GENERALIZED GORENSTEIN DIMENSION OVER
GROUP RINGS

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Abstract. Let \((R, \mathfrak{m})\) be a commutative noetherian local ring
and \(G\) be a finite group. It is proved that if \(R\) admits a dualizing
module, then the group ring \(RG\) has a dualizing bimodule, as
well. Moreover, it is shown that a finitely generated \(RG\)-module
\(M\) has generalized Gorenstein dimension zero if and only if it has
generalized Gorenstein dimension zero as an \(R\)-module.

1. Introduction

Semi-dualizing modules arise naturally in the investigations of vari-
ous duality theories in commutative algebras. One instance of this,
is Grothendieck and Hartshorne’s local duality wherein a canonical
module, or more generally a dualizing complex, is employed to study
local cohomology [13]. Another instance is Auslander and Bridger’s
methodical study of duality properties with respect to a rank one free
module that gives rise to the Gorenstein dimension [3]. Foxby [10],
Golod [12] and Vasconcelos [16] independently initiated the study of
semi-dualizing modules (under different names) over commutative no-
etherian local rings. This notion then has been the subject of several
expositions; see, for example, [8, 11, 17]. A finitely generated \(R\)-module
\(C\) is said to be semi-dualizing, provided that \(R \cong \text{Hom}_R(C, C)\) and
\(\text{Ext}^i_R(C, C) = 0\), for any \(i > 0\). A rank one free module and a canoni-
cal module, are examples of semi-dualizing modules. The definition
of semi-dualizing modules has been extended in various settings by
many authors. In particular, Araya, Takahashi and Yoshino [1] have extended this definition to a pair of non-commutative, but noetherian rings, while White [18] extended to the non-noetherian, but commutative settings. This definition is defined for a pair of associative rings by Holm and White in [14]. Associated to a semi-dualizing $R$-module $C$, of most importance is the notion of $G_C$-dimension of a finitely generated module, which is studied by several authors; see [12, 8, 18]. As this invariant is a generalization of Gorenstein dimension (which is defined by Auslander and Bridger [3]), hence it is called a generalized Gorenstein dimension.

In this paper, we study generalized Gorenstein dimension of finitely generated modules over noetherian algebras. To be precise, assume that $(R, \mathfrak{m})$ is a commutative noetherian local ring and $\Lambda$ is a noetherian $R$-algebra. Assume that $\omega$ is a semi-dualizing $\Lambda$-bimodule; see Definition 2.1. Among others, it is proved that if $\omega$ is dualizing, in the sense that it has finite injective dimension as a left and as a right $\Lambda$-module, then any finitely generated left $\Lambda$-module $M$ has finite generalized Gorenstein dimension; see Proposition 2.9. This result, enables us to infer that any finitely generated left $\Lambda$-module admits a right $G_{\omega}(\Lambda)$-approximation, where $G_{\omega}(\Lambda)$ stands for the class of all left $\Lambda$-modules with generalized Gorenstein dimension zero; see Corollary 2.11. Section 3 is devoted to study the generalized Gorenstein dimension over a group algebra $R\Gamma$, where $\Gamma$ is a finite group. At first, it will turn out that if $R$ admits a dualizing module $\omega$, then $R\Gamma \otimes_R \omega$ is a dualizing bimodule for $R\Gamma$; see Proposition 3.2. The main result of this paper asserts that if $M$ is a finitely generated (left) $RT$-module, then $M$ has generalized Gorenstein dimension zero if and only if it has generalized Gorenstein dimension zero as an $R$-module; see Theorem 3.5.

Throughout this paper, $(R, \mathfrak{m})$ denotes a commutative noetherian local ring and $\Lambda$ is a noetherian $R$-algebra. All modules will be considered are finitely generated left modules. Also, right $\Lambda$-modules will be identified with left modules over the opposite algebra $\Lambda^{\text{op}}$.

2. Generalized Gorenstein dimension of modules

In this section, by defining the notion of semi-dualizing bimodule for the noetherian algebra $\Lambda$, we investigate the generalized Gorenstein dimension of finitely generated $\Lambda$-modules. It is proved that if $\Lambda$ admits a dualizing bimodule, then generalized Gorenstein dimension of each $\Lambda$-module is finite.
Definition 2.1. A finitely generated \( \Lambda \)-bimodule \( \omega \) is called semi-dualizing for \( \Lambda \), if:

1. The homothety morphisms \( \Lambda \to \text{Hom}_{\Lambda^{op}}(\omega, \omega) \) and \( \Lambda^{op} \to \text{Hom}_{\Lambda}(\omega, \omega) \) are bijections,
2. \( \text{Ext}^i_{\Lambda}(\omega, \omega) = 0 = \text{Ext}^{i}_{\Lambda^{op}}(\omega, \omega) \), for all \( i > 0 \).

Definition 2.2. Let \( \omega \) be a semi-dualizing bimodule for \( \Lambda \). Recall from [4] that a \( \Lambda \)-module \( M \) is said to have generalized Gorenstein dimension zero (with respect to \( \omega \)), denoted by \( \text{G}_\omega \dim M = 0 \), if the following conditions are satisfied: (1) \( M \) is \( \omega \)-reflexive, that is, the canonical \( \Lambda \)-homomorphism \( M \to \text{Hom}_{\Lambda^{op}}(M^*, \omega) \) is an isomorphism and (2) \( \text{Ext}^i_{\Lambda}(M, \omega) = 0 = \text{Ext}^{i}_{\Lambda^{op}}(M^*, \omega) \), for all \( i > 0 \), whereas \( (-)^* = \text{Hom}_{\Lambda}(\cdot, \omega) \).

It follows from the definition that any finitely generated projective \( \Lambda \)-module and also any module belonging to \( \text{add}\omega \) have generalized Gorenstein dimension zero.

Here, by \( \text{add}\omega \) we mean the subcategory of all direct summands of finite direct sums of copies of \( \omega \). We use \( G_\omega(\Lambda) \) to denote the full subcategory of \( \text{mod}\Lambda \) consisting of all modules with generalized Gorenstein dimension zero. It is easy to see that the class of all modules in \( G_\omega(\Lambda) \) is closed under direct summands and finite direct sums. Moreover, according to [15, Lemma 2.4(1)], \( G_\omega(\Lambda) \) is projectively resolving; that is, it contains projective modules, and in any short exact sequence of \( \Lambda \)-modules:

\[
0 \to M' \to M \to M'' \to 0 \quad \text{with} \quad M'', M' \in G_\omega(\Lambda),
\]
we have \( M \in G_\omega(\Lambda) \) if and only if \( M' \in G_\omega(\Lambda) \).

Definition 2.3. Let \( M \) be a non-zero \( \Lambda \)-module. We write \( G_\omega - \dim_\Lambda M = 0 \), if it belongs to \( G_\omega(\Lambda) \). It is clear that \( \omega \) and \( \Lambda \) always belong to \( G_\omega(\Lambda) \). We say that \( G_\omega \)-dimension of \( M \) is \( n \geq 0 \), if \( n \) is the least integer for which there exists an exact sequence of \( \Lambda \)-modules:

\[
0 \to G_n \to \cdots \to G_0 \to M \to 0,
\]
where \( GG_i \in G_\omega(\Lambda) \), for any \( i \). If there is no such a resolution of finite length, we set \( G_\omega - \dim_\Lambda M = \infty \). By convention, \( G_\omega - \dim_\Lambda 0 = -\infty \).

The following result will be used later.

Lemma 2.4. Let \( M \) be a \( \Lambda \)-module. Then \( M \) is in \( G_\omega(\Lambda) \) if and only if there exists an exact sequence of \( \Lambda \)-modules

\[
P_\bullet : \cdots \to P_1 \to P_0 \to \omega_{-1} \to \omega_{-2} \to \cdots,
\]
where \( \omega_i \in \text{add}\omega \) and \( P_j \)s are projective \( \Lambda \)-modules, for each \( i, j \), such that \( P_\bullet \) remains exact after applying the functor \( \text{Hom}_{\Lambda}(\cdot, \omega) \), and \( M = \ker(\omega_{-1} \to \omega_{-2}) \).
Proof. Assume that $G - \dim \Lambda M = 0$. Consider the following (finitely generated) projective resolution of the $\Lambda^\text{op}$-module $\text{Hom}_\Lambda(M, \omega)$,

$$\cdots \to P_2 \to P_1 \to P_0 \to \text{Hom}_\Lambda(M, \omega) \to 0.$$ 

As $\text{Ext}^i_{\Lambda^\text{op}}(M^*, \omega) = 0$ for any $i > 0$, applying the functor $\text{Hom}_{\Lambda^\text{op}}(-, \omega)$ on the above sequence, yields the following exact sequence of $\Lambda$-modules;

$$0 \to M \to \text{Hom}_{\Lambda^\text{op}}(P_0, \omega) \to \text{Hom}_{\Lambda^\text{op}}(P_1, \omega) \to \text{Hom}_{\Lambda^\text{op}}(P_2, \omega) \to \cdots.$$ 

Therefore, by splicing this acyclic complex with the projective resolution of the $\Lambda$-module $M$, one obtains the desired acyclic complex. We should note that, it can be easily seen that this complex is $\text{Hom}_{\Lambda^\text{op}}(-, \omega)$-exact. Conversely, assume that we are given an acyclic and $\text{Hom}_{\Lambda^\text{op}}(-, \omega)$-exact complex $P_\bullet$ satisfying the mentioned conditions, and $M$ is its zeroth syzygy. We want to prove that $G - \dim \Lambda M = 0$. Let us first show that $M$ is $\omega$-reflexive. Since $P_\bullet$ is $\text{Hom}_{\Lambda^\text{op}}(-, \omega)$-exact, hence $\text{Hom}_\Lambda(P_\bullet, \omega) = P_\bullet^*$ is acyclic and again reflexivity of each $\omega_i$ implies that $\text{Hom}_{\Lambda^\text{op}}(P_\bullet^*, \omega) \cong P_\bullet$. In particular, the complex $\text{Hom}_{\Lambda^\text{op}}(P_\bullet^*, \omega)$ is exact; hence $\text{Hom}_{\Lambda^\text{op}}(M^*, \omega) \cong M$. Applying the functor $\text{Hom}_{\Lambda^\text{op}}(-, \omega)$ on the short exact sequence of $\Lambda$-modules; $0 \to L \to P_0 \to M \to 0$, yields the following exact sequence:

$$0 \to \text{Hom}_\Lambda(M, \omega) \to \text{Hom}_\Lambda(P_0, \omega) \to \text{Hom}_\Lambda(L, \omega) \to 0.$$ 

Thus, in view of the fact $H^i(\text{Hom}_\Lambda(P_\bullet, \omega)) = 0$, we deduce that $\text{Ext}^i_{\Lambda}(M, \omega) = 0$, for all $i > 0$. Furthermore, by using the same argument, one deduces that $\text{Ext}^i_{\Lambda^\text{op}}(M^*, \omega) = 0$, for all $i > 0$. So the proof is complete.

2.5. For a subcategory $\mathcal{X}$ of $\text{mod}_\Lambda$, we let $\widehat{\mathcal{X}}$ denote the category whose objects are the modules $M$ for which there exists an exact sequence of $\Lambda$-modules; $0 \to X_n \to \cdots \to X_0 \to M \to 0$, where $X_i \in \mathcal{X}$ for all $0 \leq i \leq n$.

**Proposition 2.6.** Let $M$ be a $\Lambda$-module of finite $G_\omega$-dimension. Then the following conditions are satisfied:

1. There exists a short exact sequence of $\Lambda$-modules; $0 \to L \to G \to M \to 0$, whenever $L \in \text{add} \omega$ and $G_\omega - \dim \Lambda G = 0$.

2. There exists an exact sequence of $\Lambda$-modules; $0 \to M \to \omega' \to G' \to 0$, whenever $\omega' \in \text{add} \omega$ and $G_\omega - \dim \Lambda G' = 0$.

**Proof.** (1). This follows from the argument given in the proof of [15, Proposition 3.4].

(2). In view of part (1), there exists an exact sequence $0 \to \omega_1 \to G \to M \to 0$, where $\omega_1 \in \text{add} \omega$ and $G_\omega - \dim \Lambda G = 0$. According
to Lemma 2.4, there exists an $\Lambda$-monomorphism $G \to \omega_0$ in which
$\omega_0 \in \text{add}\omega$. Consider the following push-out diagram along the maps
$G \to M$ and $G \to \omega_0$;

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \omega_1 & G & M & 0 \\
\downarrow & & \downarrow & \downarrow \\
0 & \omega_1 & \omega_0 & L & 0 \\
\downarrow & & \downarrow & \downarrow \\
G' & \cong & G' \\
\downarrow & \downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

Now, the exact sequence $0 \to M \to L \to G' \to 0$ would be the desired sequence, and the proof is complete.

As a consequence of the above proposition, we are able to state the following result.

**Corollary 2.7.** Let $I$ be an injective $\Lambda$-module. If $I$ has finite $G_\omega$-
dimension, then $I \in \text{add}\omega$.

**Proof.** According to the above proposition, there exists an exact se-
quence of $\Lambda$-modules; $0 \to I \to L \to G \to 0$, in which $L \in \text{add}\omega$
and $G \in G_\omega(\Lambda)$. Now, $I$ being injective yields that this sequence is
indeed split. So the result follows.

**Definition 2.8.** Let $\omega$ be a semi-dualizing bimodule for $\Lambda$. Following
Enochs et al. [9], we say that $\omega$ is a dualizing bimodule, if it has finite
injective dimension as both $\Lambda$ and $\Lambda^{op}$-module.

**Proposition 2.9.** Let $\omega$ be a dualizing bimodule for $\Lambda$. Then for any
$\Lambda$-module $M$, $G_\omega - \dim_\Lambda M < \infty$.

**Proof.** Assume that $\text{id}_\Lambda \omega \leq n$, for some positive integer $n$. Take the
following exact sequence of $\Lambda$-modules;

\[
0 \to \Omega^n M \to P_{n-1} \to \cdots \to P_0 \to M \to 0,
\]
in which for any $i$, $P_i$ is projective. As $P_i \in G_\omega(\Lambda)$, we only need
to show that $\Omega^n M \in G_\omega(\Lambda)$. Since $\text{id}_\Lambda \omega \leq n$, one may obtain that
$\text{Ext}^i_R(\Omega^n M, \omega) \cong \text{Ext}^{i+n}_R(M, \omega) = 0$, for any $i > 0$. So, applying
the functor $\text{Hom}_\Lambda(\cdot, \omega)$ to the projective resolution of $\Omega^n M; \cdots \to Q_1 \to Q_0 \to \Omega^n M \to 0$, gives rise to the following exact sequence
of $\Lambda^{op}$-modules;

\[
0 \to \text{Hom}_\Lambda(\Omega^n M, \omega) \to \text{Hom}_\Lambda(Q_0, \omega) \to \text{Hom}_\Lambda(Q_1, \omega) \to \cdots .
\]
As $\text{id}_{\Lambda^\omega} \omega < \infty$, by applying the functor $\text{Hom}_{\Lambda^\omega}(-, \omega)$ to the latter sequence, one obtains the following exact sequence of $\Lambda$-modules:

$$\cdots \to \text{Hom}_{\Lambda^\omega}(\text{Hom}_\Lambda(Q_0, \omega), \omega) \to \text{Hom}_{\Lambda^\omega}(\text{Hom}_\Lambda(\Omega^n M, \omega), \omega) \to 0,$$

implying the $\Lambda$-isomorphism $M \cong \text{Hom}_{\Lambda^\omega}(\text{Hom}_\Lambda(\Omega^n M, \omega), \omega)$. Moreover, it is clear that $\text{Ext}^i_{\Lambda^\omega}(\text{Hom}_\Lambda(\Omega^n M, \omega), \omega) = 0$, for all $i > 0$. Thus, $\Omega^n M \in G_\omega(\Lambda)$, as needed.

2.10. Let $M$ be a $\Lambda$-module. A $\Lambda$-homomorphism $f : G \to M$ is in $G_\omega(\Lambda)$ is said to be a right $G_\omega(\Lambda)$-approximation, if the map $\text{Hom}_\Lambda(G', f) : \text{Hom}_\Lambda(G', G) \to \text{Hom}_\Lambda(G', M)$ is surjective, for any $\Lambda$-module $G' \in G_\omega(\Lambda)$.

Corollary 2.11. Let $\omega$ be a dualizing bimodule for $\Lambda$. Then any $\Lambda$-module admits a right $G_\omega(\Lambda)$-approximation.

Proof. Assume that $M$ is an arbitrary $\Lambda$-module. By virtue of Proposition 2.6 (1), there exists a short exact sequence of $\Lambda$-modules; $0 \to L \to G \to M \to 0$, whenever $L \in \text{add}_{\omega} G_\omega$ and $G_\omega$-dim $G = 0$. Suppose that $N \in G_\omega(\Lambda)$ is arbitrary. As $L \in \text{add}_\omega$, we have $\text{Ext}^i_\Lambda(N, L) = 0$, implying that $\text{Hom}_\Lambda(N, G) \to \text{Hom}_\Lambda(N, M)$ is an epimorphism. This means that $G \to M$ is a left $G_\omega(\Lambda)$-approximation of $M$. So the proof is complete.

3. Generalized Gorenstein dimension over group rings

This section is devoted to study the generalized Gorenstein dimension of modules over group rings. First, we show that having a dualizing bimodule ascends from $R$ to $R\Gamma$, whenever $R$ is a commutative noetherian local ring and $\Gamma$ is a finite group. Our main result in this section indicates that, if $M$ is a finitely generated $R\Gamma$-module, then generalized Gorenstein dimensions of $M$ over $R$ and $R\Gamma$, are identical. We should point out that the notion of Gorenstein dimension of modules over group rings has been studied first in [2, 5].

3.1. One should observe that, since $\Gamma$ is a finite group, hence $R\Gamma$ is a noetherian $R$-algebra. We also point out that, as $R\Gamma$ is isomorphic with the opposite ring $(R\Gamma)^{op}$, the distinction between left and right modules is redundant.

Proposition 3.2. Let $\omega$ be a dualizing $R$-module and let $\Gamma$ be a finite group. Then $R\Gamma \otimes_R \omega$ is a dualizing $R\Gamma$-bimodule.

Proof. Since $R\Gamma$ is a faithfully flat $R$-module, for any $i \geq 0$, one may have the following isomorphism:

$$\text{Ext}^i_{R\Gamma}(R\Gamma \otimes_R \omega, R\Gamma \otimes_R \omega) \cong R\Gamma \otimes_R \text{Ext}^i_R(\omega, \omega).$$
By the hypothesis, \( \operatorname{Ext}^i_R(\omega, \omega) = 0 \) for any \( i > 0 \), implying that the right hand side of the above isomorphism will be zero. Consequently, using again the faithfully flatness of \( R \Gamma \) over \( R \), we conclude that

\[
\operatorname{Hom}_{R \Gamma}(R \Gamma \otimes_R \omega, R \Gamma \otimes_R \omega) \cong R \Gamma \otimes_R \operatorname{Hom}_R(\omega, \omega) \cong R \Gamma \otimes_R R \cong R \Gamma.
\]

So, \( R \Gamma \otimes_R \omega \) is a semi-dualizing \( R \)-bimodule. Moreover, assume that \( 0 \to \omega \to I^0 \to \cdots \to I^t \to 0 \) is a (minimal) injective resolution of \( \omega \). Since \( R \Gamma \) is a free \( R \)-module, applying the functor \( R \Gamma \otimes_R - \) to this sequence gives rise to the following exact sequence of \( R \)-modules:

\[
0 \to R \Gamma \otimes_R \omega \to R \Gamma \otimes_R I^0 \to \cdots \to R \Gamma \otimes_R I^t \to 0.
\]

Hence, as \( \Gamma \) is a finite group, by making use of [6, Proposition III.5.9], we infer that \( R \Gamma \otimes_R I^i \cong \operatorname{Hom}_R(R \Gamma, I^i) \) for any \( 0 \leq i \leq t \), and so \( R \Gamma \otimes_R I^i \) is an injective \( R \Gamma \)-module. This, in turn, means that \( R \Gamma \otimes_R \omega \) has finite injective dimension. Thus, \( R \Gamma \otimes_R \omega \) is a dualizing \( R \Gamma \)-bimodule, as needed.

**Example 3.3.** (1) Let \((R, m)\) be a complete Cohen-Macaulay local ring. According to [7, Corollary 3.3.8], \( R \) admits a canonical module, say \( \omega \). Clearly, \( \omega \) is a dualizing \( R \)-module in our sense. So, by Proposition 3.2, \( R \Gamma \otimes_R \omega \) is a dualizing \( R \Gamma \)-bimodule.

(2) Assume that \((R, m)\) is a Gorenstein local ring. So, \( R \) is a dualizing \( R \)-module. Thus, by the above proposition, \( R \Gamma \otimes_R R \cong R \Gamma \) is a dualizing \( R \Gamma \)-bimodule. In particular, \( R \Gamma \) is a (not necessarily commutative) Gorenstein ring.

**Lemma 3.4.** Let \( M \) be an \( R \)-module and \( \Gamma \) be a finite group. If \( G_\omega = \dim_R M = 0 \), then \( G_{R \Gamma \otimes_R \omega} = \dim_{R \Gamma} R \Gamma \otimes_R M = 0 \).

**Proof.** Since \( G_\omega = \dim_R M = 0 \), in view of Lemma 2.4, there exists an acyclic complex of \( R \)-modules;

\[
P_\bullet : \cdots \to P_1 \to P_0 \to \omega_{-1} \to \omega_{-2} \to \cdots,
\]

such that \( M = \operatorname{Coker}(P_1 \to P_0) \) and the the functor \( \operatorname{Hom}_R(-, \omega) \) leaves \( P_\bullet \) exact. According to the fact that \( R \Gamma \) is a (finitely generated) free \( R \)-module, one may obtain the following acyclic complex of \( R \Gamma \)-modules;

\[
P_\bullet : \cdots \to R \Gamma \otimes_R P_1 \to R \Gamma \otimes_R P_0 \to R \Gamma \otimes_R \omega_{-1} \to R \Gamma \otimes_R \omega_{-2} \to \cdots,
\]

in which \( R \Gamma \otimes_R P_1 \) is projective and \( R \Gamma \otimes_R \omega_i \in \operatorname{add}(R \Gamma \otimes_R \omega) \). Moreover, we have the following isomorphism of \( R \Gamma \)-modules;

\[
\operatorname{Hom}_{R \Gamma}(R \Gamma \otimes_R P_\bullet, R \Gamma \otimes_R \omega) \cong R \Gamma \otimes_R \operatorname{Hom}_R(P_\bullet, \omega).
\]
Since, by the hypothesis, \( \text{Hom}_R(P, \omega) \) is acyclic, the same is true for \( R^\Gamma \otimes_R \text{Hom}_R(P, \omega) \), implying that the right hand side of the above isomorphism is acyclic. So, the proof is complete. \( \square \)

**Theorem 3.5.** Let \( \Gamma \) be a finite group and \( M \) be an \( R^\Gamma \)-module. Then \( G_\omega - \dim_R M = 0 \) if and only if \( G_{R^\Gamma \otimes_R \omega} - \dim_{R^\Gamma} M = 0 \).

**Proof.** Let us first prove the ‘sufficiency’. Since \( G_{R^\Gamma \otimes_R \omega} - \dim_{R^\Gamma} M = 0 \), in view of Lemma 2.4, there exists an exact sequence of \( \Lambda \)-modules

\[
P_\bullet : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \omega_1 \rightarrow \omega_2 \rightarrow \cdots,
\]

where each \( \omega_i \) belongs to \( \text{add} \omega \) and all \( P_j \)'s are projective \( \Lambda \)-module, such that \( P_\bullet \) remains exact after applying the functor \( \text{Hom}_\Lambda (-, \omega) \), and \( M = \text{Ker}(\omega_1 \rightarrow \omega_2) \). As \( R^\Gamma \) is a finitely generated free \( R \)-module, we get that any projective \( R^\Gamma \)-module is also projective as an \( R \)-module, and any module in \( \text{add}(R^\Gamma \otimes_R \omega) \) also belongs to \( \text{add}(\omega) \). So, it remains to show that the complex \( \text{Hom}_R(P_\bullet, \omega) \) is exact. To see this, consider the following isomorphisms:

\[
\text{Hom}_R(P_\bullet, \omega) \cong \text{Hom}_R(R^\Gamma \otimes_R P_\bullet, \omega) \\
\cong \text{Hom}_{R^\Gamma}(P_\bullet, \text{Hom}_R(R^\Gamma, \omega)) \\
\cong \text{Hom}_{R^\Gamma}(P_\bullet, R^\Gamma \otimes_R \omega).
\]

We should point out that the first isomorphism holds trivially, the second one is the adjointness of \( \text{Hom} \) and \( \otimes \), and the validity of the last isomorphism comes from [6, Proposition III.5.9]. By our assumption, the last complex is acyclic, implying that the same is true for the complex \( \text{Hom}_R(P_\bullet, \omega) \), as needed. Next, we want to prove the ‘necessity’. Assume that \( G_\omega - \dim_R M = 0 \). In view of Lemma 2.4, there exists a short exact sequence of \( R \)-modules \( 0 \rightarrow M \rightarrow \omega_1 \rightarrow L \rightarrow 0 \), in which \( G_\omega - \dim_R L = 0 \) and \( \omega_1 \in \text{add}(\omega) \). Consider the following commutative diagram of \( R^\Gamma \)-modules with exact rows;

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M & \rightarrow & R^\Gamma \otimes_R M & \rightarrow & L' & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{\theta} & & \downarrow{\varphi} & & & & \\
0 & \rightarrow & M & \rightarrow & R^\Gamma \otimes_R \omega_1 & \rightarrow & L'' & \rightarrow & 0.
\end{array}
\]

Clearly, \( \varphi \) is a monomorphism and \( \text{Coker} \varphi = R^\Gamma \otimes_R L \). It is known that the exact sequence \( 0 \rightarrow M \rightarrow R^\Gamma \otimes_R M \rightarrow L' \rightarrow 0 \) splits over \( R \). Thus, \( L' \cong \bigoplus_{i=1}^{t-1} M \) as \( R \)-modules, where \( t = |\Gamma| \), the order of \( \Gamma \). Consequently, the hypothesis made on \( M \) implies that \( G_\omega - \dim_R L' = 0 \). On the other hand, since \( G_\omega - \dim_R L = 0 \), it is not hard to see that \( G_{R^\Gamma \otimes_R \omega} - \dim_{R^\Gamma}(R^\Gamma \otimes_R L) = 0 \) and so \( G_\omega - \dim_R(R^\Gamma \otimes_R L) = 0 \), thanks to the fact that \( R^\Gamma \otimes_R L \cong \bigoplus_{i=1}^{t-1} L \), as \( R \)-modules. We claim that \( G_\omega - \dim_R L'' = 0 \). The exact sequence

\[
0 \rightarrow R^\Gamma \otimes_R M \rightarrow R^\Gamma \otimes_R \omega_1 \rightarrow R^\Gamma \otimes_R L \rightarrow 0,
\]
splits over $R$ as follows:

$$0 \rightarrow \bigoplus_{i=1}^{t} M \rightarrow \bigoplus_{i=1}^{t} \omega_{-1} \rightarrow \bigoplus_{i=1}^{t} L \rightarrow 0. \quad (1)$$

Similarly, the short exact sequence $0 \rightarrow L' \rightarrow L'' \rightarrow R\Gamma \otimes_R L \rightarrow 0$, which splits over $R$, can be written as follows, when viewed as an exact sequence of $R$-modules;

$$0 \rightarrow \bigoplus_{i=1}^{t} M \rightarrow L'' \rightarrow \bigoplus_{i=1}^{t} L \rightarrow 0. \quad (2)$$

Considering the exact sequences (1) and (2), we conclude that $L'' \cong \bigoplus_{i=1}^{t} \omega_{-1} \oplus L$, as $R$-modules. Hence, $G_{\omega} - \dim_R L'' = 0$. In fact, we obtain the short exact sequence $0 \rightarrow M \rightarrow R\Gamma \otimes_R \omega_{-1} \rightarrow L'' \rightarrow 0$, in which $G_{\omega} - \dim_R L'' = 0$. By repeating this manner, one gets the following exact sequence of $RT$-modules

$$\omega_{i} : 0 \rightarrow R\Gamma \otimes_R \omega_{-1} \rightarrow R\Gamma \otimes_R \omega_{-2} \rightarrow \cdots,$$

where $\omega_{i} \in \text{add}(\omega)$ for all $i$, and every cosyzygy of $\omega_{i}$ belongs to $G_{\omega}(R)$. Now, we show that $\omega_{i}$ is $\text{Hom}_{RT}(-, R\Gamma \otimes_R \omega)$-exact. Consider the following isomorphisms:

$$\text{Hom}_{RT}(\omega_{i}, R\Gamma \otimes_R \omega) \cong \text{Hom}_{R}(R\Gamma \otimes_R \omega, \omega) \cong \text{Hom}_{R}(\omega_{i}, \omega).$$

By making use of the fact that every cosyzygy of $\omega_{i}$ belongs to $G_{\omega}(R)$, we conclude that the right hand side is exact, which gives the claim. Hence, in order to complete the proof, it suffices to find a left resolution of modules in $\text{add}((R\Gamma \otimes_R \omega) \oplus R\Gamma')$ for $M$, which is $\text{Hom}_{RT}(-, R\Gamma \otimes_R \omega)$-exact. Take the following projective resolution of $RT$-module $M$;

$$P_{\bullet} : \cdots \rightarrow P_{1} \rightarrow P_{0} \rightarrow M \rightarrow 0,$$

where each $P_{i}$ is finitely generated. By replacing $P_{\bullet}$ with $\omega_{i}$ in the above isomorphisms and invoking the fact that $G_{\omega} - \dim_R M = 0$, one deduces that $\text{Hom}_{RT}(-, R\Gamma \otimes_R \omega)$ leaves the sequence $P_{\bullet}$ exact. Now, splicing $\omega_{i}$ and $P_{\bullet}$ and Lemma 2.4 completes the proof. □

The above theorem immediately gives rise to the following result.

**Corollary 3.6.** Let $R$ be a commutative ring with a semi-dualizing module $\omega$. Let $\Gamma$ be a finite group and $M$ be an $RT$-module. Then $G_{\omega} - \dim_R M = G_{RT \otimes R \omega} - \dim_{RT} M$.

**Proof.** We first show that $G_{\omega} - \dim_R M \leq G_{RT \otimes R \omega} - \dim_{RT} M$. If $G_{RT \otimes R \omega} - \dim_{RT} M = \infty$, there is nothing to prove. So, assume that $G_{RT \otimes R \omega} - \dim_{RT} M$ is finite, say $t$. Hence, there exists an exact sequence of $RT$-modules; $0 \rightarrow G_{t} \rightarrow \cdots \rightarrow G_{0} \rightarrow M \rightarrow 0$, whenever $G_{RT \otimes R \omega} - \dim_{RT} G_{i} = 0$ for any $i$. According to Theorem 3.5, $G_{\omega} - \dim_R G_{i} = 0$, implying that $G_{\omega} - \dim_R M \leq t$. Next, we want to show the reverse inequality. To that end, we may assume that $G_{\omega} -
dim$_R M = n$, for some non-negative integer $n$. Take an exact sequence of $R$-modules; $0 \to \Omega^n M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$, where each $P_i$ is projective. Since $RT$ is a free $R$-module, any $P_i$ is projective over $R$. Hence, by the hypothesis, we get $G_\omega - \dim_R \Omega^n M = 0$.

Using again of Theorem 3.5, gives rise to the equality $G_{RT\otimes_R \omega} - \dim_{RT} \Omega^n M = 0$. Consequently, $G_{RT\otimes_R \omega} - \dim_{RT} M \leq n$, and the proof is complete.

Let $\Gamma$ be a finite group and $\omega$ be a semi-dualizing $R$-module. We denote by $X$ (resp., $Y$) the subcategory of $\text{mod} R$ (resp., $\text{mod} \Gamma$) consisting of all modules of finite $G_\omega$-dimension (resp., $G_{RT\otimes_R \omega}$-dimension).

**Proposition 3.7.** With the above notations, one has the equality

$$\sup \{G_\omega - \dim_R M \mid M \in X\} = \sup \{G_{RT\otimes_R \omega} - \dim_{RT} M \mid M \in Y\}.$$

**Proof.** Let us first show the inequality

$$\sup \{G_\omega - \dim_R M \mid M \in X\} \leq \sup \{G_{RT\otimes_R \omega} - \dim_{RT} M \mid M \in Y\}.$$  

If $\sup \{G_{RT\otimes_R \omega} - \dim_{RT} M \mid M \in Y\} = \infty$, there is nothing to prove. So, assume that $\sup \{G_{RT\otimes_R \omega} - \dim_{RT} M \mid M \in Y\} \exists$ finite, say $t$. Taking an arbitrary object $X \in X$, there is an exact sequence of $R$-modules; $0 \to G_n \to \cdots \to G_0 \to X \to 0$ in which for any $i$, $G_\omega - \dim_R G_i = 0$. Apply the functor $RT \otimes_R -$ to this, in order to obtain the exact sequence of $RT$-modules; $0 \to RT \otimes_R G_n \to \cdots \to RT \otimes_R G_0 \to RT \otimes_R X \to 0$. In view of Lemma 3.4, $G_{RT\otimes_R \omega} - \dim_{RT} RT \otimes_R G_i = 0$, implying that $RT \otimes_R X \in Y$. Hence, our assumption yields that $G_{RT\otimes_R \omega} - \dim_{RT} RT \otimes_R X \leq t$, and so by making use of Corollary 3.6, we infer that $G_\omega - \dim_{RT} RT \otimes_R X \leq t$ as well. Now, $X$ being an $R$-direct summand of $RT \otimes_R X$ implies that $G_\omega - \dim_{RT} X \leq t$. Consequently, $\sup \{G_\omega - \dim_R M \mid M \in X\} \leq t$. Next, we will show that $\sup \{G_{RT\otimes_R \omega} - \dim_{RT} M \mid M \in Y\} \leq \sup \{G_\omega - \dim_R M \mid M \in X\}$. To do this, we may assume that $\sup \{G_\omega - \dim_R M \mid M \in X\} = s < \infty$. Take an arbitrary $RT$-module $M$ with finite $G_{RT\otimes_R \omega}$-dimension. According to Corollary 3.6, we have that $G_{RT\otimes_R \omega} - \dim_{RT} M = G_\omega - \dim_{RT} M \leq s$. This, in particular, gives the desired result.

**Example 3.8.** Let $(R, m)$ be a complete Cohen-Macaulay local ring with canonical module $\omega$. According to Proposition 3.2, $RT \otimes_R \omega$ is a dualizing $RT$-bimodule. Now, by making use of Proposition 2.9 in conjunction with Corollary 3.6, one may deduce that for any finitely generated $RT$-module $M$, $G_{RT\otimes_R \omega} - \dim_{RT} M \leq n$, where $n = \text{id}_{RT\otimes_R \omega}$.

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بعد گرنشتاین تعمیم یافته رون حلقه گروهدای

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 فرض کنید $(R, m)$ یک حلقه جابجایی نوتری و موضوعی بوده و $\Gamma$ یک گروه متناهی باشد. ثابت می‌کنیم که اگر $R$ دارای مدول دوگانی باشد، آنگاه حلقه گروه $RT$ نیز دارای مدول دوگانی می‌باشد. به علاوه، ثابت می‌کنیم که $M$-مدول با تولید متناهی $\Gamma$ رون $R$-مدول، برابر با صفر باشد. اگر و تنها اگر بعد گرنشتاین تعمیم یافته $M$ به عنوان $R$-مدول، برابر با صفر باشد.

 کلمات کلیدی: دومدول‌های نوتری، بعد گرنشتاین تعمیم یافته، حلقه گروهدای

 GENERALIZED GORENSTEIN DIMENSION OVER GROUP RINGS

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 کلمات کلیدی: دومدول‌های نوتری، بعد گرنشتاین تعمیم یافته، حلقه گروهدای