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ADMITTING CENTER MAPS ON MULTIPLICATIVE METRIC SPACE

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Abstract. In this work, we investigate admitting center map on multiplicative metric space and establish some fixed point theorems for such maps. We modify the Banach contraction principle and the Caristi's fixed point theorem for M-contraction admitting center maps and we prove some useful theorems. Our results on multiplicative metric space improve and modify some existing fixed point theorems in the literature.

1. Introduction and Preliminaries

Let $(X, \|.\|)$ be a Banach space and C be a subset of X. The map $T : C \to X$ is admitting center map, if there exists $y_0 \in X$ such that for each $x \in C$, we have

$$\|Tx - y_0\| \leq \|x - y_0\|.$$

The point $y_0 \in X$ is said to be a center of T.

The fixed point results for admitting center maps are very useful in the system of equations. In 1972, Michael Grossman and Robert Katz [14] established a new calculus called non-Newtonian calculus also termed as multiplicative calculus. Florack and Van Assen [13] obtained the idea of multiplicative calculus in biomedical image analysis.

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Bashirov et al. [7] showed the efficiency of multiplicative calculus over the Newtonian calculus. They established that multiplicative calculus is more effective than Newtonian calculus for modeling various problems. By defining multiplicative distance, they provide the base of multiplicative metric space. Özavsar and Cevikel [16] introduced the notion of multiplicative contraction mapping. They proved the well known Banach contraction principle for such contraction in multiplicative metric space. The Banach contraction principle has result in nonlinear analysis. Generalization of the Banach contraction principle is one of the important branch of research. The Banach theorem has many generalizations (see [8, 9, 11, 12]). Rome and Sarwar [18] established several generalizations of the Banach contraction principle and proved Cantor intersection theorem in multiplicative metric space. For various definitions of multiplicative calculus we refer the reader to [1, 6, 7, 14, 15, 16, 17, 19]. The main ideas of this work is depended on the references [4, 8, 16].

In this work, we prove some fixed point results in multiplicative metric space. We introduce the concept of admitting center map on multiplicative metric space and establish some new fixed point theorems for such maps. Our results improve and modify some existing fixed point results in the literature.

Definition 1.1. [6] Let X be a non-empty set. A map $d : X \times X \to [1, \infty)$ is said to be a multiplicative metric on X if the following conditions are satisfied:

(1) $d(x, y) \geq 1$, for all $x, y \in X$,
(2) $d(x, y) = 1$ if and only if $x = y$,
(3) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(4) $d(x, z) \leq d(x, y).d(y, z)$, for all $x, y, z \in X$.

The pair (X, d) is called *multiplicative metric space*.

In multiplicative metric space, always $d(x, y) \geq 1$ and triangle inequality is obtained by product instead of adding, indeed, the roles of subtraction and addition move to division and multiplication. In multiplicative metric space some proofs become easier, for example see [6, Section 4]. In [10], Cevik et al. proved completion theorem for multiplicative metric space. In [2], M. Abbas et al. proved the fixed point result for mappings satisfying rational contractive condition in the setup of multiplicative metric space. In [16], Özavsar and Cevikel considered some topological properties of multiplicative metric space and introduced multiplicative contraction map and obtained some fixed point results.

Example 1.2. [6] The mapping $d : (0, \infty) \times (0, \infty) \to [1, \infty)$ defined by $d(x, y) = |\frac{x}{y}|^*$ where

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1, \end{cases}$$

is a multiplicative metric on $\mathbb{R}^+$.

In the following, we recall the concepts of multiplicative Cauchy sequence, multiplicative convergent sequence and complete multiplicative space.

Definition 1.3. [1, 15, 16] A sequence (x_n) in a multiplicative metric space (X, d) is said to be a multiplicative Cauchy sequence if for all $\epsilon > 1$, there exits a positive integer n_0 such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.

A sequence (x_n) in X is multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$ [16]. A sequence (x_n) in X multiplicative converges to $x \in X$, if for all $\epsilon > 1$ there exists a positive integer n_0 such that $d(x_n, x) < \epsilon$, for all $n \geq n_0$.

Definition 1.4. [1, 15, 16] A multiplicative metric space (X, d) is said to be complete if any multiplicative Cauchy sequence in X converges to a point of X.

Definition 1.5. Let (X, d) be a multiplicative metric space. A map $f : X \to X$ is called multiplicative Lipschitz if there exists a real constant $\lambda > 0$ such that $d(f(x_1), f(x_2)) \leq d(x_1, x_2)^\lambda$, for all $x_1, x_2 \in X$.

A multiplicative Lipschitz map f is said to be multiplicative contraction if $\lambda < 1$. A map f is called multiplicative nonexpansive if $\lambda = 1$. A map f is said to be multiplicative contractive, if for all $x_1, x_2 \in X$, with $x_1 \neq x_2$, we have $d(f(x_1), f(x_2)) < d(x_1, x_2)$ [1, 15, 16].

2. Main Results

In this section, we modify the Banach contraction principle and the Caristi's fixed point theorem for M-contraction admitting center maps and we prove some useful theorems.

We extend the definition of multiplicative Lipschitz for admitting center maps.

Definition 2.1. Let (X, d) be a multiplicative metric space and $C \subseteq X$. A map $f : C \to X$ is called *multiplicative Lipschitz admitting center* (abbreviated as *M-Lipschitz admitting center*) if there exist $y_0 \in X$ and $\lambda > 0$ such that for all $x \in C$ that $x \neq f(x)$, we have $d(f(x), y_0) \leq$

$d(x, y_0)^\lambda$ and $y_0 \in X$ is said to be a multiplicative center (abbreviated as M-center) of f. A multiplicative Lipschitz admitting center map f is said to be M-contraction admitting center if $0 < \lambda < 1$. The map f is called M-nonexpansive admitting center if $\lambda = 1$. A map f is said to be multiplicative contractive admitting center (abbreviated as M-contractive admitting center) if for all $x \in C$ that $x \neq f(x)$, we have $d(f(x), y_0) < d(x, y_0)$.

2.1. **Modified of the Banach Contraction Principle.** The Banach contraction principle is one of the earliest and most important results in fixed point theory [5]. A large number of authors have improved, generalized and extended this classical result in nonlinear analysis.

In this section, we set $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$ and we prove the Banach contraction principle for the class of all M-contraction admitting center maps. We recall that $T : X \to X$ is a contraction map, if for all $x, y \in X$, we have

$$d(Tx, Ty) \leq kd(x, y), \text{ where } 0 < k < 1. \tag{2.1}$$

According to the Banach contraction principle, any map T satisfying (2.1) has a unique fixed point in complete metric space X.

Let (X, d) be a multiplicative metric space, $x \in X$ and $\epsilon > 1$. As defined in [16], a set

$$B_\epsilon(x) = \{y \in X | d(x, y) < \epsilon\},$$

is called multiplicative open ball of radius ϵ with center x.

Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for any $\epsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, then f is called M-continuous at $x \in X$ [16].

In the next theorem, we modify the Banach contraction principle [3, Theorem 4.1.5, p.178] for M-contraction admitting center maps in complete multiplicative metric space..

Theorem 2.2. *Let (X, d) be a complete multiplicative metric space and let $f : X \to X$ be M-continuous, M-contraction admitting center with M-Lipschitz constant L. Let $y_0 \in X$ be the M-center of f. Then f has a fixed point $u \in X$ and for some $x \in X$, we have $\lim_{n\to\infty} f^n(x) = u$, with*

$$d(f^n(x), u) \leq d(y_0, f(x))^{L^{n-1}}.$$

Proof. Let $x \in X$. We first show that $(f^n(x))$ is a Cauchy sequence. For $n \in \mathbb{N}$ and $y_0 \in X$, we have

$$d(f^n(x), y_0) \leq d(f^{n-1}(x), y_0)^L \leq \cdots \leq d(f(x), y_0)^{L^{n-1}}.$$

Thus for $m > n$ where $n \in \mathbb{N}$,

$$\begin{aligned} d(f^n(x), f^m(x)) &\leq d(f^n(x), y_0)d(y_0, f^m(x)) \\ &\leq d(f(x), y_0)^{L^{n-1}} d(f(x), y_0)^{L^{m-1}} \\ &\leq d(f(x), y_0)^{L^{n-1}+L^{m-1}}. \end{aligned}$$

Since f is M-contraction, it follows that $0 < L < 1$. Hence for $m > n$, as $m, n \to \infty$, we have

$$d(f^n(x), f^m(x)) \leq d(f(x), y_0)^{L^{n-1}+L^{m-1}} \to 1. \tag{2.2}$$

This shows that $(f^n(x))$ is a Cauchy sequence in X. As X is complete there exists $u \in X$ with $\lim_{n\to\infty} f^n(x) = u$. As f is M-continuous, we have $u = \lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f(f^n(x)) = f(u)$. So u is a fixed point of f. Now, letting $m \to \infty$ in (2.2) implies $d(f^n(x), u) \leq d(y_0, f(x))^{L^{n-1}}$. □

2.2. **Modified of the Caristi's theorem.** Let X be a multiplicative space and $f : X \to (-\infty, \infty]$ be a function. Then f is said to be proper if there exists $x \in X$ such that $f(x) < \infty$.

Let X be a topological space and $f : X \to (-\infty, \infty]$ be a proper function. We recall that f is said to be lower semicontinuous (l.s.c.) at $x_0 \in X$ if for $x_n \to x_0$, we have $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$, where (x_n) is a sequence in X tends to x_0. Also, f is said to be lower semicontinuous on X if it is lower semicontinuous at each point of X. The Caristi's theorem [8] is a modification of the ϵ-variational principle of Ekeland [12]. We want to modify the Caristi's fixed point theorem in complete multiplicative metric space. To this end, we need the following proposition.

In the next proposition, we modify [3, Proposition 4.1.1] and [8] in complete multiplicative metric space.

Proposition 2.3. *Let (X, d) be a complete multiplicative metric space and $\varphi : X \to [1, \infty]$ be a lower semicontinuous function. Suppose that (x_n) is a sequence in X and $y_0 \in X$ such that*

$$d(x_n, y_0) \leq \frac{\varphi(x_{n-1})}{\varphi(x_n)}, \qquad \text{for all } n \in \mathbb{N}.$$

Then the sequence (x_n) is M-converge to a point $v \in X$ and $d(x_n, v) \leq \left[\frac{\varphi(x_{n-1})}{\varphi(v)}\right]^2$, moreover $\varphi(v) \leq \varphi(x_n)$, for all $n \in \mathbb{N}_0$.

Proof. By assumptions, we have $1 \leq d(x_n, y_0) \leq \frac{\varphi(x_{n-1})}{\varphi(x_n)}$, it follows that $(\varphi(x_n))$ is a decreasing sequence. Moreover, for any $m \in \mathbb{N}$, we have

$$\begin{aligned}\prod_{n=1}^{m} d(x_n, x_{n+1}) &= d(x_1, x_2)d(x_2, x_3)\cdots d(x_m, x_{m+1})\\ &\leq d(x_1, y_0)d(y_0, x_2)d(x_2, y_0)d(y_0, x_3)\cdots d(x_m, y_0)d(y_0, x_{m+1})\\ &= d(x_1, y_0)d(y_0, x_2)^2 d(y_0, x_3)^2\cdots d(x_m, y_0)^2 d(y_0, x_{m+1})\\ &\leq \frac{\varphi(x_0)}{\varphi(x_1)}\left[\frac{\varphi(x_1)}{\varphi(x_2)}\right]^2\cdots\left[\frac{\varphi(x_{m-1})}{\varphi(x_m)}\right]^2\frac{\varphi(x_m)}{\varphi(x_{m+1})}\\ &= \frac{\varphi(x_0)\varphi(x_1)}{\varphi(x_m)\varphi(x_{m+1})}\\ &\leq \frac{\varphi(x_0)\varphi(x_1)}{\varphi(x_{m+1})^2}\\ &\leq \frac{\varphi(x_0)\varphi(x_1)}{(\inf_{n\in\mathbb{N}_0}\varphi(x_n))^2}.\end{aligned}$$

Letting $m \to \infty$, we have

$$\prod_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty.$$

Since all numbers in this multiplication are greater than or equal one, they are convergent to one, this implies that (x_n) is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $\lim_{n\to\infty} x_n = v$. Let $m, n \in \mathbb{N}$ with $m > n$. Then

$$\begin{aligned}d(x_n, x_m) &\leq d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})\ldots d(x_{m-1}, x_m)\\ &\leq d(x_n, y_0)d(y_0, x_{n+1})d(x_{n+1}, y_0)d(y_0, x_{n+2})\\ &\quad\ldots d(x_{m-1}, y_0)d(y_0, x_m)\\ &= d(x_n, y_0)d(y_0, x_{n+1})^2 d(y_0, x_{n+2})^2\ldots d(x_{m-1}, y_0)^2 d(y_0, x_m)\\ &\leq \frac{\varphi(x_{n-1})}{\varphi(x_n)}\left[\frac{\varphi(x_n)}{\varphi(x_{n+1})}\right]^2\cdots\left[\frac{\varphi(x_{m-2})}{\varphi(x_{m-1})}\right]^2\left[\frac{\varphi(x_{m-1})}{\varphi(x_m)}\right]\\ &= \frac{\varphi(x_{n-1})\varphi(x_n)}{\varphi(x_{m-1})\varphi(x_m)}\\ &\leq \left[\frac{\varphi(x_{n-1})}{\varphi(x_m)}\right]^2.\end{aligned}$$

Let m tends to infinity, we have

$$d(x_n, v) \le \left[\frac{\varphi(x_{n-1})}{\varphi(v)}\right]^2, \text{ for all } n \in \mathbb{N}.$$

□

In the next theorem, we modify [3, Theorem 4.1.2] and [8] in complete multiplicative metric space. We apply Proposition 2.3 in the next theorem.

Theorem 2.4. *Let (X, d) be a complete multiplicative metric space and $\varphi : X \to [1, \infty]$ be a proper and lower semicontinuous function. Let $y_0 \in X$ such that for any $u \in X$ that $\inf_{x\in X} \varphi(x) < \varphi(u)$, there is $v \in X$ such that*

$$y_0 \neq v \text{ and } d(y_0, v) \le \frac{\varphi(u)}{\varphi(v)}.$$

Then there is an $x_0 \in X$, with $\varphi(x_0) = \inf_{x\in X} \varphi(x)$.

Proof. By assumption there is $u_0 \in X$ with $\varphi(u_0) < \infty$. If $\inf_{x\in X} \varphi(x) = \varphi(u_0)$, then we are done. So we suppose contrary to our claim, that $\inf_{x\in X} \varphi(x) < \varphi(y)$, for any $y \in X$. So $\inf_{x\in X} \varphi(x) < \varphi(u_0)$ and by assumption there is $u_1 \in X$ such that $u_1 \neq y_0$ and $d(y_0, u_1) \le \frac{\varphi(u_0)}{\varphi(u_1)}$. Inductively, we define a sequence (u_n) in X, starting with u_0. Suppose that we choose $u_{n-1} \in X$. Put

$$E_n := \left\{ t \in X \ : \ d(y_0, t) \le \frac{\varphi(u_{n-1})}{\varphi(t)} \right\}.$$

As $\inf_{x\in X} \varphi(x) < \varphi(u_{n-1})$, then there is $t_0 \in X$ such that $t_0 \neq y_0$ and

$$1 < d(y_0, t_0) \le \frac{\varphi(u_{n-1})}{\varphi(t_0)},$$

so $t_0 \in E_n$ and $E_n \neq \emptyset$. Also, we have

$$\frac{\varphi(u_{n-1})}{\varphi(t_0)} \le \frac{\varphi(u_{n-1})}{\inf_{t\in E_n} \varphi(t)}.$$

Therefore, $1 < \frac{\varphi(u_{n-1})}{\inf_{t\in E_n} \varphi(t)}$. Hence, there is $u_n \in E_n$ such that

$$\varphi(u_n) < \inf_{t\in E_n} \varphi(t) \left(\frac{\varphi(u_{n-1})}{\inf_{t\in E_n} \varphi(t)}\right)^{\frac{1}{2}} = \left[\inf_{t\in E_n} \varphi(t)\varphi(u_{n-1})\right]^{\frac{1}{2}}. \qquad (2.3)$$

Since $u_n \in E_n$, so we have

$$d(y_0, u_n) \leq \frac{\varphi(u_{n-1})}{\varphi(u_n)}.$$

Proposition 2.3 implies that $u_n \to v \in X$, and $d(u_n, v) \leq [\frac{\varphi(u_{n-1})}{\varphi(v)}]^2$. By assumption, since $\inf_{x \in X} \varphi(x) < \varphi(v)$, there is a $z \in X$ such that $z \neq y_0$ and $1 < d(y_0, z) \leq \frac{\varphi(v)}{\varphi(z)}$. So we have

$$\begin{aligned} \varphi(z) &\leq \frac{\varphi(v)}{d(y_0, z)} \\ &\leq \frac{\varphi(v)}{d(y_0, z)} . \frac{\varphi(u_{n-1})}{\varphi(v)} \\ &= \frac{\varphi(u_{n-1})}{d(y_0, z)}. \end{aligned}$$

This follows that $z \in E_n$. The inequality (2.3) implies that

$$\frac{\varphi(u_n)^2}{\varphi(u_{n-1})} \leq \inf_{t \in E_n} \varphi(t) \leq \varphi(z).$$

Thus $\varphi(z) < \varphi(v) \leq \lim_{n\to\infty} \varphi(u_n) \leq \varphi(z)$, which is a contradiction. Thus, there is $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$. □

In [8], Caristi proved the following theorem.

Theorem 2.5. *Let (X, d) be complete metric space and $\varphi : X \to (-\infty, \infty]$ be proper bounded below and lower semicontinuous function. Let $T : X \to X$ be a map such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \text{ for all } x \in X,$$

then there exists a point $v \in X$ such that $v = Tv$ and $\varphi(v) < \infty$.

Now we are in a position to obtain a modified version of the Caristi's fixed point theorem in multiplicative metric space.

Theorem 2.6. (***Modified of the Caristi's fixed point theorem***) *Let (X, d) be complete multiplicative metric space and $\varphi : X \to [1, \infty]$ be a proper and lower semicontinuous function. If $T : X \to X$ is a map and $y_0 \in X$ such that for all $x \in X$*

$$x \neq Tx \Longrightarrow 1 < d(y_0, Tx) \leq \frac{\varphi(x)}{\varphi(Tx)}, \tag{2.4}$$

then there exists $v \in X$ such that $v = Tv$.

Proof. On the contrary, suppose that $v \neq Tv$ for all $v \in X$. Since φ is proper, there exists $u \in X$ such that $\varphi(u) < \infty$. Let

$$K := \left\{ x \in X \;:\; d(y_0, x) \leq \frac{\varphi(u)}{\varphi(x)} \right\}.$$

Then by hypothesis $Tu \in K$ and so K is a non-empty closed subset of X. We show that K is invariant under T. For any $x \in K$, we have $x \neq Tx$ and

$$1 \leq d(y_0, x) \leq \frac{\varphi(u)}{\varphi(x)},$$

and hence (2.4) implies

$$\begin{aligned} \varphi(Tx) &\leq \frac{\varphi(x)}{d(y_0, Tx)} \\ &\leq \frac{\varphi(x)}{d(y_0, Tx)} . \frac{\varphi(u)}{\varphi(x)} \\ &= \frac{\varphi(u)}{d(y_0, Tx)}, \end{aligned}$$

which follows that $Tx \in K$. Hence for any $x \in K$ we have

(i) $\inf\limits_{y \in K} \varphi(y) \leq \varphi(Tx) < \varphi(x)$,

(ii) there exists $w \in K$ such that

$$w \neq y_0 \text{ and } d(y_0, w) \leq \frac{\varphi(x)}{\varphi(w)}.$$

Then by Theorem 2.4, there exists an $x_0 \in K$ with $\varphi(x_0) = \inf\limits_{x \in K} \varphi(x)$. Which contradicts (i). This proves the existence of a fixed point for T. □

In the following, we modify the Banach contraction principle ([16, Theorem 3.2] and [3, Theorem 4.1.5]) in multiplicative metric space.

Theorem 2.7. (***Modified of the Banach contraction principle***) *Let (X, d) be complete multiplicative metric space, K be a closed subset of X and $T : K \to K$ be continuous M-contraction admitting center map at $y_0 \in X$. Then for arbitrary $x_0 \in K$, the Picard iteration process defined by*

$$x_{n+1} = Tx_n, \quad \text{for } n \in \mathbb{N}_0,$$

converges to a fixed point $v \in K$ of T.

Proof. Let λ be an M-contraction constant with $0 < \lambda < 1$. Define $\varphi : K \to [1, \infty)$ by $\varphi(x) = d(y_0, Tx)^{\frac{1}{1-\lambda}}$, for $x \in K$. Hence φ is a continuous function. Let $x_0 \in K$. We define a sequence (x_n) in K by

$$x_n = T^n x_0, \quad \text{for } n \in \mathbb{N}.$$

If there exists $n \in \mathbb{N}_0$, such that $x_{n+1} = x_n$, then x_n is a fixed point of T. Therefore the result trivially holds. So we can assume that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}_0$. As T is an M-contraction admitting center map, we have

$$d(y_0, x_{n+1}) \leq d(y_0, x_n)^{\lambda}, \text{ for all } n \in \mathbb{N},$$

which implies that

$$\frac{d(y_0, x_n)}{d(y_0, x_n)^{\lambda}} \leq \frac{d(y_0, x_n)}{d(y_0, x_{n+1})}.$$

Hence

$$d(y_0, x_n) \leq \left[\frac{d(y_0, x_n)}{d(y_0, x_{n+1})}\right]^{\frac{1}{1-\lambda}} = \frac{\varphi(x_{n-1})}{\varphi(x_n)}.$$

Now, Proposition 2.3 implies that there exists $v \in K$ such that

$$\lim_{n\to\infty} x_n = v.$$

Since T is continuous and $x_{n+1} = Tx_n$, it follows that $v = Tv$. □

In the next theorem, we obtain a modified of the Boyd and Wong's fixed point theorem [3, Theorem 4.1.12] in multiplicative metric space.

Theorem 2.8. (***Modified of the Boyd and Wong's fixed point theorem***) *Let (X, d) be complete multiplicative metric space, $y_0 \in X$ and $T : X \to X$ be a continuous map that satisfies*

$$d(Tx, y_0) \leq \psi(d(x, y_0)), \text{ for all } x \in X, \ x \neq y_0. \tag{2.5}$$

where $\psi : [1, +\infty) \to [1, +\infty)$ is the upper semicontinuous function from the right (i.e., $\lambda_i \downarrow \lambda \geq 1 \Rightarrow \limsup_{i\to\infty} \psi(\lambda_i) \leq \psi(\lambda)$) such that $\psi(t) < t$ for each $t > 1$. Then T has a unique fixed point $v \in X$. Moreover, for each $x \in X$, $\lim_{n\to\infty} T^n x = v$.

Proof. Fix $x, y_0 \in X$ and define a sequence (x_n) in X by $x_n = T^n x$, $n \in \mathbb{N}_0$. Set $d_n := d(x_n, y_0)$. We divide the proof into three parts:

Part 1. $\lim_{n\to\infty} d_n = 1$.

Note that

$$d_{n+1} = d(x_{n+1}, y_0) = d(Tx_n, y_0) \leq \psi(d_n) < d_n, \quad n \in \mathbb{N}_0.$$

Hence (d_n) is a decreasing sequence and bounded below. Hence $\lim_{n\to\infty} d_n$ exists. Let $\lim_{n\to\infty} d_n = \delta \geq 1$. Assume that $\delta > 1$. By the right continuity of ψ,

$$\delta = \lim_{n\to\infty} d_{n+1} \leq \limsup_{n\to\infty} \psi(d_n) \leq \psi(\delta) < \delta,$$

which is impossible, so $\delta = 1$.

Part 2. The sequence (x_n) is a Cauchy sequence.
Assume that (x_n) is not Cauchy. Then there exists $\epsilon > 1$ such that for any $k \in \mathbb{N}$, there are integers $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$d(x_{n_k}, x_{m_k}) \geq \epsilon.$$

Now letting $k \to \infty$, Part 1 implies that

$$\epsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, y_0)d(y_0, x_{m_k}) = d_{n_k} d_{m_k} \to 1,$$

so we have $\epsilon \leq 1$, which contradicts $\epsilon > 1$. Therefore (x_n) is a Cauchy sequence in X.

Part 3. The existence and uniqueness of fixed point.
As (x_n) is a Cauchy sequence and X is complete, then there is $v \in X$, such that $\lim_{n\to\infty} x_n = v$. By continuity of T and $x_n = T^n x$, we have $v = Tv$. For the uniqueness of fixed point, we show that $v = y_0$. Suppose that v is a fixed point of T and $v \neq y_0$. Then by (2.5) and assumption, we have

$$\begin{aligned} d(v, y_0) &= d(Tv, y_0) \\ &\leq \psi(d(v, y_0)) \\ &< d(v, y_0). \end{aligned}$$

That is a contradiction. □

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ADMITTING CENTER MAPS ON MULTIPLICATIVE METRIC SPACE

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نگاشت های مرکزپذیر روی فضای متریک ضربی

محمدحسین لباف قاسمی، نها افتخاری و علی بیاتی اشکفتکی
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در این مقاله، نگاشت مرکزپذیر روی فضای متریک ضربی را بررسی کرده و برخی قضایای نقطه ثابت را برای چنین نگاشت‌ها ثابت می‌کنیم. اصل انقباض باناخ و قضیه نقطه ثابت کریستی را برای نگاشت‌های مرکزپذیر M− انقباض بیان و ثابت می کنیم. نتایج به دست آمده در این مقاله، در فضای متریک ضربی، برخی قضایای نقطه ثابت معمولی را ارتقا می‌بخشد.

کلمات کلیدی: نگاشت مرکزپذیر، فضای متریک ضربی، نگاشت مرکزپذیر انقباض−M