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ON SEMI MAXIMAL FILTERS IN BL-ALGEBRAS

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Abstract. In this paper, first we study the semi maximal filters in linear BL-algebras and we prove that any semi maximal filter is a primary filter. Then, we investigate the radical of semi maximal filters in BL-algebras. Moreover, we determine the relationship between this filters and other types of filters in BL-algebras and Gödel algebra. Specially, we prove that in a Gödel algebra, any fantastic filter is a semi maximal filter and any semi maximal filter is an (n-fold) positive implicative filter. Also, in a BL-algebra, any semi maximal and implicative filter is a positive implicative filter. Finally, we give an answer to the open problem in [S. Motamed, L. Torkzadeh, A. Borumand Saeid and N. Mohtashamnia, Radical of filters in BL-algebras, Math. Log. Quart. 57, No. 2, (2011), 166-179].

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [8] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The

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second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. Most familiar example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. In 1958, Chang [4] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [19] introduced the notion of an implicative filter and a Boolean filter and proved that these notions are equivalent in BL-algebras. Boolean filters are an important class of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. Haveshki et al. [10], continued an algebraic analysis of BL-algebras, and they introduced n-fold (positive) implicative BL-algebras and n-fold (positive) implicative filters of BL-algebras. The notion of n-fold fantastic BL-algebras and n-fold fantastic filters of BL-algebras had defined by Lele et al. [13]. S. Motamed, L. Torkzadeh, A. Borumand saeid and N. Mohtashamnia, [15], introduced the notion of radical of filters in BL-algebras and they stated and proved some theorems that determine relationship between this notion and other types of filters of a BL-algebra. Moreover, they introduced semi maximal filter in BL-algebras. In this paper, in section 2 we give some definitions and theorems which are needed in the rest of the paper. In section 3 we study semi maximal filters in linear BL-algebras and we state and prove some theorems that determine relationship between this filter and other types of filters in linear BL-algebras. Also, in a particular case, we prove that every semi maximal filter of linear BL-algebra is a prime, maximal and primary filter. In section 4 we study radical of filters and semi maximal filters in integral BL-algebras and we state and prove some theorems. Continuously in section 4 we study connection between semi maximal filters and (n-fold)fantastic filters in BL-algebras. Also, in [15], there is still an open problem which we will answer it in this section.

2. Preliminaries

In this section, we state some definitions and theorems which will be used in the sections as follows:

Definition 2.1. [8] A *BL-algebra* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that
$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
$(BL4)$ $x \wedge y = x \odot (x \rightarrow y)$,
$(BL5)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n \in \mathbb{N}$, when $\mathbb{N}$ is natural numbers and $x^0 = 1$.

A BL-algebra L is called a *Gödel algebra* if $x^2 = x \odot x = x$, for all $x \in L$ and L is called an *MV-algebra* if, $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$. Note that an operation of addition is defined in an MV-algebra L by setting $x \oplus y = (x^- \odot y^-)^-$, for all $x, y \in L$. Also, we denote $nx = \overbrace{x \oplus ... \oplus x}^{n-times}$, when $n \in \mathbb{N}$ and $x \in L$. In any BL-algebra L the following hold:
$(BL6)$ $x \leq y$ if and only if $x \to y = 1$.
$(BL7)$ $1 \to x = x$ and $x \to x = 1$.
$(BL8)$ $x \wedge y \leq x, y$ and $x^{---} = x^-$.
$(BL9)$ $x^- = 1$ if and only if $x = 0$.
$(BL10)$ $x \to (y \to z) = y \to (x \to z) = x \odot y \to z$.
$(BL11)$ $x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$.
$(BL12)$ $(x \vee y)^n = x^n \vee y^n$.
for all $x, y, z \in L$ and $n \in \mathbb{N}$ (See [5, 6, 8]).

We briefly review some types of filters and related theorems that, we refer the reader to [9, 10, 14, 17, 18, 19], for more details.

Definition 2.2. Let L be a BL-algebra and F be a non-empty subset of L. Then
(i) F is called a *filter* of L, if $x \odot y \in F$, for all $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$. Also, a filter F of L is called a proper filter, if $F \neq L$.
(ii) F is called a *maximal filter* of L, if it is a proper filter and is not properly contained in any other proper filter of L.
(iii) F is called a *primary filter*, if it is a proper filter and for all $x, y \in L$, $(x \odot y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N} \cup \{0\}$.
(iv) F is called an *n-fold implicative filter* of L, if $1 \in F$ and for all $x, y, z \in L$, $x^n \to (y \to z) \in F$ and $x^n \to y \in F$ imply $x^n \to z \in F$.
(v) F is called an *n-fold positive implicative filter* of L, if $1 \in F$ and for all $x, y, z \in L$, $x \to ((y^n \to z) \to y) \in F$ and $x \in F$ imply $y \in F$.
(vi) F is called an *n-fold fantastic filter*, if $1 \in F$ and for all $x, y, z \in L$, $z \to (y \to x) \in F$ and $z \in F$ imply $(((x^n \to y) \to y) \to x) \in F$.
(vii) F is called a *normal filter*, if for all $x, y, z \in L$, $z \to (y \to x) \in F$ and $z \in F$ imply that $(((x \to y) \to y) \to x) \in F$.

Let L be a BL-algebra and $x \in L$. The order of x, in symbols $ord(x)$, is the smallest positive integer number n such that $x^n = 0$(or x is a nilpotent element). We say is $ord(x) = \infty$, if no such n exist that

$x^n = 0$. Also, an element x of L is called a *unity* element of L if and only if for all $n \in \mathbb{N}$, $(x^n)^-$ is a nilpotent element of L (See [6, 15]).

Definition 2.3. [3, 7, 10, 13, 20] Let L be a BL-algebra. Then
(i) L is called an *n-fold positive implicative BL-algebra*, if $(x^n \to 0) \to x = x$, for all $x \in L$.
(ii) L is called a *local* BL-algebra, if it has an unique maximal filter.
(iii) L is called a *simple* BL-algebra, if L is non-trivial, and $\{1\}$ is its only proper filter.
(iv) L is called an *integral BL-algebra*, if $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$.

Theorem 2.4. (Gödel negation) A BL-algebra L is an integral BL-algebra if and only if $x \to 0 = 0$ or $x \to 0 = 1$, for all $x \in L$.

Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ which is defined by

$$x \equiv_F y \ \text{ if and only if } x \to y \in F \ \text{ and } y \to x \in F$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on L/F, the set of all congruence classes of L, as follows:

$$[x]\cdot[y] = [x\odot y],\ [x] \rightharpoonup [y] = [x \to y],\ [x]\sqcup[y] = [x\vee y],\ [x]\sqcap[y] = [x\wedge y].$$

Then $(L/F, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F (See [8]).

Theorem 2.5. [2, 10, 12, 13, 14] Let F be a filter of BL-algebra L. Then
(i) F is a fantastic filter of L if and only if $((x \to 0) \to 0) \to x \in F$, for all $x \in L$.
(ii) F is an n-fold positive implicative filter of L if and only if $(x^n)^- \to x \in F$ imply $x \in F$, for all $x \in L$.
(iii) F is a fantastic filter of L if and only if $x \to u \in F$ and $u \to x \in F$ imply $((x \to y) \to y) \to u \in F$, for all $x, y, u \in L$.
(iv) F is a fantastic filter of L if and only if F is a normal filter.
(v) A proper filter F is a maximal filter if and only if $\forall x \notin F$, $\exists n \in \mathbb{N}$ such that $(x^n)^- \in F$.

Theorem 2.6. [10, 13, 14, 20] Let F be a filter of BL-algebra L. Then
(i) F is a positive implicative filter if and only if L/F is a positive implicative BL-algebra.
(ii) F is a fantastic filter if and only if L/F is an MV-algebra.
(iii) F is a implicative filter if and only if L/F is a Gödel algebra.
(iv) If F is an n-fold (positive)implicative filter of L, then F is an $(n+1)$-fold (positive)implicative filter of L.
(v) F is an n-fold positive implicative filter if and only if F is an n-fold

implicative and n-fold fantastic filter.
(vi) F is a primary filter if and only if L/F is a local BL-algebra.
(vii) If F is an n-fold fantastic filter of L, then F is an $(n+1)$-fold fantastic filter of L.

The following theorems and definitions are from [15] and the reader can refer to it, for more details.

Let F be a proper filter of BL-algebra L. The intersection of all maximal filters of L which contain F is called the radical of F and it is denoted by $Rad(F)$. If $F = L$, then we put $Rad(L) = L$.

Theorem 2.7. A BL-algebra L is a semi simple BL-algebra if and only if $Rad(L) = \{x \in L \mid (x^n)^- \leq x, for\, any\ n \in \mathbb{N}\} = \{1\}$ and for a filter F of BL-algebra L, $Rad(F) = \{x \in L \mid (x^n)^- \to x \in F, for\, any\ n \in \mathbb{N}\}$. Also, let F be a proper filter of BL-algebra L and $x \in L$. Then $x \in Rad(F)$ if and only if $x^- \to x^n \in F$, for all $n \in \mathbb{N}$.

Note that, $Rad(\{1\})$ is the same as $Rad(L)$ which is defined in [5]. Thus, BL-algebra L is semi simple if and only if $Rad(\{1\}) = \{1\}$ (See [15]).

Theorem 2.8. Let F and G be filters of BL-algebra L. Then
(i) If $F \subseteq G$, then $Rad(F) \subseteq Rad(G)$.
(ii) $Rad(F) = L$ if and only if $F = L$.
(iii) $F \subseteq Rad(F)$.
(iv) $D_s(L) \subseteq Rad(F)$, where $D_s(L) = \{x \in L \mid x^- = 0\}$.
(v) $D_s(L) \subseteq F$ if and only if L/F is an MV-algebra.
(vi) $D_s(L) = \{1\}$ if and only if L is an MV-algebra.

Theorem 2.9. Let L be a linear BL-algebra and F be a filter of BL. Then
(i) If $x \notin Rad(F)$, then x is a nilpotent element of L.
(ii) $Rad(F) = \{x \in L \mid ord(x) = \infty\}$.

Let F be a filter of BL-algebra L. If $Rad(F) = F$, then F is called a semi maximal filter of L. We can represent semi maximal filter F of L by $F = \{x \in L \mid (x^n)^- \to x \in F, for\, any\ n \in \mathbb{N}\}$.

Theorem 2.10. Let F be a filter of BL-algebra L. The following are equivalent:
(i) F is a semi maximal filter,
(ii) $\{1\}/F$ is a semi maximal filter of L/F.

Theorem 2.11. Let F be a filter of BL-algebra L. The following are equivalent:
(i) L/F is a semi simple BL-algebra,
(ii) F is a semi maximal filter.

Let L be an MV-algebra. The intersection of all maximal ideals of L is called the radical of L and it is denoted by $Rad_{mv}(L)$. Now, if $Rad_{mv}(L) = \{0\}$, then L is called a semi simple MV-algebra (See [1]).

Theorem 2.12. [1] Let L be an MV-algebra. Then L is a semi simple MV-algebra if and only if for each $x \in L$, $nx \leq x^-$ for all $n \in \mathbb{N}$, implies $x = 0$.

From now on, in this paper $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or simply) L is a BL-algebra, unless otherwise state.

3. Semi maximal filters in linear BL-algebras

In this section, we study the notion of semi maximal filters in linear BL-algebras and we state and prove some theorems that determine relationship between this filter and other types of filter in linear BL-algebras.

Note. From now on, in this section, we let L be a linear BL-algebra, unless otherwise state.

Theorem 3.1. Let F be a semi maximal filter of L. Then F is a primary filter.

Proof. Let F be a semi maximal filter of L and $(x \odot y)^- \in F$, for $x, y \in L$. We will show that $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N} \cup \{0\}$. If $x \in F$, then $x \odot (x \odot y)^- \in F$. Since, by $(BL10)$ and $(BL4)$,

$$\begin{aligned} x \odot (x \odot y)^- &= x \odot ((x \odot y) \rightarrow 0) \\ &= x \odot (x \rightarrow (y \rightarrow 0)) \\ &= x \odot (x \rightarrow y^-) \\ &= x \wedge y^- \end{aligned}$$

we get $x \wedge y^- \in F$ and so by $(BL8)$, $y^- \in F$. By the similar way, if $y \in F$, then $x^- \in F$. Hence, F is a primary filter. Now, let $x \notin F$ and $y \notin F$. Since F is a semi maximal filter, $Rad(F) = F$ and so $x, y \notin Rad(F)$. Now, by Theorem 2.9(i), x and y are nilpotent elements of L. Then there exists $n \in \mathbb{N}$, such that $x^n = 0$ and $y^n = 0$ and so $(x^n)^- = 1 \in F$ and $(y^n)^- = 1 \in F$. Therefore, F is a primary filter of L. $\square$

The following example shows that if L is not a linear BL-algebra, then the Theorem 3.1, may not be correct. Moreover, the converse of Theorem 3.1, is not correct in general.

Example 3.2. [15] (i) Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq 1$, $0 \leq a \leq d \leq 1$ and $0 \leq c \leq d \leq 1$. Define $\odot$ and $\rightarrow$ as follow:

Table 1. Product

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

Table 2. Implication

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra which is not a linear BL-algebra. It is clear that $F = \{1\}$ is a semi maximal filter, while it is not a primary filter. Since $(b \odot c)^- = 1 \in F$, but $b^n = b$ and $c^n = c$, for all $n \in \mathbb{N}$. Then $(b^n)^- = b^- = c \notin \{1\}$ and $(c^n)^- = c^- = b \notin \{1\}$. Therefore, F is not a primary filter.

(ii) Let $L = \{0, a, b, 1\}$, where $0 \leq a \leq b \leq 1$. Define $\odot$ and $\rightarrow$ as follow:

Table 3. Product

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table 4. Implication

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a linear BL-algebra. It is clear $F = \{1, b\}$ is a primary and prime filter of L while it is not a semi maximal filter. Because, $Rad(F) = \{1, a, b\}$ and so $F \subset Rad(F)$.

Corollary 3.3. Let F be a semi maximal filter of L. Then L/F is a local BL-algebra.

Proof. It follows from Theorem 2.6(vi) and Theorem 3.1. □

Theorem 3.4. Let F be a proper filter of L. Then $Rad(F)$ is a prime filter.

Proof. Let F be a proper filter of L and $x \vee y \in Rad(F)$, for $x, y \in L$. If $x \notin Rad(F)$ and $y \notin Rad(F)$, then by Theorem 2.9(i), x and y are nilpotent elements of L and so there exists $n \in \mathbb{N}$, such that $x^n = 0$ and $y^n = 0$. By $(BL12)$, $(x \vee y)^n = x^n \vee y^n = 0$. Now, since $Rad(F)$ is a filter of L and $x \vee y \in Rad(F)$, then $0 = (x \vee y)^n \in Rad(F)$ and so $Rad(F) = L$. Therefore, by Theorem 2.8(ii), $F = L$ which is a contradiction. Hence, $x \in Rad(F)$ or $y \in Rad(F)$. Thus, $Rad(F)$ is a prime filter. □

The following example shows that if L is not a linear BL-algebra, then the Theorem 3.4, may not be correct.

Example 3.5. Let $F = \{1\}$, in Example 3.2(i). It is easy to check that $Rad(F) = F$, but $Rad(F)$ is not a prime filter. Since $d \vee b = 1 \in Rad(F)$, but $d \notin Rad(F)$ and $b \notin Rad(F)$.

Corollary 3.6. Every semi maximal filter of L is a prime filter.

Proof. It follows from Theorem 3.4. □

Theorem 3.7. Let F be a semi maximal filter of L. Then F is a maximal filter.

Proof. Let F be a semi maximal filter of L. Then $Rad(F) = F$. Now, if $x \notin F = Rad(F)$, then by Theorem 2.9(i), x is a nilpotent element of L and so there exists $n \in \mathbb{N}$, such that $x^n = 0$. Hence, by $(BL9)$, $(x^n)^- = 1 \in F$. Therefore, by Theorem 2.5(v), F is a maximal filter of L. □

The following example shows that if L is not a linear BL-algebra, then the Theorem 3.7, may not be correct.

Example 3.8. Let $F = \{1\}$, in Example 3.2. It is easy to check that F is a semi maximal filter and $Rad(F) = F$, but $Rad(F)$ is not a maximal filter. Since $F = \{1\} \subset \{1, b\}$ and $\{1, b\}$ is a proper filter of L.

Corollary 3.9. Let G be a filter of L and $F \subseteq G$ where F is a semi maximal filter of L. Then $F = G$ is a semi maximal filter of L, too.

Proof. Let $F \subseteq G$ and F be a semi maximal filter of L. Then by Theorem 3.7, F is a maximal filter and since $F \subseteq G \subseteq L$, then $F = G$ or $G = L$. Now, since in this case $Rad(G) = G$, then G is a semi maximal filter of L, too. □

Note that, if L is not a linear BL-algebra, then Corollary 3.9, may not be correct. Let $F = \{1, d\}$ and $G = \{1, d, c\}$ in Example 3.2. It is easy to check that F and G are semi maximal filters, but $F \subset G \subset L$.

Definition 3.10. Let L be a BL-algebra. Then we define the nilpotent elements of L by

$$Nil(L) = \{x \in L \mid x^m = 0, \text{ for some } m \in \mathbb{N}\}.$$

Example 3.11. Let L be a BL-algebra as in Example 3.2(i). Then $Nil(L) = \{0, a\}$.

Theorem 3.12. Let F be a filter of BL-algebra L. Then
(i) $Nil(L) \cap Rad(F) \subseteq F$.
(ii) If L is a linear BL-algebra, then $Nil(L) \cap Rad(F) = \emptyset$.

Proof. (i) Let $x \in Nil(L) \cap Rad(F)$. Then there exists $m \in \mathbb{N}$, such that $x^m = 0$ and since $x \in Rad(F)$, then $(x^n)^- \to x \in F$, for any $n \in \mathbb{N}$. Hence, by $(BL7)$, $x = 1 \to x = 0^- \to x = (x^m)^- \to x \in F$. Therefore, $x \in F$ and so $Nil(L) \cap Rad(F) \subseteq F$.
(ii) Let L be a linear BL-algebra. Then by Theorem 2.9(ii), $Rad(F) = \{x \mid ord(x) = \infty\}$. Hence, $Nil(L) \cap Rad(F) = \emptyset$. □

4. Radical of filters in integral BL-algebras

In this section we study radical of filters and semi maximal filters in integral BL-algebras and we state and prove some theorems.

Theorem 4.1. Let L be a BL-algebra. Then L is an integral BL-algebra if and only if $Rad(L) = L\backslash\{0\}$.

Proof. Let L be an integral BL-algebra. Then by Theorem 2.4, $x^- = 0$, for all $0 \neq x \in L$. Now, let $x \in Rad(L)$. Since for any $n \in \mathbb{N}$, $(x^n)^- \leq x$, we get $x \neq 0$. Now, if $x = 0$, then $1 = (0^n)^- \leq 0$ which is impossible. Hence, $Rad(L) \subseteq L\backslash\{0\}$. If $x \in L\backslash\{0\}$, since L is an integral BL-algebra, then $x^n \neq 0$, for any $n \in \mathbb{N}$. By Theorem 2.4, $(x^n)^- = 0$. Hence, $0 = (x^n)^- \leq x$, for any $n \in \mathbb{N}$. Thus, $x \in Rad(L)$ and so $Rad(L) = L\backslash\{0\}$. Conversely, let $Rad(L) = L\backslash\{0\}$. If $x \to 0 = 0$, for all $0 \neq x \in L$, then by Theorem 2.4, L is an integral BL-algebra. If there exists $x \in L\backslash\{0\} = Rad(L)$ where $x^- = x \to 0 \neq 0$, then $x^- \in L\backslash\{0\} = Rad(L)$. Hence, $0 = x \odot x^- \in Rad(L) = L\backslash\{0\}$, which is a contradiction. Therefore, L is an integral BL-algebra. □

Corollary 4.2. A BL-algebra L is an integral BL-algebra if and only if $Rad(\{1\}) = L\backslash\{0\}$.

Proof. It follows from Theorem 2.7. □

Theorem 4.3. Let L be an integral BL-algebra and F be a filter of L. Then
(i) $Rad(F) = L\backslash\{0\}$.
(ii) $Rad(F)$ is a maximal filter.
(iii) If F is a semi maximal filter, then $F = L\backslash\{0\}$.

Proof. (i) Since $\{1\} \subseteq F \subseteq L$, by Theorem 2.8$(i)$, $Rad(\{1\}) \subseteq Rad(F) \subseteq Rad(L)$. Also, since L is an integral BL-algebra, by Theorem 4.1, $Rad(\{1\}) = Rad(L) = L\backslash\{0\}$. Hence, $Rad(F) = L\backslash\{0\}$.
(ii) It is clear by (i).
(iii) It is clear by (i). □

Corollary 4.4. Let G be a filter of integral BL-algebra L and $F \subseteq G$ where F is a semi maximal filter of L. Then G is a semi maximal filter of L, too.

Proof. Let $F \subseteq G$ and F be a semi maximal filter of integral BL-algebra L. Then by Theorem 4.3, $F = L\backslash\{0\}$ and since $F \subseteq G \subseteq L$, then $F = G = L\backslash\{0\}$ or $G = L$. Now, since in this case, $Rad(G) = G$, then so G is a semi maximal filter of L, too. □

Theorem 4.5. Let L be an integral BL-algebra. Then $L\backslash\{0\} = \{x \mid x \text{ is a unity element of } L\}$.

Proof. Let L be an integral BL-algebra and $x \in L\backslash\{0\}$. Then $x^n \neq 0$, for all $n \in \mathbb{N}$. Now, by Theorem 2.4, $(x^n)^- = 0$. Hence, $(x^n)^-$ is a nilpotent element for all $n \in \mathbb{N}$. Hence, x is a unity element of L and so $L\backslash\{0\} \subseteq \{x \mid x \text{ is a unity element of } L\}$. Moreover, if x is a unity element of L, then $x \neq 0$. Since if $x = 0$, then by $(BL9)$, $(x^n)^- = 1$ which is impossible. Therefore, $\{x \mid x \text{ is a unity element of } L\} \subseteq L\backslash\{0\}$ and so $L\backslash\{0\} = \{x \mid x \text{ is a unity element of } L\}$. □

Corollary 4.6. If L is an integral BL-algebra, then

$$Rad(\{1\}) = \{x \mid x \text{ is a unity element of } L\}$$

Proof. It follows from Corollary 4.2 and Theorem 4.5. □

Example 4.7. [11] Let $L = \{0, a, b, c, 1\}$. Define $\wedge$,$\vee$,$\odot$ and $\rightarrow$ on L as follows:

Table 5. Join

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

Table 6. Meet

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Table 7. Product

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Table 8. Implication

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is an integral BL-algebra. Also,

$$Rad(\{1\}) = \{x \mid x \text{ is a unity element of } L\} = L \backslash \{0\} = \{1, a, b, c\}.$$

5. Relation between semi maximal filter and some other filters in BL-algebras

In this section we study relation between semi maximal filter and some other filters such as and (n-fold) fantastic filters in BL-algebras. Also, in [15], there is still an open problem which we will answer it in this section.

Theorem 5.1. Let F be a fantastic filter of Gödel algebra L. Then F is a semi maximal filter of L.

Proof. Let F be a fantastic filter of Gödel algebra L and $x \in Rad(F)$. Then $(x^n)^- \to x \in F$, for all $n \in \mathbb{N}$. Now, since F is a fantastic filter, then by Theorem 2.5(i), $x^{--} \to x \in F$. If $n = 1$, then by Theorem 2.5(iii), $[(x^{--} \to x^-) \to x^-] \to x \in F$. Now, by $(BL10)$

$$\begin{aligned}
[(x^{--} \to x^-) \to x^-] \to x &= [((x \to x^{---}) \to (x \to 0)] \to x \\
&= [((x \to x^-) \to (x \to 0)] \to x, \text{ by } (BL8) \\
&= [((x^2 \to 0) \to (x \to 0)] \to x, \text{ by } (BL10)
\end{aligned}$$

Therefore, $[((x^2 \to 0) \to (x \to 0)] \to x \in F$. Since, L is a Gödel algebra, then

$$\begin{aligned}
[((x^2 \to 0) \to (x \to 0)] \to x &= [((x \to 0) \to (x \to 0)] \to x \\
&= 1 \to x, \text{ by } (BL7) \\
&= x, \text{ by } (BL7)
\end{aligned}$$

Then $x \in F$ and so $Rad(F) \subseteq F$. Now, since by Theorem 2.8(iii), $F \subseteq Rad(F)$, then $Rad(F) = F$ and so F is a semi maximal filter of L. $\square$

Theorem 5.2. Let F be an n-fold fantastic filter and n-fold implicative filter of L. Then F is a semi maximal filter of L.

Proof. Let F be an n-fold fantastic and n-fold implicative filter of L. Then by Theorem 2.6(v), F is an n-fold positive implicative filter of L. If $x \in Rad(F)$, then $(x^m)^- \to x \in F$, for any $m \in \mathbb{N}$. Now, let $m = n$. Then $(x^n)^- \to x \in F$ and since F is an n-fold positive implicative filter of L, then by Theorem 2.5(ii), $x \in F$. Hence, $Rad(F) \subseteq F$. Now, since by Theorem 2.8(iii), $F \subseteq Rad(F)$, Then $Rad(F) = F$ and so F is a semi maximal filter of L. $\square$

Corollary 5.3. If F is an n-fold positive implicative filter of L, then F is a semi maximal filter of L.

Proof. It follows from Theorem 2.6(v) and Theorem 5.2. □

The following example shows that the converse of Corollary 5.3, is not correct in general.

Example 5.4. Let $F = \{1\}$, in Example 3.2. It is easy to check that F is a semi maximal filter and $Rad(F) = F$, but F is not a 2-positive implicative filter. Since $(b^2)^- \to b = c \to b = b \notin F$.

Theorem 5.5. Let F be a semi maximal filter of Gödel algebra L. Then F is a positive implicative filter of L.

Proof. Let L be a Gödel algebra and $x^- \to x \in F$, for $x \in L$. Then $x^2 = x$ and so $x^n = x$, for any $n \in \mathbb{N}$. Hence, $(x^n)^- \to x = x^- \to x \in F$, for any $n \in \mathbb{N}$. Therefore, $x \in Rad(F)$. Now, since F is a semi maximal filter of L, then $Rad(F) = F$ and so $x \in F$. Therefore, F is a positive implicative filter. □

Corollary 5.6. If F is a semi maximal filter of Gödel algebra L, then F is an n-fold positive implicative filter of L.

Proof. It follows from Theorem 2.6(iv) and Theorem 5.5. □

Theorem 5.7. Let F be a semi maximal and implicative filter of BL-algebra L. Then F is a positive implicative filter of L.

Proof. Let F be a semi maximal and implicative filter of BL-algebra L. Then by Theorem 2.6(iii), L/F is a Gödel algebra. Also, since F is a semi maximal filter, then by Theorem 2.10, $\{1\}/F$ is a semi maximal filter of L/F. Now, by Theorem 5.5, $\{1\}/F$ is a positive implicative filter of L/F. Therefore, by Theorem 2.6(i), L/F is a positive implicative BL-algebra and so by Theorem 2.6(i), F is a positive implicative filter. □

Note. In [15], there is an example which shows a semi maximal filter may not be a fantastic filter. But, this example is not true. In the following, we show that this filter is not a semi maximal filter.

Example 5.8. Let $L = [0, 1]$. Define $\odot$ and $\to$ as follow:

$$x \to y = \begin{cases} 1, & x \leq y, \\ y, & y < x. \end{cases}$$

and $x \odot y = min\{x, y\}$. Then L is a BL-algebra. Now, let $F = [\frac{1}{2}, 1]$. Then F is a filter. Now, consider $z = 1, y = \frac{1}{3}, x = \frac{1}{5}$ in Definition 2.2(vii). Since $1 \to ((\frac{1}{3} \to \frac{1}{5}) \to \frac{1}{5}) = 1 \in F$, $1 \in F$ and $(\frac{1}{5} \to \frac{1}{3}) \to \frac{1}{3} = 1 \to \frac{1}{3} = \frac{1}{3} \notin F$, then F is not a normal filter and so by Theorem

2.5(iv), F it is not a fantastic filter. Also, since $x \odot x = min\{x, x\} = x$, then L is a Gödel algebra. Hence,

$Rad(F) = \{x \in L \mid (x^n)^- \to x \in F, \forall n \in \mathbb{N}\} = \{x \in L \mid x^- \to x \in F\}$.

Now, if $x = 0$, then by $(BL7)$, $x^- \to x = 1 \to 0 = 0 \notin [\frac{1}{2}, 1] = F$. Therefore, $0 \notin Rad(F)$ and so $Rad(F) \subseteq (0, 1]$. Also, if $x \neq 0$ and $x \in (0, 1]$, then $x^- = x \to 0 = 0$ and so $x^- \to x = 0 \to x = 1 \in [\frac{1}{2}, 1] = F$. Hence, $x \in Rad(F)$ and so $(0, 1] \subseteq Rad(F)$. Therefore, $Rad(F) = (0, 1]$ and F is not a semi maximal filter.

In the following, we study relation between semi maximal filters and fantastic filters which is an open problem in [15].

Theorem 5.9. Let F be a semi maximal filter of BL-algebra L. Then F is a fantastic filter of L.

Proof. Let F be a semi maximal filter of L. Then $Rad(F) = F$ and so, by Theorem 2.8(iv), $D_s(L) \subseteq Rad(F) = F$. Hence, by Theorem 2.8$(v)$, L/F is an MV-algebra and so by Theorem 2.6(ii), F is a fantastic filter of L. □

Corollary 5.10. If F is a semi maximal filter of BL-algebra L, then F is an n-fold fantastic filter of L.

Proof. It follows from Theorem 2.6(vii) and Theorem 5.9. □

Theorem 5.11. Let L be a semi simple BL-algebra. Then L is an MV-algebra.

Proof. Let L be a semi simple BL-algebra. Then $Rad(\{1\}) = \{1\}$ and so $\{1\}$ is a semi maximal filter. Hence, by Theorem 5.9, $\{1\}$ is a fantastic filter. Therefore, by Theorem 2.6(ii), L is an MV-algebra. Moreover, by Theorem 2.8(vi), $D_s(L) = \{1\}$. □

Theorem 5.12. Let F be a filter of BL-algebra L. Then the following conditions are equivalent:
(i) F is an n-fold positive implicative filter,
(ii) F is an n-fold implicative and n-fold fantastic filter,
(iii) F is an n-fold implicative and semi maximal filter.

Proof. $(i) \Leftrightarrow (ii)$: By Theorem 2.6(v), the proof is clear.
$(ii) \Rightarrow (iii)$: By Theorem 5.2, the proof is clear.
$(iii) \Rightarrow (ii)$: By Corollary 5.10, the proof is clear. □

Corollary 5.13. Let F be a filter of BL-algebra L. Then the following conditions are equivalent:
(i) F is a positive implicative filter,

(ii) F is a implicative and fantastic filter ,
(iii) F is a implicative and semi maximal filter.

Proof. Let $n = 1$ in Theorem 5.12. Then the proof is clear. □

For the converse of Theorem 5.9, firstly we state and prove some theorem and proposition and finally we answer the following question which is an open problem in [15]:
Open problem. [15] Let F be a fantastic of L. Is F a semi maximal filter of L, without the condition in the Corollary 5.13?

Proposition 5.14. Let L be an MV-algebra. Then $Rad_{mv}(L) = \{0\}$ if and only if $Rad(L) = Rad(\{1\}) = \{1\}$.

Proof. Let $Rad_{mv}(L) = \{0\}$ and $x \in Rad(L) = Rad(\{1\})$. Then $(x^n)^- \leq x$, for any $n \in \mathbb{N}$ and so by $(BL11)$, $x^- \leq (x^n)^{--} = x^n$. Since L is an MV-algebra, then $x^n = (\overbrace{x^- \oplus ... \oplus x^-}^{n-times})^-$ and so $x^- \leq (\overbrace{x^- \oplus ... \oplus x^-}^{n-times})^- = (nx^-)^-$. Now, by $(BL11)$, $(nx^-)^{--} \leq x^{--}$ and so $(nx^-) \leq (x^-)^-$. Hence, by Theorem 2.12, $x^- = 0$ and so by $(BL9)$, $x = 1$. Therefore, $Rad(L) = Rad(\{1\}) = \{1\}$. Conversely, Let $Rad(L) = Rad(\{1\}) = \{1\}$ and $x \in Rad_{mv}(L)$ such that $nx \leq x^-$, for all $n \in \mathbb{N}$. Since, $nx = nx^{--} = ((x^-)^n)^-$, then $((x^-)^n)^- \leq x^-$, for all $n \in \mathbb{N}$. Hence, $x^- \in Rad(L) = Rad(\{1\}) = \{1\}$ and so $x^- = 1$. Therefore, by $(BL9)$, $x = 0$ and so $Rad_{mv}(L) = \{0\}$. □

Corollary 5.15. Let L be an MV-algebra. Then L is a semi simple MV-algebra if and only if L is a semi simple BL-algebra.

Proof. It holds by Proposition 5.14. □

The following definition and proposition show that an MV-algebra may not be a semi simple MV-algebra. Therefore, by Corollary 5.15, a BL-algebra may not be a semi simple BL-algebra.

Definition 5.16. [16] The free product $A_1 \amalg A_2$ of two MV-algebras A_1 and A_2 is an MV-algebra $A = A_1 \amalg A_2$ together with one-one homomorphisms $\mu_i : A_i \to A$ having the following universal property: $\mu_i(A_1) \cup \mu_2(A_2)$ generates A, and for any MV-algebra E and homomorphisms $\xi_i : A_i \to E$, there is a (necessarily unique) homomorphism $\xi : A \to E$ such that $\xi_1 = \xi\mu_1$ and $\xi_2 = \xi\mu_2$.

Proposition 5.17. [16]$(Proposition 7.3)$ Let A_ξ denote the MV-subalgebra of $[0, 1]$ generated by an irrational $\xi \in [0, 1]$. Then $A_\xi \amalg A_\xi$ is not totally ordered and is not semi simple MV-subalgebra.

Corollary 5.18. Trivial filter $\{1\}$ of MV-algebra $A_\xi \amalg A_\xi$ is not a semi maximal filter.

Proof. Let trivial filter $\{1\}$ of MV-algebra $A_\xi \amalg A_\xi$ is a semi maximal filter. Then by Theorem 2.12, $L/\{1\}$ is a semi simple BL-algebra and so L is a semi simple BL-algebra. But, by Theorem 5.17 it is impossible. Therefore, trivial filter $\{1\}$ of MV-algebra $A_\xi \amalg A_\xi$ is not a semi maximal filter.

□

Answer to the open problem: If F is a fantastic filter, then F may not be a semi maximal filter, in the another word the converse of Theorem 5.9, is not correct in general. Let $F = \{1\}$ trivial filter in MV-algebra $A_\xi \amalg A_\xi$. Then by Theorem 2.6(ii), F is a fantastic filter and by Corollary 5.18, it is not a semi maximal filter.

6. Conclusion

The results of this paper will be devoted to study the radical of filters and semi maximal filters. In this paper, we study semi maximal filters in linear BL-algebras and integral BL-algebras. Moreover, we proved that every semi maximal filter is a fantastic filter but the converse is not true in general. It was an open problem in [15], that we answered it.

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ON SEMI MAXIMAL FILTERS IN BL-ALGEBRAS

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نتایجی بر فیلترهای نیم‌ماکسیمال در BL-جبرها

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در این مقاله ابتدا فیلترهای نیم‌ماکسیمال را درBL-جبرهای خطی مطالعه می‌کنیم و ثابت می‌کنیم که هر فیلتر نیم‌ماکسیمال، یک فیلتر اولیه است. سپس رادیکال فیلترهای نیم‌ماکسیمال را مورد بررسی قرار می‌دهیم. به‌علاوه، رابطه بین این فیلترها و سایر انواع فیلترها را در BL-جبرها و جبر گودل مشخص می‌کنیم. به طور خاص، ثابت می‌کنیم که در یک جبر گودل هر فیلتر، یک فیلتر نیم‌ماکسیمال است و هر فیلتر نیم‌ماکسیمال نیز یک فیلتر استلزامی مثبت (n-لایه) است. همچنین، در یک BL-جبر، هر فیلتر نیم ماکسیمال استلزامی، یک فیلتر استلزامی مثبت است. در پایان نیز به یک مسئله باز که توسط برومند سعید و محتشم‌نیا مطرح شده است پاسخ می‌دهیم.

کلمات کلیدی: BL-جبر(نیم‌ساده)، جبر گودل، فیلتر نیم‌ماکسیمال، رادیکال فیلتر.