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SOME REMARKS ON ALMOST UNISERIAL RINGS AND MODULES

H. R. DORBIDI

Abstract. In this paper, we study almost uniserial rings and modules. An $R-$module M is called almost uniserial if any two non-isomorphic submodules of M are linearly ordered by inclusion. A ring R is an almost left uniserial ring if ${}_RR$ is almost uniserial. We give some necessary and sufficient condition for an Artinian ring to be almost left uniserial.

1. Introduction

In this paper, all rings have identity elements and all modules are unitary left modules. A left R-module M is called *uniserial* if its submodules are linearly ordered by inclusion. A ring R is called *left uniserial* if ${}_RR$ is uniserial. The notion of almost left uniserial ring is defined in [1], as a straightforward common generalization of left uniserial rings and left principal ideal domains. A ring R is called *almost left uniserial* if any two non-isomorphic left ideals of R are comparable. We note that each left uniserial ring is almost left uniserial, but the converse is not true, in general. For instance, any principal left ideal domain is almost left uniserial, but is not necessarily a left uniserial ring. We say that a left R-module M is almost uniserial if any two non-isomorphic submodules are linearly ordered by inclusion. It is clear that every submodule of an almost uniserial module is almost uniserial. But this is not true for quotient of uniserial modules. In this paper, we give some examples of quotient of almost uniserial modules which are almost uniserial. In [1], a structure theorem for commutative Artinian

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almost uniserial rings is obtained. We generalize this result by using the socle series of the ring R to the non-commutative case. Also, we prove that in a commutative almost left uniserial ring, the ideal $Nil(R)$ is a prime ideal.

Here we give some notions and definitions. A ring R is a *local* ring if R has a unique left maximal ideal. The *Jacobson* radical of R is denoted by $J(R)$, and the set of all nilpotent elements of a ring R is denoted by $Nil(R)$. For an $R-$module M, the *Socle* series of M is defined inductively as follows: $Soc_0(M) = 0$ and $Soc_1(M) = Soc(M)$ is the sum of all simple submodules of M. Also, for any ordinal i, we have $Soc_{i+1}(M)/Soc_i(M) = Soc(M/Soc_i(M))$ and for a limit ordinal α, we have $Soc_\alpha(M) = \bigcup_{\beta<\alpha} Soc_\beta(M)$. If $S \subseteq M$, we denote the left *annihilator* of S in R by $ann(S)$. The *length* of a module M is denoted by $l(M)$. An R-module M is called *indecomposable* if it is non-zero and can not be written as a direct sum of two non-zero submodules. We recall that an R-module M is called *uniform* if the intersection of any two non-zero submodules is non-zero. A submodule N of M is said to be an *essential* submodule of M if $N \bigcap K \neq (0)$ for each non-zero submodule K of M. A submodule N of M is said to be *fully invariant* if $f(N) \subseteq N$, for each endomorphism $f \in End(M)$.

2. Main results

Before stating our main results, we need some results of [1].

Proposition 2.1. [1, Proposition 2.1.] *Let M be an almost uniserial module. Then, either M is an indecomposable module or M is a direct sum of two isomorphic simple modules. Moreover, every finitely generated submodule of M is at most two-generated and the set of all non-cyclic submodules of M is a chain.*

Corollary 2.2. [1, Corollary 2.2.] *Let M be an almost uniserial module. Then one of the following two conditions hold:*

(1) *Either M is a uniform module, or*
(2) *$Soc(M)$ is a direct sum of two isomorphic simple modules and is an essential sub-module of M.*

Proposition 2.3. [1, Proposition 2.3.] *Let R be an almost left uniserial ring. Then ${}_RR$ is indecomposable or $R \cong M_2(D)$, where D is a division ring. In both cases, R is an indecomposable ring, every finitely generated left ideal of R is at most two-generated, and the set of all non-cyclic left ideals of R is a chain.*

Remark 2.4. It is clear that if M is an almost uniserial module and $N \leq M$, then N is also an almost uniserial module. But the quotient modules of an almost uniserial module are not necessarily almost uniserial. For example, $\mathbb{Z}$ is almost uniserial but $\mathbb{Z}_6 \cong \mathbb{Z}_2 \bigoplus \mathbb{Z}_3$ is not almost uniserial, by Proposition 2.1.

Remark 2.5. Let M be an almost uniserial module and $N \leq M$. If N is not essential in M, then there exists a submodule K of M such that $N \bigcap K = 0$. Since $N \bigoplus K$ is an almost uniserial module, so $N \cong K$ and N is a simple module. Hence, every submodule of M is essential or simple.

Lemma 2.6. *Let M be an almost uniserial $R-$module. If $N, K \subseteq M$ and N is not cyclic, then N and K are comparable.*

Proof. Assume $K \nsubseteq N$. If $k \in K \backslash N$, then $Rk \nsubseteq N$ and $Rk \ncong N$. So, $N \subseteq Rk$ as desired. □

Definition 2.7. Let R be an almost left uniserial ring. If R is a left uniserial ring or $R \cong M_2(D)$ for some division ring D, we say R is type (1), otherwise R is type (2).

The following lemma gives some properties of socle series. Its proof is easy and uses *transfinite* induction. So we omit the proof.

Lemma 2.8. *Let M and N be two $R-$modules and α, β be two ordinal numbers. Then the following holds:*

1. *$Soc_\alpha(M)$ is a fully invariant submodule of M;*
2. *If $f : M \to N$ is an isomorphism, then $f(Soc_\alpha(M)) = Soc_\alpha(N)$;*
3. *If $N \leqslant M$, then $Soc_\alpha(N) \leqslant Soc_\alpha(M)$;*
4. *If $N \leqslant M$ and $Soc_\alpha(M) \leqslant N$, then $Soc_\alpha(N) = Soc_\alpha(M)$. In particular, $Soc_\alpha(Soc_\alpha(M)) = Soc_\alpha(M)$;*
5. *$Soc_\alpha(M/Soc_\beta(M)) = Soc_{\alpha+\beta}(M)/Soc_\beta(M)$.*

Lemma 2.9. *Let M be an $R-$module. If $N \leqslant M$ and $N \cong Soc_\alpha(M)$, then $N \leq Soc_\alpha(M)$.*

Proof. Let $K = Soc_\alpha(M)$. So, $Soc_\alpha(K) = K$, by Lemma 2.8(4). Since $N \cong K$, so $N = Soc_\alpha(N) \subseteq Soc_\alpha(M)$, by Lemma 2.8(3). □

Corollary 2.10. *Assume M is an almost uniserial module. Then any submodule N of M is comparable to $Soc_\alpha(M)$ for each ordinal number α.*

Proof. If $N \cong Soc_\alpha(M)$, then $N \subseteq Soc_\alpha(M)$, by Lemma 2.9. □

Corollary 2.11. *Assume M is an almost uniserial module. If $N/Soc_\alpha(M)$ is a simple module for some ordinal number α, then N is a cyclic module.*

Proof. Let $a \in N \backslash Soc_\alpha(M)$. Then, $Soc_\alpha(M) \subsetneq Ra$, by Corollary 2.10. So $Ra = N$. $\square$

Theorem 2.12. *If M is an almost uniserial module, then $M/Soc_\alpha(M)$ is an almost uniserial module for each ordinal number α.*

Proof. Let α be an ordinal number and $K/Soc_\alpha(M)$ and $N/Soc_\alpha(M)$ be two submodules of $M/Soc_\alpha(M)$. So, $Soc_\alpha(M) = Soc_\alpha(N) = Soc_\alpha(K)$, by Lemma 2.8(4). If N and K are comparable, then $K/Soc_\alpha(M)$ and $N/Soc_\alpha(M)$ are comparable. Otherwise, let $f : N \to K$ be an isomorphism. So $Soc_\alpha(M) = Soc_\alpha(K) = f(Soc_\alpha(N)) = f(Soc_\alpha(M))$, by Lemma 2.8(2). Hence, f gives rise to an isomorphism $\widetilde{f} : N/Soc_\alpha(M) \to K/Soc_\alpha(M)$. $\square$

Remark 2.13. Let R be a commutative ring and M be an almost left uniserial $R-$module. Then for each $m, n \in M$ the ideals $ann(m)$ and $ann(n)$ are comparable.

Lemma 2.14. *Let R be a commutative ring. If $ann(a) \subseteq ann(b)$ and $a^n = 0$ for some $n \in \mathbb{N}$, then $b^i a^{n-i} = 0$, for each $0 \leq i \leq n$. In particular, $b^n = 0$ and $(a+b)^n = 0$.*

Proof. We proceed by induction on i. If $i = 0$, then $a^n = 0$. If $0 \leq i < n$ and $b^i a^{n-i} = 0$, then $b^i a^{n-i-1} \in ann(a) \subseteq ann(b)$. So, $b^{i+1} a^{n-(i+1)} = 0$. $\square$

Lemma 2.15. *Let R be a commutative ring and I is an ideal of R. Suppose that $f : I \to R$ is an $R-$homomorphism. Then,*

(1) *If $a^n = 0$, then $f(a)^n = 0$;*
(2) *If f is injective, then $a^n = 0$ if and only if $f(a)^n = 0$.*

Proof. It suffices to prove that $f^i(a^i) = f(a)^i$. For $i = 1$ this is trivial. If $f^i(a^i) = f(a)^i$, then $f^{i+1}(a^{i+1}) = f(f^i(a^{i+1})) = f(af^i(a^i)) = f^i(a^i)f(a) = f(a)^{i+1}$. $\square$

Lemma 2.16. *Let R be a commutative almost uniserial ring. For each $n \in \mathbb{N}$, if $I_n = \{a \in R : a^n = 0\}$, then I_n is an ideal of R and $I_n^n = 0$.*

Proof. Let $a, b \in I_n$. Since $ann(a)$ and $ann(b)$ are comparable, by Remark 2.13, so $a + b \in I_n$, by Lemma 2.14. It is clear that $rI_n \subseteq I_n$ for each $r \in R$. Thus, I is an ideal of R. Let $a_1, \ldots, a_n \in I_n$. Without loss of generality, we can assume that $ann(a_1) \subseteq \cdots \subseteq ann(a_n)$. Since $a_1^n = 0$, so $a_1^{n-1} a_2 = 0$. By iterating this process and inserting a_i's, we get $a_1 \ldots a_n = 0$, as needed. $\square$

Lemma 2.17. *Let R be a commutative almost uniserial ring. Then every ideal J is comparable to any ideal $I \in \{I_n : n \in \mathbb{N}\} \cup Nil(R)$.*

Proof. The proof follows by Lemma 2.15 and Lemma 2.16. □

Remark 2.18. Let R be a commutative almost uniserial ring. If $ab = 0$, then $a^2 = 0$ or $b^2 = 0$.

Recall that a ring R is called *reduced* if it has no non-zero nilpotent element.

Theorem 2.19. *Let R be a commutative almost uniserial ring. Then $Nil(R)$ is a prime ideal. In particular, R is an integral domain if and only if R is a reduced ring.*

Proof. If $ab \in Nil(R)$, then $a^n b^n = (ab)^n = 0$, for some $n \in \mathbb{N}$. So, $a^{2n} = 0$ or $b^{2n} = 0$, by remark 2.18. Thus, $a \in Nil(R)$ or $b \in Nil(R)$. □

Theorem 2.20. *Let R be a commutative almost uniserial ring and $I_n = \{a \in R : a^n = 0\}$, for each $n \in \mathbb{N}$. Then, R/I is a commutative almost uniserial ring for any ideal $I \in \{I_n : n \in \mathbb{N}\} \cup Nil(R)$.*

Proof. Let $I \in \{I_n : n \in \mathbb{N}\} \bigcup \{Nil(R)\}$ and K/I and N/I be two ideals of R/I. If N and K are comparable, then K/I and N/I are also comparable. Otherwise, let $f : N \to K$ be an $R-$isomorphism. So, $f(I) = I$, by Lemma 2.15. Thus, f gives rise to an isomorphism $\widetilde{f} : N/I \to K/I$. □

3. Local Artinian almost left uniserial rings

In this section, we study local Artinian almost left uniserial rings. Specially, we generalize the results of [1] for commutative local Artinian almost uniserial rings.

Theorem 3.1. *Let (R, m) be a local Artinian almost left uniserial ring. Assume k is the least integer number such that $Soc_k(R) = R$. Then, $m^i = Soc_{k-i}(R)$ for each $0 \le i \le k$. In particular, $m^k = 0$.*

Proof. We proceed by induction on i. If $i = 0$, then $m^0 = R = Soc_k(R)$. Since $R = Soc_k(R)$, so $R/Soc_{k-1}(R)$ is a local semi-simple almost left uniserial ring, by Theorem 2.12. Thus, $R/Soc_{k-1}(R)$ is a division ring by Artin-Wedderburn Theorem and Proposition 2.3. Hence, $m = Soc_{k-1}(R)$. Assume $m^i = Soc_{k-i}(R)$. Since the socle of any $R-$module is annihilated by ideal m, so $m^{i+1} = mSoc_{k-i}(R) \subseteq Soc_{k-(i+1)}(R)$. Hence, $l(\frac{Soc_{k-(i+1)}(R)}{m^{i+1}}) < l(\frac{m^i}{m^{i+1}})$. Since m^i is at most two generated and m^i/m^{i+1} is a vector space over the division ring R/m, so $l(m^i/m^{i+1}) \le 2$. If $l(\frac{Soc_{k-i}(R)}{Soc_{k-(i+1)}(R)}) = 1$, then $m^i = Soc_{k-i}(R)$ is a cyclic module by, Corollary 2.11. So, $l(m^i/m^{i+1}) = 1$. Hence,

$l(\frac{Soc_{k-(i+1)}(R)}{m^{i+1}}) = 0 < 1 = l(\frac{m^i}{m^{i+1}})$. So, $Soc_{k-(i+1)}(R) = m^{i+1}$. If $l(\frac{Soc_{k-i}(R)}{Soc_{k-(i+1)}(R)}) = 2$, then $l(m^i/m^{i+1}) = 2$. Hence, $m^{i+1} = Soc_{k-(i+1)}(R)$, and the proof is complete. □

We need the following theorem [2, Theorem 9] for the proof of our next theorem.

Theorem 3.2. *Let R be a left Artinian ring. Then R is a left uniserial ring if and only if R is a local ring and $J(R)$ is a principal ideal as a left ideal.*

The following theorem is proved in [1, Proposition 2.7.] by a different method.

Theorem 3.3. *Let R be a left Artinian principal left ideal ring. If R is almost left uniserial, then it is of type* (1)*.*

Proof. If $R \ncong M_2(D)$, then ${}_RR$ is indecomposable by Proposition 2.3. Hence, R has no nontrivial idempotent. Since $J(R)$ is nilpotent, so idempotents of semi-simple ring $R/J(R)$ can be lifted. Hence, $R/J(R)$ is a division ring and R is a local ring. Thus, it is uniserial by Theorem 3.2. □

Lemma 3.4. *Let M be an $R-$module. If the set of two-generated non-cyclic submodules of M is a chain, then every finitely generated submodule of M is at most two-generated.*

Proof. By contrary, assume M has a submodule $N = \langle x, y, z\rangle$ which can not be generated by two elements. So, $N_1 = \langle x, y\rangle$ and $N_2 = \langle y, z\rangle$ are two-generated non-cyclic submodules of M. So, they are comparable. Hence, $N = N_1 + N_2$ is two generated, which is a contradiction. □

Remark 3.5. Let (R, m) be a local ring. If $N = \langle x, y\rangle$ is a cyclic module, then $N = Rx$ or $N = Ry$, by Nakayama's Lemma.

Theorem 3.6. *Let (R, m) be a local left Artinian ring. Assume that the set of two-generated non-cyclic ideals of R is $S = \{m^i : 1 \leq i < k\}$, where k is the nilpotency index of m. Then every ideal of R is at most two-generated and each ideal $I \notin S$ is cyclic and is between m^i and m^{i+1} for some i. In particular, every left ideal is comparable to each m^i.*

Proof. Since S is chain, so every ideal of R is at most two-generated, by Lemma 3.4. Hence, each ideal $I \notin S$ is a cyclic ideal. Let $Ra \notin S$ be a cyclic ideal of R and i be the maximum integer such that $Ra \subseteq m^i$. So, $a \notin m^{i+1}$. Let $x \in m^{i+1}$ and $J = \langle a, x\rangle$. It is clear that $J \nsubseteq m^{i+1}$ and

$J \subseteq m^i$. If $J = m^i$, then m^i/m^{i+1} is a cyclic module. So, $m^i = Ra$ is a cyclic module, which is a contradiction. Thus, $J \notin S$. Hence, J is a cyclic ideal. So, $J = Ra$, by remark 3.5. This implies that $m^{i+1} \subseteq Ra$. □

Theorem 3.7. *Let (R, m) be a local left Artinian ring. Assume that the set of two-generated non-cyclic ideals of R is $S = \{m^i : 1 \leq i < k\}$, where k is a nilpotency index of m. Then, R is an almost left uniserial ring of type (2).*

Proof. First note that $l(m^i/m^{i+1}) = 2$. So, $l(R) = 2k + 1$ and $l(m^i) = 2k - 2i$. Since every ideal is comparable to m^i for each $1 \leq i \leq k$, by Theorem 3.6, so the only ideals of even length are m^i's. Let I and J be two incomparable left ideals of R. So there exist an integer i such that $m^{i+1} \subsetneqq I, J \subsetneqq m^i$. Also, I and J are cyclic. If $I = Ra$ and $J = Rb$, then $l(Ra) = l(Rb) = 2k - 2i - 1$. So, $l(ann(a)) = l(ann(b)) = 2i + 2$. Hence, $ann(a) = ann(b) = m^{k-i-1}$ and $Ra \cong Rb$. □

Theorem 3.8. *Let (R, m) be a local left Artinian ring. If R is an almost left uniserial ring of type (2), then the set of two-generated non-cyclic ideals of R is contained in $\{m^i : 1 \leq i < k\}$, where k is the nilpotency index of m and every other ideal is cyclic and is between m^i and m^{i+1} for some i.*

Proof. Since R is a local Artinian almost left uniserial ring, so $m^i = Soc_{k-i}(R)$ by Theorem 3.1. Hence, every ideal is between m^i and m^{i+1} for some i, by Theorem 2.10. Also, every ideal of R is at most two-generated, by Proposition 2.1. Assume I is a left ideal which is strictly between m^i and m^{i+1}. Let $a \in I \backslash m^{i+1}$. Since $Ra \nsubseteq m^{i+1}$, so $m^{i+1} \subsetneqq Ra$. Hence, $l(Ra/m^{i+1}) = l(I/m^{i+1}) = 1$. So, $I = Ra$ is a cyclic ideal. □

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یادداشتی بر حلقه‌ها و مدول‌های تقریباً تک زنجیر

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در این مقاله، ما حلقه‌ها و مدول‌های تقریباً تک زنجیر را مورد مطالعه قرار می‌دهیم. یک R-مدول M تقریباً تک زنجیر نامیده می‌شود هرگاه هر دو زیرمدول غیر یکریخت آن با رابطه شمول قابل مقایسه باشند. یک حلقه R یک حلقه تقریباً تک زنجیر چپ است هرگاه R به عنوان R-مدول چپ تقریباً تک زنجیر باشد. ما در این مقاله شرایط لازم و کافی برای اینکه یک حلقه آرتینی، تقریباً تک زنجیر چپ باشد را ارائه می‌دهیم.

کلمات کلیدی: حلقه‌های تقریباً تک زنجیر، مدول‌های تقریباً تک زنجیر، جمع زیرمدول‌های ساده یک مدول.