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WOVEN FRAMES IN TENSOR PRODUCT OF HILBERT SPACES

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Abstract. The tensor product is the fundamental ingredient for extending one-dimensional techniques of filtering and compression in signal preprocessing to higher dimensions. Woven frames play a crucial role in signal preprocessing and distributed data processing. Motivated by these facts, we have investigated the tensor product of woven frames and presented some of their properties. Besides, we have studied some effects of operators on woven frames in the tensor products of Hilbert spaces.

1. Introduction

Packets of data in communication networks are sequences of information bits, surrounded by timing, addressing and error control, which assure that the packet has been delivered to a receiver without any error from a source. The packet will not be delivered if it contains errors. In this case, lost packets would be retransmitted, which would take longer than the main transfer. In many applications, retransmission of lost packets is not feasible and the potential for the large delay is unacceptable [4].

If a lost packet was independent of the other transmitted data, the information would be entirely lost to the receiver. However, if there were dependencies between transmitted packets, one could have partial or completed recovery despite losses. This leads to the use of *frames* for encoding. Frame theory has significant applications in different areas

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like *filter bank theory* [9], *sampling theory* [13], *wireless communication*, *internet coding* [10], *digital signal* and *image processing* [11] and more.

Signal processing methods are defined as changes of the bases for vectors or functions of one variable and therefore cannot be directly applied to higher dimensional data like images. The tensor product can be used to generalize the filtering and compression techniques from audio to images [2]. Furthermore, the concept of *woven frames* in Hilbert spaces has been introduced by Bemrose et al. [1] in one of the applications of frames in signal processing and wireless sensor networks, which requires distributed processing under different frames.

Two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for a Hilbert space $\mathbf{H}$ are *woven* if there are constants $0 < A \leq B < \infty$ so that for every subset σ of countable index set I, the family $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathbf{H}$ with frame bounds A and B. The *woven frames* have potential applications in the preprocessing of signals using Gabor frames. Casazza and Lynch in [5], have reviewed the fundamental properties of weaving frames. They have shown the connection between frames and projections and considered a *weaving equivalent* of an unconditional basis for weaving Riesz bases. In [4], the concept of weaving Hilbert space frames extended to the Banach spaces and *woven Schauder frames* have been introduced and studied. Deepshikha and Vashisht have studied *continuous woven frames* for Hilbert spaces regarding the measure space in [12].

This paper focuses on the study of the properties of the tensor product of woven frames, which has been organized as follows: In Section 2, we have given basic definitions of frames, the tensor products, and woven frames. In Section 3, we have provided some structural results of woven frames on the tensor product of two Hilbert spaces and given some examples.

2. Preliminaries

We begin with a brief overview of the basic definitions related to the frames, the tensor product of frames and woven frames. We refer to [6, 7, 8] for more information. Throughout the paper, I and J can represent a finite or countably infinite index set and $\mathbf{H}$, $\mathbf{K}$ will denote either finite or infinite dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$, respectively.

2.1. ***Frames.*** A *frame* is an overcomplete family of vectors with specific properties permitting it to act approximately as a basis.

Definition 2.1. A family of vectors $\Phi = \{\varphi_i\}_{i\in I}$ in $\mathbf{H}$ is said to be a frame if there are constants $0 < A \leq B < \infty$ so that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, \varphi_i\rangle|^2 \leq B\|f\|^2, \qquad f \in \mathbf{H}.$$

The numbers A and B are called *lower* and *upper* frame bounds respectively. A frame is called *tight* if $A = B$, when $A = B = 1$ it is a *Parseval frame*. If only right hand side inequality is assumed, then it is called a *B-Bessel sequence*. The values $\{\langle f, \varphi_i\rangle\}_{i\in I}$ are called the *frame coefficients* of the vector $f \in \mathbf{H}$ with respect to the frame Φ.

2.2. ***Woven Frames.*** In *distributed signal processing* when two sets $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ of linear measurements with steady recovery are given, each set is a frame tagged by a node $i \in I$. A signal F at each node measures either with φ_i or ψ_i, so the gathered information is the set of numbers $\{\langle F, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle F, \psi_i\rangle\}_{i\in\sigma^c}$ for some subset $\sigma \subset I$. This leads to the definition of a new concept in the frame theory called " *woven frames*". Many useful and interesting results of woven frames are obtained in the literature; we refer to [1, 3, 5, 12] for more information in this subject.

Definition 2.2. Two frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ for a Hilbert space $\mathbf{H}$ are called *woven* if there exist constants $0 < A \leq B < \infty$ so that for any partition $\sigma \subset I$ the *weaving* $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame with bounds A, B. Furthermore, they are called *weakly woven* if $\{\varphi_i\}_{i\in\sigma} \cup \{\psi_i\}_{i\in\sigma^c}$ is a frame for $\mathbf{H}$.

Bemrose et al. [1] have proven that weakly woven frames and woven frames are essentially equivalent. The next example introduces two woven frames for Hilbert space $\mathbb{R}^2$. Weaving frame bounds can be obtained easily.

Example 2.3. Let $\{e_i\}_{i=1^2}$ be an orthonormal basis of $\mathbb{R}^2$ and let Φ and Ψ be the sets

$$\Phi = \{\varphi_i\}_{i=1}^{3} = \{2e_2, 3e_1, 2e_1 + 3e_2\}$$

and

$$\Psi = \{\psi_i\}_{i=1}^{3} = \{e_1, e_2, 3e_1 + e_2\}.$$

Then, Φ is a frame for Euclidean space $\mathbb{R}^2$ with lower and upper (not necessarily optimal) bounds 4 and 22 and Ψ is a frame with bounds 1 and 19.

The frames Φ and Ψ constitute woven frames. For example, if we assume that $\sigma_1 = \{1, 2\}$ then for any f

$$4\|f\|^2 \leq \sum_{i\in\sigma_1} |\langle f, \varphi_i\rangle|^2 + \sum_{i\in\sigma_1^c} |\langle f, \psi_i\rangle|^2 \leq 27\|f\|^2,$$

so $\{\varphi_i\}_{i\in\sigma_1}\bigcup\{\psi_i\}_{i\in\sigma_1^c}$ is a frame with lower and upper bounds $A_1=4$ and $B_1=27$, respectively. Now, if we take

$$A=max\{A_i;1\leq i\leq 8\}$$

and

$$B=min\{B_i;1\leq i\leq 8\},$$

then Φ and Ψ are woven frames with universal bounds A and B, respectively.

Woven frames are preserved under a bounded invertible operator [1].

Proposition 2.4. *Assume that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven frames for $\boldsymbol{H}$ with frame bounds A, B and F is an invertible operator on $\boldsymbol{H}$. Then $\{F\varphi_i\}_{i\in I}$ and $\{F\psi_i\}_{i\in I}$ are also woven with bounds $A\,\|F^{-1}\|^{-2}$ and $B\,\|F\|^2$.*

2.3. ***Tensor Product of Frames.*** The tensor product has been highly regarded in recent decades. For instance, it suggests a natural language for expressing the algorithms of digital signal processing based on matrix factorization. Some tensor product properties, which are required in studying this section, are presented below [7, 8].

Definition 2.5. Let $\mathbf{H}$ and $\mathbf{K}$ be complex separable Hilbert spaces. An operator T from $\mathbf{K}$ to $\mathbf{H}$ is said to be antilinear if

$$T(af+bg)=\overline{a}T(f)+\overline{b}T(g),\qquad f,g\in\mathbf{K},$$

for all complex numbers a,b where $\overline{a},\overline{b}$ are the complex conjugate of a and b respectively. The adjoint of a bounded antilinear map T is an antilinear map defined by

$$\langle T^*f,g\rangle=\langle Tg,f\rangle,\qquad f\in\mathbf{H}\text{ and }g\in\mathbf{K}.$$

Definition 2.6. The tensor product of Hilbert spaces $\mathbf{H}$ and $\mathbf{K}$ is the set $\mathbf{H}\otimes\mathbf{K}$ of all bounded antilinear maps $T:K\to H$ such that $\sum_j\|Tu_j\|^2<\infty$ for every orthonormal basis $\{u_j\}_{j\in J}$ of $\mathbf{K}$.

For every $T\in\mathbf{H}\otimes\mathbf{K}$, the norm of T is defined by

$$|||T|||^2=\textstyle\sum_j\|Tu_j\|^2.$$

Moreover, by the Parseval identity

$$\sum_{i\in I}\|\,T^*e_i\,\|^2=\sum_{j\in J}\|\,Tu_j\,\|^2,$$

$$|||T|||=|||T^*|||.$$

The space $\mathbf{H}\otimes\mathbf{K}$ is a Hilbert space with the norm $|||\cdot|||$ and associated inner product

$$\langle Q,T\rangle=\sum_j\langle Qu_j,Tu_j\rangle.\tag{2.1}$$

Recall that for given $x \in \mathbf{H}$ and $y \in \mathbf{K}$ the antilinear map $x \otimes y$ from $\mathbf{K}$ into $\mathbf{H}$ is defined by

$$(x \otimes y)(y') = \langle y, y' \rangle x, \qquad y' \in \mathbf{K}, \tag{2.2}$$

therefore by Parseval identity and (2.1)

$$|||x \otimes y||| = \|x\|\|y\|, \tag{2.3}$$

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle, \qquad x' \in \mathbf{H}, y' \in \mathbf{K}. \tag{2.4}$$

In [8] it has been proved that if $\{x_n\}_{n\in I}$ and $\{y_m\}_{m\in J}$ are two frames for Hilbert spaces $\mathbf{H}$, $\mathbf{K}$ respectively, then $\{x_n \otimes y_m\}_{n\in I, m\in J}$ is a frame for $\mathbf{H} \otimes \mathbf{K}$. Also, if $\{T_n\}_{n\in I}$ is a frame for $\mathbf{H} \otimes \mathbf{K}$ then for each $x_0 \in \mathbf{H}$ and $y_0 \in \mathbf{K}$, $\{T_n y_0\}_{n\in I}$ and $\{T_n^* x_0\}_{n\in I}$ are respectively frames for $\mathbf{H}$, $\mathbf{K}$.

3. Tensor product of woven frames

Extending and improving the notion of woven frames in the tensor product of Hilbert spaces are discussed in this section. We have generalized some results of woven frames in the tensor product of Hilbert spaces and examined the effects of operators on them.

In the next theorem, we show that the tensor product of woven frames is a woven frame for tensor product space.

Theorem 3.1. *Suppose $\{x_n\}_{n\in I}$ and $\{x'_n\}_{n\in I}$ are woven frames for $\boldsymbol{H}$ with universal frame bounds A, B and $\{y_m\}_{m\in J}$ and $\{y'_m\}_{m\in J}$ are woven frames for $\boldsymbol{K}$ with universal frame bounds C, D.*
Then $\{x_n \otimes y_m\}_{(n,m)\in I\times J}$ and $\{x'_n \otimes y'_m\}_{(n,m)\in I\times J}$ are woven for $\boldsymbol{H} \otimes \boldsymbol{K}$ with universal frame bounds AC, BD.

Proof. By definition of woven frames, for any partition σ_1 of I, $\{x_n\}_{n\in\sigma_1} \cup \{x'_n\}_{n\in\sigma_1^c}$ is a frame with bounds A, B and for every partition σ_2 of J, $\{y_m\}_{m\in\sigma_2} \cup \{y'_m\}_{m\in\sigma_2^c}$ is a frame with bounds C, D.
Let $T \in \mathbf{H} \otimes \mathbf{K}$, similar to the proof in [8] we have

$$\langle T, x_n \otimes y_m \rangle = \langle T y_m, x_n \rangle. \tag{3.1}$$

Now let $\sigma \subset I \times J$ be any partition of $I \times J$ then

$$\sum_{(n,m)\in\sigma} |\langle T, x_n \otimes y_m \rangle|^2 + \sum_{(n,m)\in\sigma^c} |\langle T, x'_n \otimes y'_m \rangle|^2$$

$$= \sum_{(n,m)\in\sigma} |\langle T y_m, x_n \rangle|^2 + \sum_{(n,m)\in\sigma^c} |\langle T y'_m, x'_n \rangle|^2,$$

let I, J be partitioned by σ_1 and σ_2 respectively. Since $\{x_n\}_{n\in I}$ and $\{x'_n\}_{n\in I}$ are woven frames, therefore

$$\begin{aligned}
&\sum_{(n,m)\in\sigma} |\langle Ty_m, x_n\rangle|^2 + \sum_{(n,m)\in\sigma^c} |\langle Ty_m^{'}, x_n^{'}\rangle|^2 \\
&= \left(\sum_{m\in\sigma_2}\sum_{n\in\sigma_1} |\langle Ty_m, x_n\rangle|^2 + \sum_{m\in\sigma_2}\sum_{n\in\sigma_1^c} |\langle Ty_m, x_n^{'}\rangle|^2\right) \\
&+ \left(\sum_{m\in\sigma_2^c}\sum_{n\in\sigma_1} |\langle Ty_m^{'}, x_n\rangle|^2 + \sum_{m\in\sigma_2^c}\sum_{n\in\sigma_1^c} |\langle Ty_m^{'}, x_n^{'}\rangle|^2\right) \\
&= \sum_{m\in\sigma_2}\left(\sum_{n\in\sigma_1} |\langle Ty_m, x_n\rangle|^2 + \sum_{n\in\sigma_1^c} |\langle Ty_m, x_n^{'}\rangle|^2\right) \\
&+ \sum_{m\in\sigma_2^c}\left(\sum_{n\in\sigma_1} |\langle Ty_m^{'}, x_n\rangle|^2 + \sum_{n\in\sigma_1^c} |\langle Ty_m^{'}, x_n^{'}\rangle|^2\right) \\
&\leq B\left(\sum_{m\in\sigma_2} ||Ty_m||^2 + \sum_{m\in\sigma_2^c} ||Ty_m^{'}||^2\right).
\end{aligned}$$

Moreover, since $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathbf{H}$, then by the Parseval identity

$$\begin{aligned}
&B\left(\sum_{m\in\sigma_2} \|Ty_m\|^2 + \sum_{m\in\sigma_2^c} \|Ty_m^{'}\|^2\right) \\
&= B\left(\sum_{m\in\sigma_2}\sum_{i\in I} |\langle Ty_m, e_i\rangle|^2 + \sum_{m\in\sigma_2^c}\sum_{i\in I} |\langle Ty_m^{'}, e_i\rangle|^2\right) \\
&= B\sum_{i\in I}\left(\sum_{m\in\sigma_2} |\langle T^*e_i, y_m\rangle|^2 + \sum_{m\in\sigma_2^c} |\langle T^*e_i, y_m^{'}\rangle|^2\right),
\end{aligned}$$

frames $\{y_m\}_{m\in J}$ and $\{y_m^{'}\}_{m\in J}$ are woven hence by definition of the adjoint of a map and Parseval identity, we conclude

$$B\sum_{i\in I}\left(\sum_{m\in\sigma_2}|\langle T^*e_i,y_m\rangle|^2+\sum_{m\in\sigma_2^c}|\langle T^*e_i,y_m^{'}\rangle|^2\right)$$
$$\leq BD\sum_{i\in I}||T^*e_i||^2 = BD\sum_{j\in J}||Tu_j||^2 = BD\,|||\ T\ |||^2\ .$$

Similarly,

$$\sum_{(n,m)\in\sigma}|\langle T,x_n\otimes y_m\rangle|^2+\sum_{(n,m)\in\sigma^c}|\langle T,x_n^{'}\otimes y_m^{'}\rangle|^2\geq AC\,|||\ T\ |||^2\ .$$

□

An example of the tensor product of woven frames is given below.

Example 3.2. Let $\{e_i\}_{i=1}^3$ be an orthonormal basis for $\mathbb{R}^3$. Two frames

$$\{x_n\}_{n=1}^6=\{e_1,\frac{e_2}{\sqrt{2}},\frac{e_2}{\sqrt{2}},\frac{e_3}{\sqrt{3}},\frac{e_3}{\sqrt{3}},\frac{e_3}{\sqrt{3}}\}$$

and

$$\{x_n^{'}\}_{n=1}^6=\{e_1,e_1,e_2,e_2,e_3,e_3\}$$

are woven for $\mathbb{R}^3$ with universal lower and upper weaving bounds A=1, $B=3$.

A simple calculation shows that two frames $\{y_m\}_{m=1}^2=\{e_1,e_2\}$ and $\{y_m^{'}\}_{m=1}^2=\{e_1+e_2,2e_1+e_2\}$ are woven frames for $\mathbb{R}^2$ with universal lower and upper bounds $C=1$, $D=6$.
According to the proof of the Theorem 3.1, we can conclude that two frames $\{x_n\otimes y_m\}_{n=1,m=1}^{6,\ 2}$ and $\{x_n^{'}\otimes y_m^{'}\}_{n=1,m=1}^{6,\ 2}$ are woven frames with universal lower and upper frame bounds $AC=1$, $BD=18$ for $\mathbb{R}^6$.

Corollary 3.3. *Assume that $\{x_n\}_{n\in I}$ and $\{x_n^{'}\}_{n\in I}$ are two woven frames for $\boldsymbol{H}$ with universal frame bounds A, B and let $\{y_m\}_{m\in J}$ and $\{y_m^{'}\}_{m\in J}$ be woven frames for $\boldsymbol{K}$ with universal frame bounds C, D. If F, G are invertible bounded operators on $\boldsymbol{H}$ and $\boldsymbol{K}$ respectively, then $\{Fx_n\otimes Gy_m\}_{(n,m)\in I\times J}$ and $\{Fx_n^{'}\otimes Gy_m^{'}\}_{(n,m)\in I\times J}$ are woven with bounds$AC\|(GF)^{-1}\|^2$ and $BD\|GF\|^2$.*

Proof. Let U be an operator in $\mathbf{H}\otimes\mathbf{K}$, applying an invertible operator to woven frames leaves them woven, also using (3.1) we have

$$\langle U,Fx_n\otimes Gy_m\rangle=\langle UGy_m,Fx_n\rangle.$$

As stated in the proof of Theorem 3.1, we have

$$\sum_{(n,m)\in\sigma}|\langle U,Fx_n\otimes Gy_m\rangle|^2+\sum_{(n,m)\in\sigma^c}|\langle U,Fx_n^{'}\otimes Gy_m^{'}\rangle|^2\leq(BD\|GF\|^2)|\|U\||^2,$$

and
$$\sum_{(n,m)\in\sigma} |\langle U, Fx_n \otimes Gy_m\rangle|^2 + \sum_{(n,m)\in\sigma^c} |\langle U, Fx'_n \otimes Gy'_m\rangle|^2 \geq (AC\|(GF)^{-1}\|^2)\||U\||^2,$$
hence the lower and upper weaving bounds are $AC\|(GF)^{-1}\|^2$ and $BD\|GF\|^2$. □

Following [5, Proposition 13], we know that every frame is woven with a copy of itself. Due to this we have the following result.

Corollary 3.4. *Let $\{x_n\}_{n\in I}$, $\{y_n\}_{n\in I}$ be woven frames for $\boldsymbol{H}$ and $\{z_m\}_{m\in J}$ be a frame for $\boldsymbol{K}$. Then $\{x_n \otimes z_m\}_{n\in I, m\in J}$ and $\{y_n \otimes z_m\}_{n\in I, m\in J}$ are woven frames for $\boldsymbol{H} \otimes \boldsymbol{K}$.*

Proof. Let $\{z_m\}_{m\in J}$ be a frame with bounds C, D and $\{x_n\}_{n\in I}$, $\{y_n\}_{n\in I}$ be woven frames with universal bounds A, B. Since a frame is always woven with a copy of itself then $\{z_m\}_{m\in J}$ is woven with itself. Theorem 3.1 implies that $\{x_n \otimes z_m\}_{n\in I, m\in J}$ and $\{y_n \otimes z_m\}_{n\in I, m\in J}$ are woven with universal frame bounds AC, BD. □

The following proposition shows that by having woven frames in $\mathbf{H} \otimes \mathbf{K}$, woven frames in $\mathbf{H}$ and $\mathbf{K}$ can be obtained.

Proposition 3.5. *Let $\{F_n\}_{n\in I}$ and $\{G_n\}_{n\in I}$ be two woven frames for $\boldsymbol{H} \otimes \boldsymbol{K}$. For each nonzero $x_0 \in \boldsymbol{H}$ and $y_0 \in \boldsymbol{K}$,*
(i) the sequences $\{F_n y_0\}_{n\in I}$ and $\{G_n y_0\}_{n\in I}$ are woven frames for $\boldsymbol{K}$ with bounds $B\|y_0\|^2$, $A\|y_0\|^2$.
(ii) the sequences $\{F_n^* x_0\}_{n\in I}$ and $\{G_n^* x_0\}_{n\in I}$ are woven frames for $\boldsymbol{H}$ with frame bounds $B\|x_0\|^2$, $A\|x_0\|^2$.

Proof. Let $\sigma \subset I$ be any partition of I and $y_0 \in \mathbf{K}$. The sequences $\{F_n\}_{n\in I}$ and $\{G_n\}_{n\in I}$ are woven frames, then there exist constants A, B such that $\{F_n\}_{n\in\sigma} \cup \{G_n\}_{n\in\sigma^c}$ is a frame for $\mathbf{H} \otimes \mathbf{K}$ with bounds A and B hence

$$A\||x \otimes y_0|\|^2 \leq \sum_{n\in\sigma} |\langle x \otimes y_0, F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle x \otimes y_0, G_n\rangle|^2 \leq B\||x \otimes y_0|\|^2.$$
By using equalities (2.3) and (3.1), we conclude that
$$A\|y_0\|^2\|x\|^2 \leq \sum_{n\in\sigma} |\langle x, F_n y_0\rangle|^2 + \sum_{n\in\sigma^c} |\langle x, G_n y_0\rangle|^2 \leq B\|y_0\|^2\|x\|^2.$$
Therefore, $\{F_n y_0\}$ and $\{G_n y_0\}$ are woven frames with universal frame bounds $B\|y_0\|^2$, $A\|y_0\|^2$. Similary, since for all $y \in \mathbf{K}$ and $U \in \mathbf{H} \otimes \mathbf{K}$
$$\langle y, U^* x_0\rangle = \langle x_0, Uy\rangle = \langle x_0 \otimes y, U\rangle,$$

we conclude that $\{F_n^* x_0\}_{n\in I}$ and $\{G_n^* x_0\}_{n\in I}$ are woven frames for Hilbert space $\mathbf{H}$ with frame bounds $B\|x_0\|^2$, $A\|x_0\|^2$. □

In the following, we have proven that the adjoint of woven frames is also woven.

Proposition 3.6. *Let $\{F_n\}_{n\in J}$ and $\{G_n\}_{n\in J}$ be two frames for the Hilbert space $\boldsymbol{H}\otimes\boldsymbol{K}$ then $\{F_n\}_{n\in J}$ and $\{G_n\}_{n\in J}$ are woven with universal frame bounds A, B if and only if $\{F_n^*\}_{n\in J}$ and $\{G_n^*\}_{n\in J}$ are woven frames for $\boldsymbol{K}\otimes\boldsymbol{H}$ with universal frame bounds A, B.*

Proof. Let $\{F_n\}_{n\in J}$ and $\{G_n\}_{n\in J}$ be two woven frames in $\mathbf{H}\otimes\mathbf{K}$ with universal frame bounds A, B then for every $T\in\mathbf{K}\otimes\mathbf{H}$ and arbitrary partition σ of J we have

$$\sum_{n\in\sigma}|\langle T,F_n^*\rangle|^2+\sum_{n\in\sigma^c}|\langle T,G_n^*\rangle|^2=\sum_{n\in\sigma}|\langle T^*,F_n\rangle|^2+\sum_{n\in\sigma^c}|\langle T^*,G_n\rangle|^2$$

since $|||T|||=|||T^*|||$ and $\{F_n\}_{n\in J}$, $\{G_n\}_{n\in J}$ are woven therefore, $\{F_n^*\}_{n\in J}$ and $\{G_n^*\}_{n\in J}$ are woven. Now let $\{F_n^*\}_{n\in J}$ and $\{G_n^*\}_{n\in J}$ be woven in $\mathbf{K}\otimes\mathbf{H}$ its enough to note that $\{F_n^{**}\}_{n\in J}$ and $\{G_n^{**}\}_{n\in J}$ are woven in $\mathbf{H}\otimes\mathbf{K}$ and $F_n^{**}=F_n$ and $G_n^{**}=G_n$. □

Next, we have shown that woven frames in tensor product spaces, with some conditions, preserved under two different bounded invertible operators.

Theorem 3.7. *Let $\{F_n\}_{n\in J}$ and $\{G_n\}_{n\in J}$ be two woven frames for the Hilbert space $\boldsymbol{H}\otimes\boldsymbol{K}$ with universal constants A, B and T_1 , T_2 be two bounded invertible operators in $L(\boldsymbol{H})$.*

If there exists constant $D>0$ such that $D<\dfrac{A(\|T_1\|^2+\|T_2\|^2)}{2}$ and for every $I\subset J$ and $T\in\boldsymbol{H}\otimes\boldsymbol{K}$,

$$\sum_{n\in I}|\langle T,T_2F_n\rangle|^2\leq D|||T|||^2$$

and

$$\sum_{n\in I^c}|\langle T,T_1G_n\rangle|^2\leq D|||T|||^2,$$

then $\{T_1F_n\}_{n\in J}$ and $\{T_2G_n\}_{n\in J}$ are woven.

Proof. Let $T \in \mathbf{H} \otimes \mathbf{K}$ and $\sigma \subset J$ be an arbitrary partition of J, then

$$
\begin{aligned}
&(\sum_{n\in\sigma} |\langle T, T_1F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle T, T_2G_n\rangle|^2 \\
&= \sum_{n\in\sigma} |\langle T_1^*T, F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle T_2^*T, G_n\rangle|^2 \\
&\leq \sum_{n\in J} |\langle T_1^*T, F_n\rangle|^2 + \sum_{n\in J} |\langle T_2^*T, G_n\rangle|^2 \\
&\leq Bmax\{\|T_1\|^2, \|T_2\|^2\}\||T|\|^2.
\end{aligned}
$$

So, upper frame bound is $Bmax\{\|T_1\|^2, \|T_2\|^2\}$. On the other hand,

$$
\begin{aligned}
\||T|\|^2 = \||T_1^{-1}T_1T|\|^2 \leq \|T_1^{-1}\|^2\||T_1^*T|\|^2 \\
&\leq \frac{\|T_1^{-1}\|^2}{A}\left(\sum_{n\in\sigma} |\langle T_1^*T, F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle T_1^*T, G_n\rangle|^2\right) \\
&\leq \frac{1}{A\|T_1\|^2}\left(\sum_{n\in\sigma} |\langle T, T_1F_n\rangle|^2 + D\||T|\|^2\right),
\end{aligned}
$$

therefore,

$$
(A\|T_1\|^2 - D)\||T|\|^2 \leq \sum_{n\in\sigma} |\langle T, T_1F_n\rangle|^2. \tag{3.2}
$$

In a similar way,

$$
(A\|T_2\|^2 - D)\||T|\|^2 \leq \sum_{n\in\sigma^c} |\langle T, T_2G_n\rangle|^2. \tag{3.3}
$$

Via (3.2) and (3.3), we conclude that

$$
\big(A(\|T_1\|^2 + \|T_2\|^2) - 2D\big)\||T|\|^2 \leq \sum_{n\in\sigma} |\langle T, T_1F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle T, T_2G_n\rangle|^2.
$$

Hence, $\{T_1F_n\}_{n\in\sigma} \cup \{T_2G_n\}_{n\in\sigma^c}$ is a frame with lower and upper frame bound $A(\|T_1\|^2 + \|T_2\|^2) - 2D$ and $Bmax\{\|T_1\|^2, \|T_2\|^2\}$. □

In general, frames may be woven without their canonical dual frames being woven [1]. In the following theorem, we have shown that by taking a condition, canonical duals of woven frames in the tensor product of Hilbert spaces can be woven.

Theorem 3.8. *Let $F = \{F_n\}_{n\in J}$ and $G = \{G_n\}_{n\in J}$ be two woven frames for the Hilbert space $\boldsymbol{H} \otimes \boldsymbol{K}$ with universal frame bounds A, B and frame operators S_F and S_G. If $\||S_F^{-1} \otimes (S_F - S_G)|\| < \sqrt{\frac{A}{B}}$ then $\{S_F^{-1}F_n\}_{n\in J}$, $\{S_G^{-1}G_n\}_{n\in J}$ are woven.*

Proof. Let $T \in \mathbf{H} \otimes \mathbf{K}$ and $\sigma \subset J$ be an arbitrary partition of J, then

$$\begin{aligned}
&\left(\sum_{n\in\sigma} |\langle T, S_F^{-1}F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle T, S_G^{-1}G_n\rangle|^2\right)^{1/2} \\
&= \left(\sum_{n\in\sigma} |\langle S_F^{-1}T, F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle S_F^{-1}T + (S_G^{-1} - S_F^{-1})T, G_n\rangle|^2\right)^{1/2} \\
&\geq \left(\sum_{n\in\sigma} |\langle S_F^{-1}T, F_n\rangle|^2 + \sum_{n\in\sigma^c} |\langle S_F^{-1}T, G_n\rangle|^2\right)^{1/2} \\
&\qquad - \left(\sum_{n\in\sigma^c} |\langle (S_G^{-1} - S_F^{-1})T, G_n\rangle|^2)\right)^{1/2} \\
&\geq \sqrt{A}|||S_F^{-1}T||| - \sqrt{B}|||(S_G^{-1} - S_F^{-1})T||| \\
&= \left(\frac{\sqrt{A}}{\|S_F\|} - \sqrt{B}\|S_G^{-1} - S_F^{-1}\|\right)|||T|||.
\end{aligned}$$

The condition $|||S_F^{-1} \otimes (S_F - S_G)||| < \sqrt{\dfrac{A}{B}}$ implies that $(\dfrac{\sqrt{A}}{\|S_F\|} - \sqrt{B}\|S_G^{-1} - S_F^{-1}\|) > 0$, thus $\{S_F^{-1}F_n\}_{n\in\sigma} \cup \{S_G^{-1}G_n\}_{n\in\sigma^c}$ is a frame. □

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WOVEN FRAMES IN TENSOR PRODUCT OF HILBERT SPACES

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قاب‌های بافتنی در ضرب تانسوری فضاهای هیلبرت

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ضرب تانسوری، عنصر اصلی در پردازش سیگنال به منظور گسترش روش‌های یک بعدی فیلتر و فشرده‌سازی به ابعاد بالاتر است. قاب‌های بافتنی در پردازش سیگنال و پردازش داده‌های توزیع شده دارای نقشی اساسی هستند. با این انگیزه، ضرب تانسوری قاب‌های بافتنی را ارائه نموده و تأثیر برخی عملگرها بر روی قاب‌های بافتنی در ضرب تانسوری فضاهای هیلبرت و سایر ویژگی‌های آنها را مورد بررسی قرار می‌دهیم.

کلمات کلیدی: قاب، قاب‌های بافتنی، ضرب تانسوری.