrm{Ker}(\omega_{-1} \longrightarrow \omega_{-2})$.

Proof. Assume that $\mathrm{G}_\omega - \dim_\Lambda M = 0$. Consider the following (finitely generated) projective resolution of the Λ^{op}-module $\mathrm{Hom}_\Lambda(M, \omega)$,

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathrm{Hom}_\Lambda(M, \omega) \longrightarrow 0.$$

As $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M^*, \omega) = 0$ for any $i > 0$, applying the functor $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-, \omega)$ on the above sequence, yields the following exact sequence of Λ-modules;

$$0 \to M \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P_0, \omega) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P_1, \omega) \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P_2, \omega) \to \cdots .$$

Therefore, by splicing this acyclic complex with the projective resolution of the Λ-module M, one obtains the desired acyclic complex. We should note that, it can be easily seen that this complex is $\mathrm{Hom}_\Lambda(-, \omega)$-exact. Conversely, assume that we are given an acyclic and $\mathrm{Hom}_\Lambda(-, \omega)$-exact complex $\mathbf{P}_\bullet$ satisfying the mentioned conditions, and M is its zeroth syzygy. We want to prove that $\mathrm{G}_\omega - \dim_\Lambda M = 0$. Let us first show that M is ω-reflexive. Since $\mathbf{P}_\bullet$ is $\mathrm{Hom}_\Lambda(-, \omega)$-exact, hence $\mathrm{Hom}_\Lambda(\mathbf{P}_\bullet, \omega) = {\mathbf{P}_\bullet}^*$ is acyclic and again reflexivity of each ω_i implies that $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathbf{P}^*_\bullet, \omega) \cong \mathbf{P}_\bullet$. In particular, the complex $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathbf{P}^*_\bullet, \omega)$ is exact; hence $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(M^*, \omega) \cong M$. Applying the functor $\mathrm{Hom}_\Lambda(-, \omega)$ on the short exact sequence of Λ-modules; $0 \longrightarrow L \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, yields the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_\Lambda(M, \omega) \longrightarrow \mathrm{Hom}_\Lambda(P_0, \omega) \longrightarrow \mathrm{Hom}_\Lambda(L, \omega) \longrightarrow 0.$$

Thus, in view of the fact $\mathrm{H}^i(\mathrm{Hom}_\Lambda(\mathbf{P}_\bullet, \omega)) = 0$, we deduce that $\mathrm{Ext}^i_\Lambda(M, \omega) = 0$, for all $i > 0$. Furthermore, by using the same argument, one deduces that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(M^*, \omega) = 0$, for all $i > 0$. So the proof is complete. □

2.5. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\Lambda$, we let $\widehat{\mathcal{X}}$ denote the category whose objects are the modules M for which there exists an exact sequence of Λ-modules; $0 \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$, where $X_i \in \mathcal{X}$ for all $0 \leq i \leq n$.

Proposition 2.6. *Let M be a Λ-module of finite G_ω-dimension. Then the following conditions are satisfied:*

(1) *There exists a short exact sequence of Λ-modules; $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$, whenever $L \in \widehat{\mathsf{add}\omega}$ and $\mathrm{G}_\omega - \dim_\Lambda G = 0$.*
(2) *There exists an exact sequence of Λ-modules; $0 \longrightarrow M \longrightarrow \omega' \longrightarrow G' \longrightarrow 0$, whenever $\omega' \in \widehat{\mathsf{add}\omega}$ and $\mathrm{G}_\omega - \dim_\Lambda G' = 0$.*

Proof. (1). This follows from the argument given in the proof of [15, Proposition 3.4].
(2). In view of part (1), there exists an exact sequence $0 \longrightarrow \omega_1 \longrightarrow G \longrightarrow M \longrightarrow 0$, where $\omega_1 \in \widehat{\mathsf{add}\omega}$ and $\mathrm{G}_\omega - \dim_\Lambda G = 0$. According

to Lemma 2.4, there exists an Λ-monomorphism $G \longrightarrow \omega_0$ in which $\omega_0 \in \mathsf{add}\omega$. Consider the following push-out diagram along the maps $G \longrightarrow M$ and $G \longrightarrow \omega_0$;

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_1 & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \omega_1 & \longrightarrow & \omega_0 & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & G' & = & G' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Now, the exact sequence $0 \longrightarrow M \longrightarrow L \longrightarrow G' \longrightarrow 0$ would be the desired sequence, and the proof is complete. □

As a consequence of the above proposition, we are able to state the following result.

Corollary 2.7. *Let I be an injective Λ-module. If I has finite G_ω-dimension, then $I \in \widehat{\mathsf{add}\omega}$.*

Proof. According to the above proposition, there exists an exact sequence of Λ-modules; $0 \longrightarrow I \longrightarrow L \longrightarrow G \longrightarrow 0$, in which $L \in \widehat{\mathsf{add}\omega}$ and $G \in \mathrm{G}_\omega(\Lambda)$. Now, I being injective yields that this sequence is indeed split. So the result follows. □

Definition 2.8. Let ω be a semi-dualizing bimodule for Λ. Following Enochs et al. [9], we say that ω is a dualizing bimodule, if it has finite injective dimension as both Λ and Λ^{op}-module.

Proposition 2.9. *Let ω be a dualizing bimodule for Λ. Then for any Λ-module M, $\mathrm{G}_\omega - \dim_\Lambda \mathrm{M} < \infty$.*

Proof. Assume that $\mathrm{id}_\Lambda \omega \leq n$, for some positive integer n. Take the following exact sequence of Λ-modules;

$$0 \longrightarrow \Omega^n M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

in which for any i, P_i is projective. As $P_i \in \mathrm{G}_\omega(\Lambda)$, we only need to show that $\Omega^n M \in \mathrm{G}_\omega(\Lambda)$. Since $\mathrm{id}_\Lambda \omega \leq n$, one may obtain that $\mathrm{Ext}^i_R(\Omega^n M, \omega) \cong \mathrm{Ext}^{i+n}_R(M, \omega) = 0$, for any $i > 0$. So, applying the functor $\mathrm{Hom}_\Lambda(-, \omega)$ to the projective resolution of $\Omega^n M$; $\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow \Omega^n M \longrightarrow 0$, gives rise to the following exact sequence of Λ^{op}-modules;

$$0 \longrightarrow \mathrm{Hom}_\Lambda(\Omega^n M, \omega) \longrightarrow \mathrm{Hom}_\Lambda(Q_0, \omega) \longrightarrow \mathrm{Hom}_\Lambda(Q_1, \omega) \longrightarrow \cdots .$$

As $\mathrm{id}_{\Lambda^{\mathrm{op}}}\omega < \infty$, by applying the functor $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-, \omega)$ to the latter sequence, one obtains the following exact sequence of Λ-modules:

$$\cdots \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(Q_0, \omega), \omega) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(\Omega^n M, \omega), \omega) \to 0,$$

implying the Λ-isomorphism $M \cong \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(M, \omega), \omega)$. Moreover, it is clear that $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(\Omega^n M, \omega), \omega) = 0$, for all $i > 0$. Thus, $\Omega^n M \in \mathrm{G}_\omega(\Lambda)$, as needed. □

2.10. Let M be a Λ-module. A Λ-homomorphism $f : G \longrightarrow M$ with G is in $\mathrm{G}_\omega(\Lambda)$ is said to be a right $\mathrm{G}_\omega(\Lambda)$-approximation, if the map $\mathrm{Hom}_\Lambda(G', f) : \mathrm{Hom}_\Lambda(G', G) \longrightarrow \mathrm{Hom}_\Lambda(G', M)$ is surjective, for any Λ-module $G' \in \mathrm{G}_\omega(\Lambda)$.

Corollary 2.11. *Let ω be a dualizing bimodule for Λ. Then any Λ-module admits a right $\mathrm{G}_\omega(\Lambda)$-approximation.*

Proof. Assume that M is an arbitrary Λ-module. By virtue of Proposition 2.6 (1), there exists a short exact sequence of Λ-modules; $0 \longrightarrow L \longrightarrow G \longrightarrow M \longrightarrow 0$, whenever $L \in \widehat{\mathsf{add}\omega}$ and $\mathrm{G}_\omega\!-\!\dim_\Lambda G = 0$. Suppose that $N \in \mathrm{G}_\omega(\Lambda)$ is arbitrary. As $L \in \widehat{\mathsf{add}\omega}$, we have $\mathrm{Ext}^1_\Lambda(N, L) = 0$, implying that $\mathrm{Hom}_\Lambda(N, G) \longrightarrow \mathrm{Hom}_\Lambda(N, M)$ is an epimorphism. This means that $G \longrightarrow M$ is a left $\mathrm{G}_\omega(\Lambda)$-approximation of M. So the proof is complete. □

3. Generalized Gorenstein dimension over group rings

This section is devoted to study the generalized Gorenstein dimension of modules over group rings. First, we show that having a dualizing bimodule ascends from R to $R\Gamma$, whenever R is a commutative noetherian local ring and Γ is a finite group. Our main result in this section indicates that, if M is a finitely generated $R\Gamma$-module, then generalized Gorenstein dimensions of M over R and $R\Gamma$, are identical. We should point out that the notion of Gorenstein dimension of modules over group rings has been studied first in [2, 5].

3.1. One should observe that, since Γ is a finite group, hence $R\Gamma$ is a noetherian R-algebra. We also point out that, as $R\Gamma$ is isomorphic with the opposite ring $(R\Gamma)^{\mathrm{op}}$, the distinction between left and right modules is redundant.

Proposition 3.2. *Let ω be a dualizing R-module and let Γ be a finite group. Then $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$-bimodule.*

Proof. Since $R\Gamma$ is a faithfully flat R-module, for any $i \geq 0$, one may have the following isomorphism:

$$\mathrm{Ext}^i_{R\Gamma}(R\Gamma \otimes_R \omega, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \mathrm{Ext}^i_R(\omega, \omega).$$

By the hypothesis, $\mathrm{Ext}^i_R(\omega,\omega) = 0$ for any $i > 0$; implying that the right hand side of the above isomorphism will be zero. Consequently, using again the faithfully flatness of $R\Gamma$ over R, we conclude that

$$\mathrm{Hom}_{R\Gamma}(R\Gamma \otimes_R \omega, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \mathrm{Hom}_R(\omega,\omega) \cong R\Gamma \otimes_R R \cong R\Gamma.$$

So, $R\Gamma \otimes_R \omega$ is a semi-dualizing $R\Gamma$-bimodule. Moreover, assume that $0 \longrightarrow \omega \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^t \longrightarrow 0$ is a (minimal) injective resolution of ω. Since $R\Gamma$ is a free R-module, applying the functor $R\Gamma \otimes_R -$ to this sequence gives rise to the following exact sequence of $R\Gamma$-modules:

$$0 \longrightarrow R\Gamma \otimes_R \omega \longrightarrow R\Gamma \otimes_R I^0 \longrightarrow \cdots \longrightarrow R\Gamma \otimes_R I^t \longrightarrow 0.$$

Hence, as Γ is a finite group, by making use of [6, Proposition III.5.9], we infer that $R\Gamma \otimes_R I^i \cong \mathrm{Hom}_R(R\Gamma, I^i)$ for any $0 \le i \le t$, and so $R\Gamma \otimes_R I^i$ is an injective $R\Gamma$-module. This, in turn, means that $R\Gamma \otimes_R \omega$ has finite injective dimension. Thus, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$-bimodule, as needed. □

Example 3.3. (1) Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring. According to [7, Corollary 3.3.8], R admits a canonical module, say ω. Clearly, ω is a dualizing R-module in our sense. So, by Proposition 3.2, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$-bimodule.
(2) Assume that $(R, \mathfrak{m})$ is a Gorenstein local ring. So, R is a dualizing R-module. Thus, by the above proposition, $R\Gamma \otimes_R R \cong R\Gamma$ is a dualizing $R\Gamma$-bimodule. In particular, $R\Gamma$ is a (not necessarily commutative) Gorenstein ring.

Lemma 3.4. *Let M be an R-module and Γ be a finite group. If $\mathrm{G}_\omega - \dim_R M = 0$, then $\mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} R\Gamma \otimes_R M = 0$.*

Proof. Since $\mathrm{G}_\omega - \dim_R M = 0$, in view of Lemma 2.4, there exists an acyclic complex of R-modules;

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \omega_{-1} \longrightarrow \omega_{-2} \longrightarrow \cdots ,$$

such that $M = \mathrm{Coker}(P_1 \longrightarrow P_0)$ and the the functor $\mathrm{Hom}_R(-,\omega)$ leaves $\mathbf{P}_\bullet$ exact. According to the fact that $R\Gamma$ is a (finitely generated) free R-module, one may obtain the following acyclic complex of $R\Gamma$-modules;

$$\mathbf{P}_\bullet : \cdots \to R\Gamma \otimes_R P_1 \to R\Gamma \otimes_R P_0 \to R\Gamma \otimes_R \omega_{-1} \to R\Gamma \otimes_R \omega_{-2} \to \cdots ,$$

in which $R\Gamma \otimes_R P_i$ is projective and $R\Gamma \otimes_R \omega_i \in \mathsf{add}(R\Gamma \otimes_R \omega)$. Moreover, we have the following isomorphism of $R\Gamma$-modules;

$$\mathrm{Hom}_{R\Gamma}(R\Gamma \otimes_R \mathbf{P}_\bullet, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \mathrm{Hom}_R(\mathbf{P}_\bullet, \omega).$$

Since, by the hypothesis, $\mathrm{Hom}_R(\mathbf{P}_\bullet,\omega)$ is acyclic, the same is true for $R\Gamma\otimes_R \mathrm{Hom}_R(\mathbf{P}_\bullet,\omega)$, implying that the right hand side of the above isomorphism is acyclic. So, the proof is complete. □

Theorem 3.5. *Let Γ be a finite group and M be an $R\Gamma$-module. Then $\mathrm{G}_\omega-\dim_R M=0$ if and only if $\mathrm{G}_{R\Gamma\otimes_R\omega}-\dim_{R\Gamma} M=0$.*

Proof. Let us first prove the 'sufficiency'. Since $\mathrm{G}_{R\Gamma\otimes_R\omega}-\dim_{R\Gamma} M=0$, in view of Lemma 2.4, there exists an exact sequence of Λ-modules

$$\mathbf{P}_\bullet:\cdots\longrightarrow P_1\longrightarrow P_0\longrightarrow \omega_{-1}\longrightarrow\omega_{-2}\longrightarrow\cdots,$$

where each ω_i belongs to $\mathsf{add}\omega$ and all P_js are projective Λ-module, such that $\mathbf{P}_\bullet$ remains exact after applying the functor $\mathrm{Hom}_\Lambda(-,\omega)$, and $M=\mathrm{Ker}(\omega_{-1}\longrightarrow\omega_{-2})$. As $R\Gamma$ is a finitely generated free R-module, we get that any projective $R\Gamma$-module is also projective as an R-module, and any module in $\mathsf{add}(R\Gamma\otimes_R\omega)$ also belongs to $\mathsf{add}(\omega)$. So, it remains to show that the complex $\mathrm{Hom}_R(\mathbf{P}_\bullet,\omega)$ is exact. To see this, consider the following isomorphisms;

$$\begin{aligned}\mathrm{Hom}_R(\mathbf{P}_\bullet,\omega)&\cong \mathrm{Hom}_R(R\Gamma\otimes_{R\Gamma}\mathbf{P}_\bullet,\omega)\\&\cong \mathrm{Hom}_{R\Gamma}(\mathbf{P}_\bullet,\mathrm{Hom}_R(R\Gamma,\omega))\\&\cong \mathrm{Hom}_{R\Gamma}(\mathbf{P}_\bullet,R\Gamma\otimes_R\omega)).\end{aligned}$$

We should point out that the first isomorphism holds trivially, the second one is the adjointness of Hom and $\otimes$, and the validity of the last isomorphism comes from [6, Proposition III.5.9]. By our assumption, the last complex is acyclic, implying that the same is true for the complex $\mathrm{Hom}_R(\mathbf{P}_\bullet,\omega)$, as needed. Next, we want to prove the 'necessity'. Assume that $\mathrm{G}_\omega-\dim_R M=0$. In view of Lemma 2.4, there exists a short exact sequence of R-modules $0\longrightarrow M\longrightarrow\omega_{-1}\longrightarrow L\longrightarrow 0$, in which $\mathrm{G}_\omega-\dim_R L=0$ and $\omega_{-1}\in\mathrm{add}(\omega)$. Consider the following commutative diagram of $R\Gamma$-modules with exact rows;

$$\begin{array}{ccccccccc}0&\longrightarrow&M&\longrightarrow&R\Gamma\otimes_R M&\longrightarrow&L'&\longrightarrow&0\\&&\downarrow{\scriptstyle id}&&\downarrow{\scriptstyle\theta}&&\downarrow{\scriptstyle\varphi}&&\\0&\longrightarrow&M&\longrightarrow&R\Gamma\otimes_R\omega_{-1}&\longrightarrow&L''&\longrightarrow&0.\end{array}$$

Clearly, φ is a monomorphism and $\mathrm{Coker}\varphi=R\Gamma\otimes_R L$. It is known that the exact sequence $0\longrightarrow M\longrightarrow R\Gamma\otimes_R M\longrightarrow L'\longrightarrow 0$ splits over R. Thus, $L'\cong\oplus_{i=1}^{t-1}M$ as R-modules, where $t=\mid\Gamma\mid$, the order of Γ. Consequently, the hypothesis made on M implies that $\mathrm{G}_\omega-\dim_R L'=0$. On the other hand, since $\mathrm{G}_\omega-\dim_R L=0$, it is not hard to see that $\mathrm{G}_{R\Gamma\otimes_R\omega}-\dim_{R\Gamma}(R\Gamma\otimes_R L)=0$ and so $\mathrm{G}_\omega-\dim_R(R\Gamma\otimes_R L)=0$, thanks to the fact that $R\Gamma\otimes_R L\cong\oplus_{i=1}^{t}L$, as R-modules. We claim that $\mathrm{G}_\omega-\dim_R L''=0$. The exact sequence

$$0\longrightarrow R\Gamma\otimes_R M\longrightarrow R\Gamma\otimes_R\omega_{-1}\longrightarrow R\Gamma\otimes_R L\longrightarrow 0,$$

splits over R as follows:

$$0 \longrightarrow \oplus_{i=1}^{t} M \longrightarrow \oplus_{i=1}^{t} \omega_{-1} \longrightarrow \oplus_{i=1}^{t} L \longrightarrow 0. \quad (1)$$

Similarly, the short exact sequence $0 \longrightarrow L' \longrightarrow L'' \longrightarrow R\Gamma \otimes_R L \longrightarrow 0$, which splits over R, can be written as follows, when viewed as an exact sequence of R-modules;

$$0 \longrightarrow \oplus_{i=1}^{t-1} M \longrightarrow L'' \longrightarrow \oplus_{i=1}^{t} L \longrightarrow 0. \quad (2)$$

Considering the exact sequences (1) and (2), we conclude that $L'' \cong \oplus_{i=1}^{t-1} \omega_{-1} \oplus L$, as R-modules. Hence, $\mathrm{G}_\omega - \dim_R L'' = 0$. In fact, we obtain the short exact sequence $0 \longrightarrow M \longrightarrow R\Gamma \otimes_R \omega_{-1} \longrightarrow L'' \longrightarrow 0$, in which $\mathrm{G}_\omega - \dim_R L'' = 0$. By repeating this manner, one gets the following exact sequence of $R\Gamma$-modules

$$\omega_\bullet : 0 \longrightarrow R\Gamma \otimes_R \omega_{-1} \longrightarrow R\Gamma \otimes_R \omega_{-2} \longrightarrow \cdots,$$

where $\omega_i \in \mathrm{add}(\omega)$ for all i, and every cosyzygy of $\omega_\bullet$ belongs to $\mathrm{G}_\omega(R)$. Now, we show that $\omega_\bullet$ is $\mathrm{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$-exact. Consider the following isomorphisms:

$$\mathrm{Hom}_{R\Gamma}(\omega_\bullet, R\Gamma \otimes_R \omega) \cong \mathrm{Hom}_R(R\Gamma \otimes_{R\Gamma} \omega_\bullet, \omega) \cong \mathrm{Hom}_R(\omega_\bullet, \omega).$$

By making use of the fact that every cosyzygy of $\omega_\bullet$ belongs to $\mathrm{G}_\omega(R)$, we conclude that the right hand side is exact, which gives the claim. Hence, in order to complete the proof, it suffices to find a left resolution of modules in $\mathrm{add}((R\Gamma \otimes_R \omega) \oplus R\Gamma)$ for M, which is $\mathrm{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$-exact. Take the following projective resolution of $R\Gamma$-module M;

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is finitely generated. By replacing $\mathbf{P}_\bullet$ with $\omega_\bullet$ in the above isomorphisms and invoking the fact that $\mathrm{G}_\omega - \dim_R M = 0$, one deduces that $\mathrm{Hom}_{R\Gamma}(-, R\Gamma \otimes_R \omega)$ leaves the sequence $\mathbf{P}_\bullet$ exact. Now, splicing $\omega_\bullet$ and $\mathbf{P}_\bullet$ and Lemma 2.4 completes the proof. □

The above theorem immediately gives rise to the following result.

Corollary 3.6. *Let R be a commutative ring with a semi-dualizing module ω. Let Γ be a finite group and M be an $R\Gamma$-module. Then $\mathrm{G}_\omega - \dim_R M = \mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$.*

Proof. We first show that $\mathrm{G}_\omega - \dim_R M \leq \mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$. If $\mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M = \infty$, there is nothing to prove. So, assume that $\mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} M$ is finite, say t. Hence, there exists an exact sequence of $R\Gamma$-modules; $0 \longrightarrow G_t \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$, whenever $\mathrm{G}_{R\Gamma \otimes_R \omega} - \dim_{R\Gamma} G_i = 0$ for any i. According to Theorem 3.5, $\mathrm{G}_\omega - \dim_R G_i = 0$, implying that $\mathrm{G}_\omega - \dim_R M \leq t$. Next, we want to show the reverse inequality. To that end, we may assume that $\mathrm{G}_\omega -$

$\dim_R M = n$, for some non-negative integer n. Take an exact sequence of $R\Gamma$-modules; $0 \longrightarrow \Omega^n M \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where each P_i is projective. Since $R\Gamma$ is a free R-module, any P_i is projective over R. Hence, by the hypothesis, we get $\mathrm{G}_\omega - \dim_R \Omega^n M = 0$. Using again of Theorem 3.5, gives rise to the equality $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} \Omega^n M = 0$. Consequently, $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \leq n$, and the proof is complete. □

Let Γ be a finite group and ω be a semi-dualizing R-module. We denote by $\mathcal{X}$ (resp., $\mathcal{Y}$) the subcategory of $\mathsf{mod}R$ (resp., $\mathsf{mod}R\Gamma$) consisting of all modules of finite G_ω-dimension (resp., $\mathrm{G}_{\mathrm{R}\Gamma\otimes_{\mathrm{R}}\omega}$-dimension).

Proposition 3.7. *With the above notations, one has the equality*

$$\sup\{\mathrm{G}_\omega - \dim_R M \mid M \in \mathcal{X}\} = \sup\{\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}.$$

Proof. Let us first show the inequality

$$\sup\{\mathrm{G}_\omega - \dim_R M \mid M \in \mathcal{X}\} \leq \sup\{\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}.$$

If $\sup\{\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\} = \infty$, there is nothing to prove. So, assume that $\sup\{\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\}$ exists finite, say t. Taking an arbitrary object $X \in \mathcal{X}$, there is an exact sequence of R-modules; $0 \longrightarrow G_n \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$ in which for any i, $\mathrm{G}_\omega - \dim_R G_i = 0$. Apply the functor $R\Gamma \otimes_R -$ to this, in order to obtain the exact sequence of $R\Gamma$-modules; $0 \longrightarrow R\Gamma \otimes_R G_n \longrightarrow \cdots \longrightarrow R\Gamma \otimes_R G_0 \longrightarrow R\Gamma \otimes_R X \longrightarrow 0$. In view of Lemma 3.4, $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} R\Gamma \otimes_R G_i = 0$, implying that $R\Gamma \otimes_R X \in \mathcal{Y}$. Hence, our assumption yields that $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} R\Gamma \otimes_R X \leq t$, and so by making use of Corollary 3.6, we infer that $\mathrm{G}_\omega - \dim_R R\Gamma \otimes_R X \leq t$ as well. Now, X being an R-direct summand of $R\Gamma \otimes_R X$ implies that $\mathrm{G}_\omega - \dim_R X \leq t$. Consequently, $\sup\{\mathrm{G}_\omega - \dim_R M \mid M \in \mathcal{X}\} \leq t$. Next, we will show that $\sup\{\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \mid M \in \mathcal{Y}\} \leq \sup\{\mathrm{G}_\omega - \dim_R M \mid M \in \mathcal{X}\}$. To do this, we may assume that $\sup\{\mathrm{G}_\omega - \dim_R M \mid M \in \mathcal{X}\} = s < \infty$. Take an arbitrary $R\Gamma$-module M with finite $\mathrm{G}_{R\Gamma\otimes_R\omega}$-dimension. According to Corollary 3.6, we have that $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M = \mathrm{G}_\omega - \dim_R M \leq s$. This, in particular, gives the desired result. □

Example 3.8. Let $(R, \mathfrak{m})$ be a complete Cohen-Macaulay local ring with canonical module ω. According to Proposition 3.2, $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$-bimodule. Now, by making use of Proposition 2.9 in conjunction with Corollary 3.6, one may deduce that for any finitely generated $R\Gamma$-module M, $\mathrm{G}_{R\Gamma\otimes_R\omega} - \dim_{R\Gamma} M \leq n$, where $n = \mathrm{id}_R\omega$.

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GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

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بعد گرنشتاین تعمیم یافته روی حلقه گروه‌ها

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فرض کنید $(R, \mathfrak{m})$ یک حلقه‌ی جابجایی نوتری و موضعی بوده و Γ یک گروه متناهی باشد. ثابت می‌کنیم که اگر R دارای مدول دوگانی باشد، آنگاه حلقه گروه $R\Gamma$ نیز دارای دومدول دوگانی می‌باشد. به‌علاوه، ثابت می‌کنیم که $R\Gamma$-مدول با تولید متناهی M دارای بعد گرنشتاین تعمیم یافته‌ی صفر است، اگر و تنها اگر بعد گرنشتاین تعمیم یافته‌ی M به عنوان R-مدول، برابر با صفر باشد.

کلمات کلیدی: دومدول‌های نیم‌دوگانی، بعد گرنشتاین تعمیم یافته، حلقه گروه‌ها.