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ON THE NORMALITY OF t-CAYLEY HYPERGRAPHS OF ABELIAN GROUPS

R. BAYAT, M. ALAEIYAN AND S. FIROUZIAN*

Abstract. A t-Cayley hypergraph $X = t\text{-}Cay(G, S)$ is called *normal* for a finite group G, if the right regular representation $R(G)$ of G is normal in the full automorphism group $Aut(X)$ of X. In this paper, we investigate the normality of t-Cayley hypergraphs of abelian groups, where $|S| \leq 4$.

1. Introduction

A *hypergraph* X is a pair (V, E), where V is a finite nonempty set and E is a finite family of nonempty subsets of V. The elements of V are called *hypervertices* or simply *vertices* and the elements of E are called *hyperedges* or simply *edges*. Two vertices u and v are *adjacent* in hypergraph X=(V, E) if there is an edge $e \in E$ such that $u, v \in e$. If for two edges $e, f \in E$ holds $e \cap f \neq 0$, we say that e and f are *adjacent*. A vertex v and an edge e are *incident* if $v \in e$. We denote by $X(v)$ the *neighborhood* of a vertex v, i.e. $X(v) = \{u \in V : \{u, v\} \in E\}$. Given $v \in V$, denote the number of edges incident with v by $d(v)$; $d(v)$ is called the *degree* of v. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or d-*regular*. The size, or the *cardinality*, $|e|$ of a hyperedge is the number of vertices in e. A hypergraph X=(V, E) is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. A hypergraph is known as *uniform* or k-*uniform* if all the edges have cardinality k. Note that an ordinary graph with no isolated vertex is a 2-uniform hypergraph.

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*Corresponding author.

A *path* of length k in a hypergraph (V, E) is an alternating sequence $(v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1})$ in which $v_i \in V$ for each $i = 1, 2, \ldots, k+1$, $e_i \in E$, $\{v_i, v_{i+1}\} \subseteq e_i$ for $i = 1, 2, \ldots, k$ and $v_i \neq v_j$ and $e_i \neq e_j$ for $i \neq j$. A hypergraph is *connected* if there is a path between every pair of vertices.

Let $X_1{=}(V_1, E_1)$ and $X_2{=}(V_2, E_2)$ be two hypergraphs. A *homomorphism* $\varphi : X_1 \to X_2$ is a map $\varphi : V_1 \to V_2$ that preserves adjacencies, that is, $\varphi(e) \in E_2$ for each $e \in E_1$. When φ is a bijection and its inverse map is also a homomorphism then φ is an *isomorphism* between the two hypergraphs and X_1 and X_2 are isomorphic.

An isomorphism from a hypergraph X onto itself is an *automorphism*. The *automorphism group* of X is denoted by $Aut(X)$. For more information about hypergraphs, the readers are referred to [3, 4].

For a group G and a subset S of G such that $1_G \notin S$ and $S = S^{-1} := \{s^{-1} | s \in S\}$, the *Cayley graph* $X = Cay(G, S)$ of G with respect to S is defined as the graph with vertex set $V(X) = G$, and edge set $E(X) = \{\{g, h\} | hg^{-1} \in S\}$.

Obviously, the Cayley graph $Cay(G, S)$ has valency $|S|$, and it easily follows that $Cay(G, S)$ is connected if and only if $G = \langle S \rangle$, that is, S generates G. For a group G, denote $R(G)$ as the right regular representation of G. Define

$$Aut(G, S) := \{\alpha \in Aut(G) | S^\alpha = S\},$$

acting naturally on G. Then, it is easy to see that each Cayley graph $X = Cay(G, S)$ admits the group $R(G).Aut(G, S)$ as a subgroup of automorphisms. Moreover (see [6]), $N_{Aut(X)}(R(G)) = R(G).Aut(G, S)$. Note that $R(G) \cong G$. So, we can identify G with $R(G) \leq Aut(X)$ for $X = Cay(G, S)$. The Cayley graph $X = Cay(G, S)$ is called *normal* if G is normal in $Aut(X)$. In this case, $Aut(X) = G.Aut(G, S)$.

Let G be a group and let S be a set of subsets $s_1, s_2, \ldots, s_n$ of $G - \{1_G\}$ such that $G = \langle \bigcup_{i=1}^n s_i \rangle$, that is, $\bigcup_{i=1}^n s_i$ generates G. A *Cayley hypergraph* $CH(G, S)$ has vertex set G and edge set $\{\{g, gs\} | g \in G, s \in S\}$, where an edge $\{g, gs\}$ is the set $\{g\} \cup \{gx | x \in s\}$. For all $s \in S$, if $|s| = 1$, then the Cayley hypergraph is a Cayley graph. Therefore, a Cayley hypergraph is a generalization of a Cayley graph [7]. Also, Lee and Kwon [7] proved that a hypergraph X is Cayley if and only if $Aut(X)$ contains a subgroup which acts regularly on the vertex set of X. For example, the hypergraph X, with

$$\begin{aligned} V(X) &= \{0, 1, 2, 3, 4, 5, 6\}, \\ E(X) &= \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\} \end{aligned}$$

is considered. This hypergraph which is called the Fano plane, is the Cayley hypergraph $X = CH(\mathbb{Z}_7, \{1, 3\})$.

In 1994, Buratti [5] introduced the concept of a t-Cayley hypergraph as follows. Let G be a finite group, S a subset of $G - \{1_G\}$ and t an integer satisfying $2 \leq t \leq$max$\{o(s)|s \in S\}$. The *t-Cayley hypergraph* $X = t\text{-}Cay(G, S)$ of G with respect to S is defined as the hypergraph with vertex set $V(X) = G$, and for $E \subseteq G$,

$$E \in E(X) \Longleftrightarrow \exists g \in G, \exists s \in S : E(X) = \{gs^i|0 \leq i \leq t-1\}.$$

Note that any 2-Cayley hypergraph is a Cayley graph and vice versa. For any $s_i \in S$, if $s_i = \{s, \ldots, s^{t-1}\}$ for some $s \in G - \{1_G\}$, then the Cayley hypergraph $CH(G, S)$ is a t-Cayley hypergraph $t - Cay(G, S)$. Hence, a Cayley hypergraph is a generalization of a t-Cayley hypergraph. In fact every t-Cayley hypergraph is a subclass of the more general Cayley hypergraphs, or *group hypergraphs* which is defined by Shee in [8].

The concept of normality of the Cayley graph is known to be of fundamental importance for the study of arc transitive graphs. So, for a given finite group G, a natural problem is to determine all the normal or non-normal Cayley graph of G. Some meaningful results in this direction, especially for the undirected Cayley graphs, have been obtained. Baik et al. [1] determined all non-normal Cayley graphs of abelian groups of valency at most 4 and later [2] dealt with valency 5. For directed Cayley graphs, Xu et al. [10] determined all non-normal Cayley graphs of abelian groups of valency at most 3. In this paper, we extended the results of [1] to Cayley hypergraphs, and classify all normal t-Cayley hypergraphs, where G is a finite abelian group and $|S| \leq 4$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $X = t\text{-}Cay(G, S)$ be a connected t- Cayley hypergraph of an abelian group G with respect to S with $|S| \leq 4$. Then X is normal except one of the following cases happens:*

(1) $X = n\text{-}Cay(\mathbb{Z}_n = \langle a \rangle, \{a, a^{-1}\})$, *where* $n \geq 2$.

(2) $X = 4\text{-}Cay(\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, a^{-1}, b\})$.

(3) $X = 6\text{-}Cay(\mathbb{Z}_6 = \langle a \rangle, \{a, a^{-1}, a^3\})$.

(4) $X = 2\text{-}Cay(\mathbb{Z}_2^3 = \langle r \rangle \times \langle s \rangle \times \langle t \rangle, \{t, tr, ts, tsr\})$.

(5) $X = 4\text{-}Cay(\mathbb{Z}_4 = \langle a \rangle, \{a, a^{-1}, a^2\}) \times K_2$, *where* $K_2 = 2\text{-}Cay(\mathbb{Z}_2 = \langle s \rangle, \{s\})$.

(6) $X = 4\text{-}Cay(\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle \times \langle s \rangle, \{a, a^{-1}, a^2 s, s\})$.

(7) $X = 4\text{-}Cay(\mathbb{Z}_4 \times \mathbb{Z}_2^2 = \langle a \rangle \times \langle r \rangle \times \langle s \rangle, \{a, a^{-1}, r, s\})$.

(8) $X = 4\text{-}Cay(\mathbb{Z}_4 = \langle a \rangle, \{a, a^{-1}\}) \times m\text{-}Cay(\mathbb{Z}_m = \langle b \rangle, \{b, b^{-1}\})$.

(9) $X = 4m\text{-}Cay(\mathbb{Z}_{4m} = \langle b \rangle, \{b, b^{-1}, b^m, b^{-m}\})$, *where* $m \geq 2$.

(10) $X = 2m\text{-}Cay(\mathbb{Z}_{4m} = \langle x \rangle, \{x^2, x^{-2}, x^m, x^{-m}\})$, *where* $m \geq 1$.

(11) $X = 4m\text{-}Cay(\mathbb{Z}_{4m} \times \mathbb{Z}_2 = \langle x \rangle \times \langle y \rangle, \{x, x^{-1}, x^m y, x^{-m} y\})$, *where* $m \geq 1$.

(12) $X = m\text{-}Cay(Z_m \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle, \{a, a^{-1}, ab, a^{-1}b\})$, $m \geq 4$.

(13) $X = n\text{-}Cay(\mathbb{Z}_n = \langle a \rangle, \{a, a^{-1}, a^3, a^{-3}\})$, *where* $n \geq 5$ *and* $n \neq 6$.

2. Preliminary Results

In this section, we introduce some preliminary results and definitions which will be needed in the subsequent section.

Lemma 2.1. *Let* $X = t\text{-}Cay(G, S)$ *be a* t*-Cayley hypergraph where* S *is a subset of* $G - \{1_G\}$. *Then* $Aut(G) \cap Aut(X) = Aut(G, S)$.

Proof. By definition we have $Aut(G, S) = \{\alpha \in Aut(G) | S^\alpha = S\}$. Suppose that $\alpha \in Aut(X) \cap Aut(G)$. The claim is $S^\alpha = S$. Now, $s \in S$ if and only if

$$\begin{aligned} & \{1, s, s^2, s^3, \ldots, s^{t-1}\} \in E(X) \\ \Leftrightarrow\ & \{1, s, s^2, s^3, \ldots, s^{t-1}\}^\alpha \in E(X) \\ \Leftrightarrow\ & \{1 = 1^\alpha, s^\alpha, (s^2)^\alpha, \ldots, (s^{t-1})^\alpha\} \in E(X) \\ \Leftrightarrow\ & s^\alpha \in S, \end{aligned}$$

therefore $S^\alpha = S$, and hence $\alpha \in Aut(G, S)$. So $Aut(G) \cap Aut(X) \leq Aut(G, S)$. Now assume $\alpha \in Aut(G, S)$, which by definition means that $\alpha \in Aut(G)$. We will have $e \in E(X)$ if and only if $\exists\ s \in\ S$ such that

$$\begin{aligned} e &= \{x, xs, xs^2, \dots, xs^{t-1}\} \in E(X) \\ &\Leftrightarrow \{x^\alpha, x^\alpha s^\alpha, x^\alpha (s^2)^\alpha, \dots, x^\alpha (s^{t-1})^\alpha\} \in E(X) \\ &\Leftrightarrow \{x^\alpha, x^\alpha s', x^\alpha (s')^2, \dots, x^\alpha (s')^{t-1}\} \in E(X), \end{aligned}$$

where $s^\alpha = s'$. Thus $\alpha \in Aut(X)$ and so $\alpha \in Aut(X) \cap Aut(G)$, which implies $Aut(G,S) \leq Aut(X) \cap Aut(G)$. □

The coming result is obtained from previous lemma. Consider $A := Aut(X)$.

Lemma 2.2. *Let $X = t\text{-}Cay(G,S)$ be a t-Cayley hypergraph of G with respect to S. Then $N_A(R(G)) = R(G).Aut(G,S)$. Furthermore, the stabilizer of 1_G in $N_A(R(G))$ is $Aut(G,S)$.*

Definition 2.3. Let $X = t\text{-}Cay(G,S)$ be a t-Cayley hypergraph of G with respect to S. Then X is called *normal* if $R(G) \lhd A$.

The following obvious result is a direct consequence of Definition 2.3 and Lemma 2.2.

Lemma 2.4. *Let $X = t\text{-}Cay(G,S)$. Then X is normal if and only if $A_1 = Aut(G,S)$, where A_1 is the stabilizer of 1_G in A.*

Proposition 2.5. *Let G be a finite group, and let S be a generating set of G not containing the identity 1_G, and α an automorphism of G. Then t-Cayley hypergraph $X = t\text{-}Cay(G,S)$ is normal if and only if $X' = t\text{-}Cay(G,S^\alpha)$ is normal.*

Proof. Let $A' = Aut(X')$. It will be shown that (1) $\alpha^{-1}A\alpha = A'$, and (2) $\alpha^{-1}R(G)\alpha = R(G)$. For the first equation, we suppose that $\alpha^{-1}\rho\alpha \in \alpha^{-1}A\alpha$, where $\rho \in A$. Now if $E' \in E(X')$, then $E' = \{xs^i | 0 \leq i \leq t-1\}$ for some $x \in G$ and $s \in S$. Therefore

$$\begin{aligned} (E')^{\alpha^{-1}\rho\alpha} &= \{(xs^i)^{\alpha^{-1}\rho\alpha} | 0 \leq i \leq t-1\} \\ &= \{x^{\alpha^{-1}}, x^{\alpha^{-1}}(s)^{\alpha^{-1}}, \dots, x^{\alpha^{-1}}(s^{t-1})^{\alpha^{-1}}\}^{\rho\alpha}. \end{aligned}$$

It follows that,

$$(E')^{\alpha^{-1}\rho\alpha} = \{y, ys', y(s')^2, \dots, y(s')^{t-1}\}^{\rho\alpha},$$

where $s' = s^{\alpha^{-1}}$ and $x^{\alpha^{-1}} = y$. Since $\rho \in A$,

$$(E')^{\alpha^{-1}\rho\alpha} = \{z, zs'', \dots, z(s'')^{t-1})\}^{\rho} \in E(X'),$$

where $s'' = (s')^\alpha$ and $y^\alpha = z$. By a similar argument $A' \subseteq \alpha^{-1}A\alpha$ and so $\alpha A\alpha^{-1} = A'$. Also it is easy to see that $\alpha^{-1}R(G)\alpha = R(G)$. Now X is normal, that is, $R(G) \lhd A$ if and only if $R(G) = \alpha^{-1}R(G)\alpha \lhd \alpha^{-1}A\alpha = A'$. □

By considering the above proposition, the following result is obtained.

Proposition 2.6. *Let G be a finite abelian group, and let S be a generating set of G not containing the identity 1_G. Assume S satisfies the condition $s,\ t,\ u,\ v \in S$ with*

$$st = uv \neq 1 \Rightarrow \{s,t\} = \{u,v\}. \tag{2.1}$$

Then the t-Cayley hypergraph is normal.

Let G and H be two groups. Given (g_1, h_1) and $(g_2, h_2) \in G \times H$ define the product by the rule: $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$. With this rule for multiplication, $G \times H$ becomes a group, called the *direct product* of G and H.

The direct product of two hypergraphs is as follows:

Definition 2.7. Let X_1 and X_2 be two hypergraphs. The *direct product* $X_1 \times X_2$ is defined as the graph with vertex set $V(X_1 \times X_2) = V(X_1) \times V(X_2)$ such that for any two vertices $x = (u_1, v_1)$ and $y = (u_2, v_2)$ in $V(X_1 \times X_2)$, $[x, y]$ is an edge in $X_1 \times X_2$ whenever the first element of all of the pairs is the same and the second element of all of the pairs be an edge in X_2, or the first elements of all of the pairs be an edge in X_1 and the second element of all of the pair is the same.

Two hypergraphs are called *relatively prime* if they have no nontrivial common direct factor. We omit the easy proof of the following lemma.

Lemma 2.8. *Let $G = G_1 \times G_2$ be the direct product of two finite groups G_1 and G_2, S_1 and S_2 subsets of G_1 and G_2, respectively, and $S = S_1 \cup S_2$ the disjoint union of S_1 and S_2. Let $t, t^{'}, t^{''}$ be integers where t=$max\{t^{'}, t^{''}\}$. Then*

(i) *t-$Cay(G,S) \cong t^{'}$-$Cay(G_1, S_1) \times t^{''}$-$Cay(G_2, S_2)$.*

(ii) *If t-$Cay(G,S)$ is normal, then $t^{'}$-$Cay(G_1, S_1)$ is also normal.*

(iii) *If $t^{'}$-$Cay(G_1, S_1)$ and $t^{''}$-$Cay(G_2, S_2)$ are both normal and relatively prime, then t-$Cay(G,S)$ is normal.*

Proposition 2.9. [5, Proposition 1.10] *A t-Cayley hypergraph $X = t$-$Cay(G,S)$ is connected if and only if S is a set of generators for G.*

Let $X = t$-$Cay(G,S)$ be a connected t-Cayley hypergraph of an abelian group G with respect to S, and T the subgroup generated by all non-involutions in S. Set $K = T \cap S$ and $J = S - K$ so that $T = \langle K \rangle$.

Let $Y = t\text{-}Cay(T, K)$. If J is independent, then $\langle J\rangle = \mathbb{Z}_2^J$, the direct product of J copies of $\mathbb{Z}_2$. So by Proposition 2.6, $t\text{-}Cay(\langle J\rangle, J)$ is normal for $\langle J\rangle$. From Lemma 2.8, we have the following.

Lemma 2.10. *If $T \cap \langle J\rangle = 1$ and J is independent, then $G = T \times \mathbb{Z}_2^J$ and $X = Y\times\ t\text{-}Cay(\langle J\rangle, J)$. Moreover, if Y is normal and relatively prime with K_2, then X is normal.*

Now, we are ready to give the proof of Theorem 1.1.

3. Proof of Theorem 1.1

By Proposition 2.6, we can assume that S does not satisfy the condition (2.1). If $S = \{a, a^{-1}\}$, where $o(a) = t$, then the permutation $(a, a^2, a^3, \ldots, a^{t-2})$ is not in $Aut(G, S)$ but in A_1 and so X is not normal. Now suppose that $|S| = 3$, then the following cases are considered:
i) $S = \{r, s, t\}$ where r, s, t are involutions. In this case G is an elementary abelian 2-group and r, s, t are not independent by our assumption. Then $G = \mathbb{Z}_2^2$ and $X = K_4$, so X is normal.
ii) $S = \{a, a^{-1}, r\}$ where r is an involution but a is not. Then $S^2 - 1 = \{a^2, ar, a^{-2}, a^{-1}r\}$. By our assumption, we have either $a^2 = a^{-2}$ or $r = a^3$. For the case of $a^2 = a^{-2}$, if $r \in \langle a\rangle$, then $G = \mathbb{Z}_4$, and $|A_1| = |Aut(G, S)| = 2$, where $|Aut(G, S)| = \langle (a, a^3)\rangle$. Therefore $X = 4\text{-}Cay(G, S)$ is normal. Otherwise, $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a\rangle \times \langle r\rangle$ and the permutation $(a, ar)(a^2, a^2r)(a^3, a^3r)$ is not in $Aut(G, S)$ but in A_1 and so $X = 4\text{-}Cay(G, S)$ is not normal, that is the case (2) in the theorem. For the case $r = a^3$, we have $G = \mathbb{Z}_6$ and the permutation $(a, a^2)(a^4, a^5)$ is not in $Aut(G, S)$ but in A_1 and so $X = 6\text{-}Cay(G, S)$ is not normal, that is the case (3).
Now we assume that $|S| = 4$, and the following cases are considered:
(i) $S = \{r, s, t, u\}$ where r, s, t, u are involutions. In this case, G is an elementary abelian 2-group and r, s, t, u are not independent by our assumption. So $G = \mathbb{Z}_2^3$, if $u = rs$, then $X = K_4 \times K_2$ and if $u = rst$, then $X = K_{4,4}$. When $X = K_4 \times K_2$ it is normal by Lemma 2.10. When $X = K_{4,4}$, since the permutation (rs, rt, st) is not in $Aut(G, S)$ but in A_1, and so it is not normal, that is the case (4) in the theorem.
(ii) $S = \{a, a^{-1}, r, s\}$ where r, s are involutions, but a is not. Then $S^2 - 1 = \{a^2, a^{-2}, ar, as, rs, a^{-1}r, a^{-1}s\}$. By our assumption, we only need to consider the case of $a^2 = a^{-2}$ and the case when $a^3 = r$ or $a^3 = s$. For the case of $a^2 = a^{-2}$, if $a^2 = r$ or $a^2 = s$, then $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. Let $Y = 4\text{-}Cay(\langle a\rangle, \{a, a^{-1}, a^2\})$. Then Y is not normal, and so $X = Y \times K_2$, $(K_2 = 2\text{-}Cay(\langle s\rangle, \{s\}))$ is not normal, that is the case (5) in the theorem. If $a^2 = rs$ again with the same reason

$X = 4\text{-}Cay(G, S = \{a, a^{-1}, a^2 s, s\})$ is not normal, that is the case (6). Otherwise $G = \mathbb{Z}_4 \times \mathbb{Z}_2^2$ and $S = S_1 \cup S_2 = \{a, a^{-1}, r\} \cup \{s\}$, again with the same reason and by Lemma 2.8, $X = 4\text{-}Cay(G, S)$ is not normal, that is the case (7) in the theorem.

(iii) $S = \{a, a^{-1}, b, b^{-1}\}$ where a, b are not involutions. First, we suppose $s^4 = 1$ for some $s \in S$. Without loss of generality, we can assume that $a^4 = 1$. If $\langle a \rangle \cap \langle b \rangle = 1$, then $G = \mathbb{Z}_4 \times \mathbb{Z}_m$ and by Lemma 2.8, $X = 4\text{-}Cay(\mathbb{Z}_4, \{a, a^{-1}\}) \times m\text{-}Cay(\mathbb{Z}_m, \{b, b^{-1}\})$. Since $Y = 4\text{-}Cay(\mathbb{Z}_4, \{a, a^{-1}\})$ is not normal and so by Lemma 2.8, X is not normal, that is the case (8). If $\langle a \rangle \cap \langle b \rangle = \langle a \rangle$, then $G = \mathbb{Z}_{4m}$ with $m \geq 2$. We may assume that $a = b^m$, then the permutation $b^i \to b^{m+i}$ where $1 \leq i \leq m-1$, is not in $Aut(G, S)$ but in A_1 and so $X = 4m\text{-}Cay(\mathbb{Z}_{4m}, \{b, b^{-1}, b^m, b^{-m}\})$ is not normal, that is the case (9). Consider the case when $\langle a \rangle \cap \langle b \rangle = \langle a^2 \rangle$. If G be cyclic, then we have $G = \mathbb{Z}_{4m} = \langle x \rangle$, for some odd integer $m > 2$. We may assume $a = x^m$ and $b = x^2$, if m is even, there is the permutation $\sigma : x^i \to x^{m+i}$, where $1 \leq i < 4m (i \neq m, 2m, 3m)$ and $\sigma(x^m) = x^m$, $\sigma(x^{2m}) = x^{2m}$, $\sigma(x^{3m}) = x^{3m}$. Such that σ is in A_1, but it is not in $Aut(G, S)$. Thus $X = 2m\text{-}Cay(\mathbb{Z}_{4m}, \{x^2, x^{-2}, x^m, x^{-m}\})$ is not normal. If m is odd, there is the permutation σ in A_1 such that $\sigma = \Pi_1^{4m-1}(x^i, x^{i+2})(x^{m+i}, x^{m+i-1})$, but this is not in $Aut(G, S)$. Thus $X = 2m\text{-}Cay(\mathbb{Z}_{4m}, \{x^2, x^{-2}, x^m, x^{-m}\})$ is not normal, that is the case (10). For non-cyclic G, we have $G = \langle x \rangle \times \langle y \rangle = \mathbb{Z}_{4m} \times \mathbb{Z}_2$ and $S = \{x, x^{-1}, x^m y, x^{-m} y\}$, where $m \geq 1$, $x = b$ and $y = ab^{-1}$, the permutation

$$\sigma = (x, x^2, \ldots, x^{2m-1}, x^{2m+1}, \ldots, x^{4m-1})$$
$$\times (y, yx, yx^2, \ldots, yx^{m-1}, yx^{m+1}, \ldots, yx^{3m-1}, yx^{3m+1}, \ldots, yx^{4m-1})$$

is not in $Aut(G, S)$ but in A_1 and so $X = 4m\text{-}Cay(G, S)$ is not normal, that is the case (11). We then assume that neither $a^4 = 1$, nor $b^4 = 1$. From our assumption, we have either (i) $s^2 = t^2$ for some different s, t in S or (ii) $s = t^3$ for some different s, t in S. For (i), without loss of generality we only need to consider the case when $a^2 = b^2$. In this case, $|G| = 2m$, $m \geq 3$. We have two cases: a generates G, or a does not generate G. In the second case, $o(a) = m$ and $G = \mathbb{Z}_m \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$. where $S = \{a, a^{-1}, ab, a^{-1}b\}$, if $m \geq 4$ then the permutation $(a, a^3)(a^2 b, a^4 b)$ is not in $Aut(G, S)$ but in A_1 and so $X = m\text{-}Cay(G, S)$, $(m \geq 4)$ is not normal. For $m = 4$ we have $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$ and $S = \{a, a^3, ab, a^3 b\}$. In this case, the permutation $(ab, a^3 b)(b, a^2 b)(a, a^3)$ is not in $Aut(G, S)$ but in A_1 and so $X = 4\text{-}Cay(G, S)$ is not normal, that is the case (12).

For (ii), it suffices to consider the case when $b = a^3$, then $G = \mathbb{Z}_n$ where

$n \geq 5$ and $X = n\text{-}Cay(\mathbb{Z}_n, \{a^1, a^{-1}, a^3, a^{-3}\})$. For $n \geq 5$, while $n = 6$ cannot happen, the permutation $(a, a^2, \ldots, a^{n-1})$ is not in $Aut(G, S)$ but in A_1 and so X is not normal, that is the case (13).

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References

1. Y. G. Baik, H. S. Sim, Y. Feng and M. Y. Xu, On the normality of Cayley graphs of abelian groups, *Algebra Colloq.*, **5** (1998), 297–304.
2. Y. G. Baik, Y. Feng and H. S. Sim, The normality of Cayley graphs of finite abelian groups with valency 5, *System Sci. Math. Sci.*, **13** (2000), 425–431.
3. C. Berge, *Graphs and Hypergraphs*, North-Holland, New York, (1976).
4. C. Berge, *Hypergraphs*, North-Holland, Amsterdam, (1989).
5. M. Buratti, Cayley, Marty and Schreier Hypergraphs, *Abh. Math. Sem. Univ. Hamburg*, **64** (1994), 151–162.
6. C. D. Godsil, On the full automorphism group of a graph, *Combinatorica*, **1** (1981), 243–256.
7. J. Lee and Y. S. Kwon, Cayley hypergraphs and Cayley hypermaps, *Discrete Math.*, **313** (2013), 540–549.
8. S. C. Shee, On group hypergraphs, *Southeast Asian Bull. Math.*, **14** (1990), 49–57.
9. M. Y. Xu, Automorphism groups and isomorphism of Cayley digraphs, *Discrete Math.*, **182** (1998), 309–319.
10. M. Y. Xu, Q. Zhang and J. X. Zhou, On the normality of directed Cayley graphs of abelian groups, *System Sci. Math. Sci.*, **25** (2005), 700–710.

Reza Bayat
Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
Email: r.bayat.tajvar@gmail.com

Mehdi Alaeiyan
Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran.
Email: alaeiyan@iust.ac.ir

Siamak Firouzian
Department of Mathematics, Payame Noor University, P.O. Box 19395-3697, Tehran, Iran.
Email: siamfirouzian@pnu.ac.ir

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ON THE NORMALITY OF t-CAYLEY HYPERGRAPHS OF ABELIAN GROUPS

R. BAYAT, M. ALAEIYAN AND S. FIROUZIAN

ابرگراف‌های t-کیلی نرمال از گروه‌های آبلی

رضا بیات[۱]، مهدی علائیان[۲] و سیامک فیروزیان[۳]
[۱]دانشکده ریاضی، دانشگاه پیام نور، تهران، ایران
[۲]دانشکده ریاضی، دانشگاه علم و صنعت، تهران، ایران
[۳]دانشکده ریاضی، دانشگاه پیام نور، تهران، ایران

یک ابرگراف t-کیلی $X = t - Cay(G, S)$ را نرمال گوییم، هرگاه نمایش منظم راست $R(G)$ از G، در گروه خودریختی‌های $Aut(X)$ از X، نرمال باشد. در این مقاله، شرایط نرمال بودن ابرگراف‌های t-کیلی از گروه‌های آبلی که $|S| \leq ۴$، مورد مطالعه قرار می‌گیرند.

کلمات کلیدی: ابرگراف، ابرگراف t-کیلی، ابرگراف t-کیلی نرمال.