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A GRAPH WHICH RECOGNIZES IDEMPOTENTS OF A COMMUTATIVE RING

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Abstract. In this paper we introduce and study a graph on the set of ideals of a commutative ring R. The vertices of this graph are non-trivial ideals of R and two distinct ideals I and J are adjacent if and only $IJ = I \cap J$. We obtain some properties of this graph and study its relation to the structure of R.

1. Introduction

The study of algebraic structures, using the properties of graph theory, tends to exciting research topic in the last years. There are many papers on assigning a graph to a ring. Also some graph structures on the set of ideals of a ring R are defined in the last decade. The intersection graph of a ring ([3]) is a graph whose its vertices are non trivial ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$. This graph is denoted by $\Gamma(R)$. Authors in [3], have characterized the rings R for which the graph $\Gamma(R)$ is connected and obtained several necessary and sufficient conditions on a ring R such that $\Gamma(R)$ is a complete graph. Also they determined the values of n for which the graph of $\mathbb{Z}_n$ is Eulerian and Hamiltonian. Akbari, et al. in [1] determined all rings whose clique number of the intersection graphs of ideals is finite. Also they showed that, if the clique number of $\Gamma(R)$ is finite, then its chromatic number is finite and if R is a reduced ring, then both are equal.

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Annihilating ideal graph ([2]) is a graph which its vertices are ideals with nonzero annihilators and two distinct vertices I and J are adjacent if and only if $IJ = 0$. This graph is denoted by $AG(R)$. It is shown that ([2]) if R is not a domain, then $AG(R)$ has ascending chain condition (respectively, descending chain condition) on vertices if and only if R is Noetherian (respectively, Artinian). Also the connectivity, the diameter and coloring of $AG(R)$ has studied in [2].

Comaximal ideal graph of a ring is defined in [6]. The vertices are ideals which are not contained in the Jacobson of R and two distinct vertices I and J are adjacent if and only if $I + J = R$. Also this graph has studied in [4].

Intersection graph ([3]) is the complement of a zero divisor graph of a semigroup and annihilating ideal graph and comaximal ideal graph are the zero divisor graph of some semigroups. So these graphs share many properties with zero divisor graphs. In this paper we study a new graph on the set of non-trivial ideals of a commutative ring. Also we study the relationship between the primary decomposition of ideals of a ring R and connectivity of new graph.

First recall some facts and notations related to this paper. Let $G = (V, E)$ be a graph. The order of G is the number of vertices of G. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the disjoint union of G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. A graph without edges is called an empty (null) graph. If every two distinct vertices of a graph of order n are adjacent, the graph is called a complete graph and is denoted by K_n. A clique of a graph G is a complete subgraph of G and clique number of G is the number of vertices in a maximum clique of G. For every vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For every vertex $v \in V(G)$, the degree of v is $|N(v)|$, i.e., the number of edges incident with v. Let $G_1, \ldots, G_k$ be some graphs. Then we consider $G_1 \times \cdots \times G_k$ as a graph whose vertex set is $V(G_1) \times \cdots \times V(G_k)$ and two vertices $(v_1, \ldots, v_k)$ and $(u_1, \ldots, u_k)$ are adjacent if and only if v_i and u_i are adjacent in G_i for each i. As usual we show the distance between two vertices v and w, by $d(v, w)$. The eccentricity $\epsilon(v)$ of a vertex v is the greatest distance between v and any other vertex. The diameter of a graph G is denoted by $diam(G)$ and is the maximum eccentricity of any vertex in the graph.

Throughout this paper all rings are commutative with unit element. A Von Neumann regular ring is a ring R such that for every $a \in R$ there exists an $x \in R$ such that $a = axa$. This implies that $(ax)^2 = ax$

and $\langle a \rangle = \langle ax \rangle$. So every principal ideal is generated by an idempotent element. If for any nonzero ideal J of R, we have $I \cap J \neq 0$ we say that I is a large ideal. A ring with a unique maximal ideal is called a local ring. We denote the set of all maximal ideals and all prime ideals of R by $Max(R)$ and $Spec(R)$, respectively. Also $M(R)$ denotes the set of minimal ideals of R. The intersection of all maximal ideals of R is called the Jacobson radical of R and is denoted by $J(R)$. The intersection of all prime ideals is the set of all nilpotent elements and is denoted by $Nil(R)$. It is clear that $Nil(R) \subseteq J(R)$. A discrete valuation ring (DVR) is a principal ideal domain (PID) with exactly one non-zero maximal ideal.

The radical of an ideal I is denoted by $r(I)$ and is defined as $\{r \in R : a^n \in I\}$. It is a standard fact that $r(I) = \bigcap_{I \subseteq P} P$. An ideal $Q \neq R$ is called a primary ideal if $ab \in Q$ implies $a \in Q$ or $b \in r(Q)$. It is easily seen that if Q is a primary ideal then $r(Q)$ is a prime ideal. Also if $r(Q)$ is a maximal ideal then Q is a primary ideal. We say that an ideal I has a primary decomposition if $I = \bigcap_{i=1}^{n} Q_i$ where Q_i are primary ideals. The set $\{r(Q_i)\}$ is denoted by $Ass(I)$ and is called the set of associated prime ideals of I. For two ideals I and J of R, we denote the set $\{r \in R : rJ \subseteq I\}$ by $(I : J)$. Also, we denote the finite field with q elements by $\mathbb{F}_q$.

In the next section we introduce a new graph on the set of ideals and study its properties. In Section 3, we study the relationship between the primary decomposition of ideals of a ring and the connectivity of the new graph.

2. Introduction to a new graph

In this section, we introduce a new graph on the set of ideals of a commutative ring R which we denote it by $\Gamma_0(R)$ and study its properties.

Definition 2.1. Let R be a commutative ring. The vertices of the graph $\Gamma_0(R)$ are non-trivial ideals of R and two distinct ideals I and J are adjacent if and only if $IJ = I \cap J$.

We also need the following definition in some cases:

Definition 2.2. Let R be a commutative ring. The vertices of the graph $\Gamma_1(R)$ are all ideals of R and two ideals I and J are adjacent if and only if $IJ = I \cap J$.

By this definition, there is a loop in the vertex I of $\Gamma_1(R)$ if and only if $I^2 = I$. Also $\{0\}$ and R adjacent to all vertices in $\Gamma_1(R)$. We denote the degrees of a vertex I in $\Gamma_0(R)$ and $\Gamma_1(R)$, by $deg_0(I)$ and $deg_1(I)$, respectively. Any loop is counted by multiplicity one in this definition.

Remark 2.3. (i) *The graph $\Gamma_0(R)$ is a null graph i.e., the ring R has only two ideals if and only if R is a field.*
(ii) *The graph $\Gamma_0(R)$ has only one vertex i.e., the ring R has only three ideals if and only if R is a local ring with a principal maximal ideal $m = Ra$ such that $a^2 = 0$. In this case m is also a minimal ideal.*

Note that the graph $\Gamma_0(R)$ contains comaximal graph and the complement of intersection graph. Also if R is a reduced ring, it contains the annihilating ideal graph. To investigate some properties of $\Gamma_0(R)$, first we state and prove the following lemma.

Lemma 2.4. *Let I and J be two ideals of a ring R.*
(i) *If $I + J = R$, then $IJ = I \cap J$.*
(ii) *If $I \cap J = 0$, then $IJ = 0$.*
(iii) *If R is a reduced ring, then $IJ = 0$ implies $I \cap J = 0 = IJ$.*
(iv) *If $IJ = J$, then $IJ = I \cap J$.*

Proof. (i) It is obvious that $IJ \subseteq I \cap J$. If $I + J = R$, then there are $i \in I$ and $j \in J$ such that $i + j = 1$. If $t \in I \cap J$, then $t = t(i + j) = ti + tj \in IJ$. So $IJ = I \cap J$.
(ii) It follows from $IJ \subseteq I \cap J$.
(iii) It is easy to see that $(I \cap J)^2 \subseteq IJ$. So $(I \cap J)^2 = 0$. Since R is a reduced ring, therefore $I \cap J = 0 = IJ$.
(iv) If $IJ = J$ then $J = IJ \subseteq I$. So $IJ = J = I \cap J$. □

By Lemma 2.4 and the definition of $\Gamma_0(R)$ we have the following corollary.

Corollary 2.5. (i) *The set of all maximal ideals of R, $Max(R)$ is a clique in $\Gamma_0(R)$.*
(ii) *The set of all minimal ideals of R, $M(R)$ is a clique in $\Gamma_0(R)$.*

We need the following well-known theorem.

Theorem 2.6. *(Nakayama's Lemma) Let M be a finitely generated R-module and I be an ideal of R. If $IM = M$, then $ann(M) \bigcap (1+I) \neq \emptyset$.*

A set $S \subseteq V$ is an independent set of a graph G, if no two vertices of S are adjacent. The following corollary which is an immediate consequence of Nakayama's Lemma, is useful for determining of independent sets of $\Gamma_0(R)$ (Corollary 2.8).

Corollary 2.7. *Let I and J be two ideals of R such that J is a finitely generated ideal and $IJ = J$.*

(i) *If* $I \subseteq J(R)$*, then* $J = 0$*.*
(ii) *If* $ann(J) = 0$*, i.e.,* J *contains a non zero divisor, then* $I = R$*.*

Corollary 2.8. (i) *If* $\{I_\alpha\}$ *is a chain of finitely generated proper ideals in* $J(R)$*, then* $\{I_\alpha\}$ *is an independent set of* $\Gamma_0(R)$*.*
(ii) *If* $\{I_\alpha\}$ *is a chain of finitely generated proper ideals in an integral domain* R *then* $\{I_\alpha\}$ *is an independent set of* $\Gamma_0(R)$*.*

Proof. (i) If $I_\alpha \subsetneqq I_\beta$ and $I_\alpha I_\beta = I_\alpha \bigcap I_\beta$ then $I_\alpha = I_\alpha I_\beta$ which is a contradiction by part (i) of Corollary 2.7 (Nakayama's Lemma).
(ii) If $I_\alpha \subsetneqq I_\beta$ and $I_\alpha I_\beta = I_\alpha \bigcap I_\beta$ then $I_\alpha = I_\alpha I_\beta$ which is a contradiction by part (ii) of Corollary 2.7. □

The following lemma is well known and let us give a proof for it.

Lemma 2.9. *Let* I *be a finitely generated idempotent ideal of a ring* R*. Then* $I = Re$ *is generated by an idempotent element.*

Proof. Since $I = I^2$, so $ann(I) \bigcap (1 - I) \neq \emptyset$ by Theorem 2.6. Hence, there is $s = 1 - e \in ann(I)$ such that $sI = 0$. This implies that $I = Ie$ and $(1 - e)e = 0$. Therefore $e = e^2$ and $I = Re$. □

Now we state and prove the following theorem, which is one of the main result of this section:

Theorem 2.10. *Let* R *be a ring such that* $\Gamma_0(R)$ *has order at least two. The vertex* I *is adjacent to any other vertices if and only if for every* $a \in I$*,* $a \in Ia$ *and* $I = I^2$*.*

Proof. First assume that vertex I is adjacent to any other vertices. We consider two cases:

Case 1) $I \bigcap Ann(I) = 0$. We have $0 \neq I^2$. If $Ra \subsetneqq I$, then $Ra = I \bigcap Ra = Ia \subseteq I^2$. So $a \in Ia$. If I is not a principal ideal, then $I = I^2$. So assume $I = Rb$ is a principal ideal. If $0 \neq I^2 \subsetneqq I$, then $I^3 = I^2$. Hence by Lemma 2.9, we have $I^2 = Re$. This implies that $I(1 - e) \subseteq Ann(I) \bigcap I = 0$. Thus $I = Ie \subseteq I^2$ and so $I = I^2$. Therefore $b \in I^2 = Ib$.

Case 2) $I \bigcap Ann(I) \neq 0$. Since $0 = IAnn(I) \neq I \bigcap Ann(I)$, so $I = Ann(I)$. If $0 \neq J \subsetneqq I$, then $0 = IJ = I \bigcap J = J$ which is a contradiction. So I is a minimal ideal. Hence $Ann(I)$ is a maximal ideal and so I is both a maximal and minimal ideal of R. If R has another maximal ideal $m \neq I$, then $0 = IAnn(I) = I^2 \subseteq m$. So $I \subseteq m$ which is a contradiction. Since every ideal of R sits in a maximal ideal, so I is the the only nontrivial ideal of R which is a contradiction.

Conversely, assume that for each $a \in I$, $a \in Ia$. If $a \in I \bigcap J$, then $a \in Ia \subseteq IJ$. So $a \in IJ$ and therefore $IJ = I \bigcap J$. □

Remark 2.11. *Theorem 2.10 states that a vertex I is adjacent to all other vertices in $\Gamma_0(R)$ if and only if the vertex I is adjacent to all principal ideals contained in I in graph $\Gamma_1(R)$. Also in this case we have a loop in the graph $\Gamma_1(R)$*

Theorem 2.12. *Let I be a finitely generated ideal of R. Then I is adjacent to all other vertices if and only if $I = Re$ is generated by an idempotent*

Proof. Let J be an ideal of R and $t \in J \cap Re$, so $t = re$ and $t = te \in Je$. Thus I is adjacent to all other vertices. Conversely, If I is adjacent to all other vertices then by Theorem 2.10, $I = I^2$. So the proof is complete by Lemma 2.9. □

Corollary 2.13. *If R has a non trivial idempotent, then $\Gamma_0(R)$ is a connected graph and $diam(\Gamma_0(R)) \leq 2$.*

The following theorem gives the structure of Γ_0-graph of Von Neumann regular ring.

Theorem 2.14. *Assume that $\Gamma_0(R)$ has at least two vertices. The ring R is a Von Neumann regular ring if and only if $\Gamma_0(R)(\Gamma_1(R))$ is a complete graph.*

Proof. If every two vertices are adjacent, then every principal ideal is generated by an idempotent by Theorem 2.12. So $\langle a \rangle = \langle e \rangle$ where e is an idempotent element and so $a = re$. Thus $ae = re^2 = re = a$ and $a = ae^2 = eae$. So R is a Von Neumann regular ring. Conversely, assume that R is a Von Neumann regular ring. If $t \in I \bigcap J$ then, $Rt = Re$ for some idempotent e. Therefore $t = te \in It \subseteq IJ$ and we have the result. □

The following theorem gives an upper bound for the diameter of $\Gamma_0(R)$ while the Jacobson radical of the ring is zero.

Theorem 2.15. *If $J(R) = 0$, then $diam(\Gamma_0(R)) \leq 2$.*

Proof. Let I, J be two distinct ideal of R. If $I \bigcap J = 0$ then I and J are adjacent by Part (ii) of Lemma 2.4. So assume that $I \bigcap I \neq 0$. Since $J(R) = 0$, there is a maximal ideal m such that $I \bigcap J \nsubseteq m$. This implies that $I + m = R = J + m$. So m is adjacent to both ideals I and J by part (i) of Lemma 2.4. □

As an application of Theorem 2.15, consider the ring of real continuous functions on a topological space X, i.e., $C(X)$. Since the ideals $M_{x_0} = \{f \in C(X) : f(x_0) = 0\}$ are maximal ideals of $C(X)$, so $J(C(X)) = 0$ and $diam(\Gamma_0(C(X))) \leq 2$.

The following theorem is stated in [5, p. 15] as an exercise.

Theorem 2.16. *Let R be a commutative ring and $f(x) = a_n x^n + \cdots + a_0 \in R[x]$.*

(i) *$f(x)$ is a nilpotent element of $R[x]$ if and only if a_i is a nilpotent element of R for each i.*
(ii) *$f(x)$ is an invertible element of $R[x]$ if and only if a_0 is an invertible element and a_i is nilpotent for each $i \geq 1$.*
(iii) *$J(R[x]) = Nil(R[x]) = Nil(R)[x]$.*

Now we state and prove the following corollary:

Corollary 2.17. *Let R be a reduced ring. Then $diam(R[x]) \leq 2$.*

Proof. Since $Nil(R) = 0$, so $J(R[x]) = Nil(R[x]) = Nil(R)[x] = 0$. So we have the result by Theorem 2.15. □

The following result state a necessary condition for an ideal to be an isolated vertex of $\Gamma_0(R)$.

Theorem 2.18. *Let R be a ring. If I is an isolated vertex of $\Gamma_0(R)$, then $I \subseteq J(R)$ and I is a large ideal.*

Proof. Let m be a maximal ideal of R. If $I \nsubseteq m$, then $I + m = R$. Hence $Im = I \cap m$ which is a contradiction. So $I \subseteq J(R)$. If $I \cap J = 0$, then I is adjacent to J which is a contradiction. So I is a large ideal and hence $J(R)$ is a large ideal. □

The following corollary is an immediate consequence of Theorem 2.18.

Corollary 2.19. *If $\Gamma_0(R)$ is the empty graph, then R is a local ring and every ideal of R is large.*

Theorem 2.20. *If (R, m) is a Noetherian local ring, then m is an isolated vertex of $\Gamma_0(R)$.*

Proof. If $I \subseteq m$, then by Nakayama's Lemma $Im \subsetneqq I = I \cap m$. Hence m is an isolated vertex. □

Corollary 2.21. *Let R be an Artinian ring such that $\Gamma_0(R)$ has at least two vertices. Then $\Gamma_0(R)$ is a connected graph if and only if R is not a local ring.*

Proof. First assume that $\Gamma_0(R)$ is a connected graph. Since every Artinian ring is Noetherian, so the proof is complete by Theorem 2.20. Conversely, assume that R is not a local ring. So $R \cong R_1 \times \cdots \times R_k$ where $k \geq 2$ by [5, Theorem 8.7]. Hence R has a non trivial idempotent. Thus $\Gamma_0(R)$ is a connected graph by Corollary 2.13. □

Now, we consider a specific ring and investigate the structure of its Γ_0-graph.

Example 2.22. Let $R = \frac{\mathbb{F}_q[X,Y]}{\langle X,Y\rangle^2}$. It is clear that R is an Artinian local ring with $m = \langle x, y\rangle$ as maximal ideal. Since $m^2 = 0$, so every non maximal ideal of R correspond to a one dimensional vector subspace of two dimensional vector space m. So R has $q+1 = \frac{q^2-1}{q-1}$ ideal of dimension one and a maximal ideal of dimension two. Also one dimensional ideals are minimal ideals. Since for every two distinct minimal ideals I and J we have $I \bigcap J = 0$, so every two distinct minimal ideals I and J are adjacent in $\Gamma_0(R)$. So $\Gamma_0(R)$ is union of a complete graph K_{q+1} and K_1, i.e., $\Gamma_0\big(\frac{\mathbb{F}_q[X,Y]}{\langle X,Y\rangle^2}\big) = K_{q+1} \cup K_1$.

The following theorem describes the Γ_0-graph for discrete valuation ring.

Theorem 2.23. *If R is a discrete valuation ring(DVR), then $\Gamma_0(R)$ is the empty graph.*

Proof. Suppose that R is a discrete valuation ring(DVR). It is well known that R is a local ring such that its maximal ideal m is principal and only ideals of R are m^i. If m^i and $m^j (j > i)$ are adjacent then $m^i m^j = m^i \cap m^j = m^j$. So Nakayama's Lemma implies that $m^j = 0$ which is a contradiction. Hence $\Gamma_0(R)$ is the empty graph. $\square$

To study the structure of $\Gamma_0(\mathbb{Z}_n)$, we need the following result.

Theorem 2.24. *Let $R \cong R_1 \times \cdots \times R_k$. Then $\Gamma_1(R) \cong \Gamma_1(R_1) \times \cdots \times \Gamma_1(R_k)$.*

Proof. It is well known that every ideal I of R is equal to $I_1 \times \cdots \times I_k$ for some ideals I_i of R_i. Also the vertices $I = I_1 \times \cdots \times I_k$ and $J = J_1 \times \cdots \times J_k$ are adjacent if and only if $I_i J_i = I_i \bigcap J_i$ for each i. So the proof is complete. $\square$

Remark 2.25. *If $R = \mathbb{Z}_{p_1^{\gamma_1}} \times \cdots \times \mathbb{Z}_{p_k^{\gamma_k}}$ (p_i's are not necessarily distinct), then ideals of R are $\langle (p_1^{\alpha_1}, \ldots, p_k^{\alpha_k})\rangle$ where $0 \le \alpha_i \le \gamma_i$. Also two ideals $I = \langle (p_1^{\alpha_1}, \ldots, p_k^{\alpha_k})\rangle$ and $J = \langle (p_1^{\beta_1}, \ldots, p_k^{\beta_k})\rangle$ are adjacent if and only if $max\{\alpha_i, \beta_i\} = min\{\alpha_i + \beta_i, \gamma_i\}$. So we can construct $\Gamma_1(R)$ as follows:*

The vertex set is $\{(\alpha_1, \ldots, \alpha_k) : 0 \le \alpha_i \le \gamma_i\}$ and two vertices $(\alpha_1, \ldots, \alpha_k)$ and $(\beta_1, \ldots, \beta_k)$ are adjacent if and only if $max\{\alpha_i, \beta_i\} = min\{\alpha_i + \beta_i, \gamma_i\}$.

Example 2.26. Consider $R = \mathbb{Z}_{20}$. We have $\mathbb{Z}_{20} = \mathbb{Z}_{2^2} \times \mathbb{Z}_5$. We shall draw $\Gamma_0(\mathbb{Z}_{20})$. The vertex set of this graph is $V(\Gamma_0(\mathbb{Z}_{20})) = \{(1,0), (2,0), (0,1), (1,1)\}$. By Remark 2.25 we have the Figure 1 for this graph:

The following theorem gives the degree of the vertices of $\Gamma_0(\mathbb{Z}_n)$.

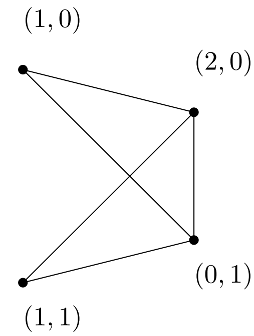

FIGURE 1. The graph $\Gamma_0(\mathbb{Z}_{20})$.

Theorem 2.27. *Let $n = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$ and $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $0 \leq \alpha_i \leq \gamma_i$ be a divisor of n. Set $R = \mathbb{Z}_n$ and $I = \langle a \rangle$. Suppose that $A(I) = \{i : 1 \leq i \leq k, 0 \leq \alpha_i \leq \gamma_i\}$. The degree of ideal I in $\Gamma_0(R)$ is*

$$deg_0(I) = \Big(2^{|A(I)|} \prod_{i \notin A(I)} (\gamma_i + 1)\Big) - 2 - \Big\lfloor \frac{1}{|A(I)| + 1} \Big\rfloor.$$

Proof. According to Remark 2.25, we do computations with $(\alpha_1, \dots, \alpha_k)$. Let $B = N_{\Gamma_1}(I)$ be the open neighborhood of the vertex I. So $N_{\Gamma_0}(I) = B \setminus \{1, I, n\}$. Assume $b = (\beta_1, \cdots, \beta_k) \in B$. Hence $max\{\alpha_i, \beta_i\} = min\{\alpha_i + \beta_i, \gamma_i\}$. This implies that if $i \notin A(I)$, then we have the result. If $i \in A(I)$ then $\beta_i = 0$ or $\beta_i = \gamma_i$. So in the first case β_i can be any number of the set $\{0, \cdots, \gamma_i\}$. In the last case $\beta_i = 0$ or $\beta_i = \gamma_i$. So we can choose b in $2^{|A(I)|} \prod_{i \notin A(I)} (\gamma_i + 1)$ ways. Two of these b correspond to $(0, \dots, 0)$ and $(\gamma_1, \dots, \gamma_k)$. If $I \in B$ then $min\{2\alpha_i, \gamma_i\} = \alpha_i$. So $\alpha_i = 0$ or $\alpha_i = \gamma_i$. Thus $A(I) = \emptyset$. Conversely, if $A(I) = \emptyset$ then for each i, $min\{2\alpha_i, \gamma_i\} = \alpha_i$. Hence $I \in B$. So we must exclude I from B in this case. So $deg_0(I) = \Big(2^{|A(I)|} \prod_{i \notin A(I)} (\gamma_i + 1)\Big) - 2 - \lfloor \frac{1}{|A(I)|+1} \rfloor$. $\square$

Example 2.28. Let $n = 36 = 2^2 3^2$. So $\mathbb{Z}_{36} \cong \mathbb{Z}_4 \times \mathbb{Z}_9$ and $\gamma_1 = \gamma_2 = 2$. If $I = 6 = 2 \times 3$ then $A(6) = \{1, 2\}$ and $|A(6)| = 2$. So $deg_0(6) = 2^2 - 2 = 2$. The neighbors of the vertex 6 are 4 and 9. If $I = 4$ then $A(4) = \emptyset$ and $|A(4)| = 0$. So $deg_0(4) = 3 \times 3 - 2 - 1 = 6$. Note that $\langle 4 \rangle = \langle 16 \rangle$ i.e. $\langle 4 \rangle$ is an idempotent ideal. So $N_{\Gamma_0}(4) = \{2, 3, 6, 9, 12, 18\}$. We have shown $\Gamma_0(\mathbb{Z}_{36})$ in Figure 2.

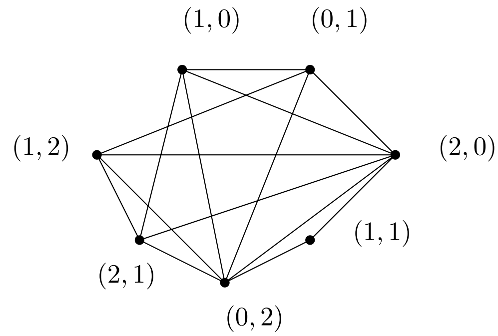

FIGURE 2. The graph $\Gamma_0(\mathbb{Z}_{36})$.

3. PRIMARY DECOMPOSITION OF IDEALS OF R AND CONNECTIVITY OF $\Gamma_0(R)$

In this section we shall study the relationship between the primary decomposition of ideals of a ring R and the connectivity of $\Gamma_0(R)$. We begin with the following result.

Lemma 3.1. *(Prime avoidance lemma[5]) Suppose that $I \subseteq \cup_{i=1}^{n} P_i$, where P_i's are prime ideals. Then $I \subseteq P_i$ for some $1 \leq i \leq n$.*

Theorem 3.2. *Let I be an ideal of the ring R. Then $I \bigcap Ra = Ia$ if and only if $(I : a) = I + ann(a)$. In particular, if $(I : a) = I$ then $I \bigcap Ra = Ia$.*

Proof. Assume that $x = ra \in Ra \bigcap I$. So $r \in (I : a)$. Hence $r = i + s$ where $i \in I$ and $s \in ann(a)$. Thus $x = ra = ia + sa = ia \in Ia$. Conversely, assume that $r \in (I : a)$. So $x = ra \in I \bigcap Ra = Ia$. hence there is an $i \in I$ such that $ra = ia$. This implies that $r - i \in ann(a)$. So $r = i + (r - i) \in I + ann(a)$. $\square$

Theorem 3.3. (i) *Let Q be a primary ideal and $a \notin r(Q) = P$. Then $(Q : a) = Q$.*
(ii) *Suppose that I has a primary decomposition. If $a \notin \bigcup_{P \in Ass(I)} P$, then $(I : a) = I$.*

Proof. (i) Let $b \in (Q : a)$. So $ba \in Q$. If $b \notin Q$ then $a \in r(Q)$ which is a contradiction.
(ii) Let $I = \bigcap_{i=1}^{n} Q_i$. Then $(I : a) = (\bigcap_{i=1}^{n} Q_i : a) = \bigcap_{i=1}^{n}(Q_i : a) = \bigcap_{i=1}^{n} Q_i = I$. $\square$

Since every prime ideal is a primary ideal, we have the following corollary.

Corollary 3.4. *Let P be a prime ideal of a ring R. If $a \notin P$, then $Ra \bigcap P = Pa = PRa$, i.e., P and Ra are adjacent.*

Corollary 3.5. *Suppose that R is not an integral domain. If $ab \notin Nil(R)$ then the distance between Ra and Rb is not more than two, i.e, $d(Ra, Rb) \leq 2$.*

Proof. Since $Nil(R)$ is the intersection of all prime ideals, so there is a prime ideal P such that $ab \notin P$. Hence $a, b \notin P$. So Ra and Rb are adjacent to P by Corollary 3.4. ☐

Theorem 3.6. *Suppose that two ideals I and J have primary decomposition. If $Max(R) \nsubseteq (Ass(I) \bigcup Ass(J))$, then $d(I, J) \leq 2$.*

Proof. Let $m \in Max(R) \backslash (Ass(I) \bigcup Ass(J))$. So there is an element $a \in m \backslash \bigcup_{P \in Ass(I) \bigcup Ass(J)} P$ by prime avoidance lemma. So I and J are adjacent to Ra by Theorem 3.3(ii) and the proof is complete. ☐

Corollary 3.7.
 (i) *Let $P, Q \in Spec(R)$ be two prime ideals. If $Max(R) \nsubseteq \{P, Q\}$ then $d(P, Q) \leq 2$.*
 (ii) *Suppose that R has at least three maximal ideal. Then for every two prime ideals P and Q, $d(P, Q) \leq 2$.*

Theorem 3.8. *Let R be a Noetherian ring with infinitely many maximal ideals. Then $diam(\Gamma_0(R)) \leq 2$.*

Proof. Let I and J be two ideals of R. It is well known that every ideal in a Noetherian ring has a primary decomposition ([5]). So the proof is complete by Theorem 3.6. ☐

The following example shows that the condition $|Max(R)| = \infty$ is necessary in the Theorem 3.8.

Example 3.9. Let $p_1, \ldots, p_k$ be distinct prime numbers. Set $S = \mathbb{Z} \backslash \bigcup_{i=1}^{k} p_i\mathbb{Z}$ and $R = S^{-1}\mathbb{Z}$. Then R is a PID with k distinct prime $p_1, \ldots, p_k$. If $I = \langle a \rangle$ and $J = \langle b \rangle$ are adjacent then $\langle a, b \rangle = R$. So if $I \subseteq J(R) = p_1 \cdots p_k R$ then I is an isolated vertex.

Here, we state and prove the following lemma to obtain the diameter of Γ_0-graph of the polynomial ring.

Lemma 3.10. *Let R be a commutative ring. The polynomial ring $R[x]$ has infinitely many maximal ideals.*

Proof. Let M be a maximal ideal of R and $F = \frac{R}{M}$ be its residue field. Since $F[x] \cong \frac{R[x]}{M[x]}$, it suffices to prove the lemma for $F[x]$. Every maximal ideal of $F[x]$ is generated by an irreducible polynomial, because,

$F[x]$ is a PID. If F is an infinite field, the set $\{\langle x-a\rangle : a \in F\}$ is an infinite set of maximal ideals. Now suppose that F is a finite field, then by a well-known result, for every $n \in \mathbb{N}$, there is an irreducible polynomial of degree n. Therefore we have the result. $\square$

By Theorem 3.8 and Lemma 3.10, we have the following corollary.

Corollary 3.11. *If R is a Noetherian ring, then diam$(\Gamma_0(R[x])) \leq 2$.*

By checking the Γ_0-graph for well-known rings, we think that the diameter of every connected component of $\Gamma_0(R)$ is not more than two. So, we end the paper by the following conjecture:

Conjecture 3.12. The diameter of every connected component of $\Gamma_0(R)$ is not more than two.

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A GRAPH WHICH RECOGNIZES IDEMPOTENTS OF A COMMUTATIVE RINGS

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گرافی که عنصر خودتوان یک حلقه جابه‌جایی را مشخص می‌کند

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در این مقاله گرافی را روی مجموعه ایده‌آل‌های حلقه جابه‌جایی R معرفی نموده و آن‌را مطالعه خواهیم کرد. رئوس این گراف، ایده‌آل‌های غیربدیهی R بوده و دو ایده‌آل I و J در آن مجاورند اگر و فقط اگر $IJ = I \cap J$. برخی از خواص این گراف را به‌دست آورده و ارتباطش با ساختار حلقه R را مطالعه خواهیم کرد.

کلمات کلیدی: گراف، قطر، حلقه، خودتوان.